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Das Herausgebergremium hatte sich überlegt, dass es grundsätzlich wünschenswert sei, „Themenhefte“ zu gestalten. Denn auf diese Weise könne eine Thematik aus verschiedenen Perspektiven beleuchtet werden und es ergebe sich ein unmittelbarerer Eindruck von deren Vielseitigkeit und Vielschichtigkeit.

In diesem – und aller Voraussicht nach auch im nächsten – Heft ist der Jahresbericht nun in der glücklichen Situation, dass die druckfertig vorliegenden Manuskripte die Gestaltung eines derartigen „Themenheftes“ gestatten. Im vorliegenden Heft finden Sie zwei Übersichtsartikel zu dem Themenkomplex „Partielle Differentialgleichungen“, der von dem einen Beitrag aus der Perspektive „geometrische Eigenschaften von Lösungen“ und von dem anderen Beitrag aus der Perspektive „Regularität und Abschätzungen von Lösungen“ betrachtet wird.

Stellen Sie sich z.B. die möglichen Eigenschwingungen einer eingespannten kreisförmigen Membran (→ Pauke) vor. Hier kann man das entsprechende Eigenwertproblem (für das Dirichletproblem der Laplacegleichung) explizit analysieren: Das räumliche Profil der Grundschwingung ist positiv und radialsymmetrisch, während man bei der ersten Oberschwingung zwei Bereiche entgegengesetzten Vorzeichens und lediglich noch Axialsymmetrie beobachtet. Die weiteren Oberschwingungen werden nun immer komplizierter und weisen sukzessive ein immer geringeres Maß an Symmetrie auf. Tobias Weth stellt diese experimentellen Beobachtungen in seinem Übersichtsartikel „Symmetry of solutions to variational problems for nonlinear elliptic equations via reflection methods“ in den Kontext einer allgemeinen und gleichermaßen relativ elementaren Theorie in dem Sinne, dass er sich auf Reflektions-

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methoden beschränkt und lediglich auf wohletablierte analytische und topologische Werkzeuge verweisen muss.

Um Existenz von Lösungen für Randwertprobleme bei partiellen Differentialgleichungen zu zeigen, besteht ein typischer Zugang darin, zunächst „a priori-Abschätzungen“ für als existent angenommene Lösungen herzuleiten. Sind diese Abschätzungen hinreichend gut, so ergeben funktionalanalytische Hilfsmittel auch sofort Existenz. Bei linearen „elliptischen“ Gleichungen (Prototyp ist die Laplacegleichung $-\Delta u = f$) heißen die grundlegenden Abschätzungen in Räumen klassisch differenzierbarer Funktionen nach Schauder und in Räumen schwach differenzierbarer Funktionen mit L^{γ}-Integralnormen nach Calderón und Zygmund. Giuseppe Mingione gibt nun einen bemerkenswerten Überblick über „Nonlinear aspects of Calderón-Zygmund theory“. Er betrachtet eine allgemeine Klasse nichtlinearer elliptischer Differentialgleichungen und stellt im L^{γ}-Kontext mittels neuer methodischer Zugänge zahlreiche aktuelle Regularitäts- und Abschätzungsaussagen zusammen, wie man sie vorher oft nur von linearen Problemen her kannte. Zum Einstieg in diese Thematik lässt er die klassischen Tatsachen aus der linearen Theorie zunächst Revue passieren.

Einen Kontrapunkt zu soviel partiellen Differentialgleichungen in diesem Heft bilden die Buchbesprechungen. Hier werden neue Monographien unter anderem aus der Stochastik, der Finanzmathematik und der symplektischen Geometrie vorgestellt. Zwar sind auch hier die partiellen Differentialgleichungen mit einem Titel präsent, der allerdings hyperbolische Gleichungen und diese zudem in zufälligen Medien zum Gegenstand hat.

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ÜBERSICHTSARTIKEL

Symmetry of Solutions to Variational Problems for Nonlinear Elliptic Equations via Reflection Methods

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Abstract We discuss some recent results on symmetry of solutions of nonlinear partial differential equations. We focus on elliptic and degenerate elliptic boundary value problems of second order with variational structure and the simple looking case where the underlying domain is radially symmetric. In this setting, we study solutions which are given as minimizers of constrained minimization problems or have low Morse index, and we examine which amount of symmetry of the data is inherited by these solutions. We highlight how the answer to this general question depends on specific assumptions on the data. The underlying techniques collected in this survey are elementary as they solely rely on hyperplane reflections and well known analytical and topological tools, but they yield surprisingly general results in situations where classical methods do not apply.

Keywords Constrained minimization problems · Morse index · Nonlinear boundary value problems · Foliated Schwarz symmetry · Local symmetry · Hyperplane reflections

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1 Introduction

It is frequently observed in nature that physical objects in low energy states have a fairly simple shape, whereas in energetically excited states their structure becomes more and more complex. However, observations of this type do not lead to a general principle but rather to a rule of thumb, and some exceptions from this rule correspond to highly interesting phenomena. As a consequence, it is important to formulate precise criteria to distinguish different levels of complexity and to check whether these criteria provide theoretical explanations of experimental observations within suitable mathematical models. The most natural way to distinguish simple and complex structures seems to be the analysis of their symmetries. Consider for example the classical fixed membrane eigenvalue problem

$$\begin{cases} -\Delta u = \lambda u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1.1}$$

in a planar bounded domain $\Omega \subset \mathbb{R}^2$. In a simplified model, the solutions u of (1.1) describe (up to sign) the amplitude of a time-periodic oscillation of a membrane fixed at the boundary $\partial\Omega$, and the corresponding eigenvalue λ corresponds to the energy of this oscillation. More precisely, the eigenvalues λ are exactly the critical values of the Dirichlet energy functional $u \mapsto \frac{1}{2}\int_\Omega |\nabla u|^2\,dx$ subject to the constraint $\int_\Omega u^2\,dx = 1$ within a suitable function space (we will be more precise below). It is well known that the eigenvalues of this problem form an unbounded increasing sequence $(\lambda_k)_k$. The first eigenvalue λ_1 is simple, and the corresponding eigenspace is generated by a function φ_1 which is positive in Ω. The uniqueness of this eigenfunction (up to a constant factor) implies that φ_1 inherits all the symmetries of the underlying domain Ω. In particular, in the case where Ω is the unit disc, φ_1 must be radial. In fact, φ_1 is then given explicitly as $\varphi_1(x) = J_0(j_0|x|)$, where J_0 is the Bessel function (of the first kind) of order zero and j_0 is its first zero point, see Fig. 1 (left) below. As a consequence, φ_1 is *Schwarz symmetric*, i.e., it is radial and decreasing in the radial variable. In a celebrated paper, Gidas, Ni and Nirenberg [49] showed in 1979 that this kind of symmetry is shared by every *positive* solution of the more general semilinear equation

$$-\Delta u = f(x, u), \quad \text{in } \Omega \tag{1.2}$$

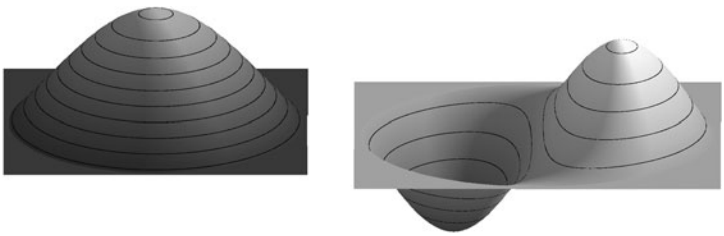

Fig. 1 First and second Dirichlet eigenfunctions of the Laplacian in a disc

which satisfies Dirichlet boundary conditions on $\partial\Omega$, provided that $\Omega \subset \mathbb{R}^n$ is a ball and the nonlinearity f is locally Lipschitz in u, only depends on the radial variable $r = |x|$ and is nonincreasing in r. We note that equations of type (1.2) arise e.g. in conformal geometry, plasma physics, nonlinear optics and mathematical biology (see e.g. [31, 38, 62, 69, 80, 81]), and they have been studied extensively in the last four decades.

The seminal result of Gidas, Ni and Nirenberg relies on the moving plane method which has its roots in earlier work of Alexandrov [3] and Serrin [75]. Despite its importance, this method fails to provide symmetry information for nodal (i.e., sign-changing) solutions and, in each of the following instances, also for positive solutions:

- $\Omega \subset \mathbb{R}^n$ is an annulus or the exterior of a ball;
- f is not locally Lipschitz in u and/or increasing in $|x|$;
- other boundary conditions are considered instead of $u = 0$ on $\partial\Omega$.

In fact, in these cases, solutions having a rather complicated shape have been constructed for suitable data, including solutions with arbitrarily many isolated local maxima. Therefore one is led to study the symmetry problem within restricted classes of solutions characterized by variational information. Let us explain this by going back to the linear eigenvalue problem (1.1). Here classical results give some information on the shape of eigenfunctions corresponding to higher eigenvalues $\lambda_k > \lambda_1$. By the well known Courant nodal domain theorem [30], every eigenfunction corresponding to λ_k has at most k nodal domains. Moreover, the eigenfunctions inherit part of the symmetry of the underlying domain Ω. In particular, assuming again that Ω is the unit disk in $\mathbb{R}^2$, we have an explicit representation of the corresponding two-dimensional eigenspace corresponding to the second eigenvalue λ_2. More precisely, every eigenfunction is a scalar multiple of one of the functions φ_e given by $\varphi_e(x) = \frac{x}{|x|} \cdot e\, J_1(j_1|x|)$ with a unit vector $e \in \mathbb{R}^2$, where $\cdot$ denotes the Euclidean inner product in $\mathbb{R}^2$, J_1 is the Bessel function of order 1 and j_1 is its first zero. Note that the function φ_e is nonradial, but it is symmetric with respect to reflection at the line $\mathbb{R}e \subset \mathbb{R}^2$, and it is decreasing in the angle $\theta = \arccos[\frac{x}{|x|} \cdot e]$ for $\theta \in (0, \pi)$. These properties can be seen as a spherical version of Schwarz symmetry along the foliation of the underlying disc by circles. Therefore this symmetry has been called *foliated Schwarz symmetry* in a large part of the literature, and we will stick to this name in the present survey.

At first glance, it seems too optimistic to expect that variational characteristics of solutions of the nonlinear problem (1.2) are similarly closely related to their geometric properties as in the case of eigenfunctions of the fixed membrane problem. Let us quickly recall the variational structure of (1.2). Assuming that f satisfies appropriate growth and regularity assumptions, (1.2) is the Euler-Lagrange equation of the energy functional

$$\Phi : H \to \mathbb{R}, \quad \Phi(u) = \int_\Omega \left(\frac{1}{2}|\nabla u|^2 - F(x,u) \right) dx,$$

which means that solutions of (1.2) are precisely the critical points of Φ. Here $F(x,u) = \int_0^u f(x,\tau)\, d\tau$, and H is a first order Sobolev space chosen in accordance with the boundary conditions. Speaking of variational properties of a solution u of (1.2), we refer to any information on the position of $u \in H$ with respect to the 'energy landscape' of Φ. Such information could be a minimax characterization of the corresponding critical Φ-value, an estimate of the Morse index of u with respect to Φ or the fact that u has been obtained by minimization subject to additional constraints. We recall that, roughly speaking, the Morse index of a critical point u of Φ is the maximal number of orthogonal directions in which, starting from u, Φ is locally strictly decreasing. As an example of the relationship between variational and geometric properties, we mention a variant of Courant's nodal domain theorem stating that—for a fairly large class of superlinear nonlinearities f—the Morse index of a solution u of (1.2) controls the number of its nodal domains. This was observed by Benci and Fortunato [14].

The relationship between variational and geometric properties has attracted growing interest in recent years but is still far from being understood. In the present survey we focus on symmetry results for (1.2) relying on the variational framework. More precisely, we consider radial domains $\Omega \subset \mathbb{R}^n$ and radially symmetric nonlinearities f, and we examine which amount of symmetry of Ω is inherited by solutions with given variational characteristics. We do this for (1.2) and in part also for the quasilinear generalization

$$-\Delta_p u = f(x, u), \quad x \in \Omega, \tag{1.3}$$

where $\Delta_p = \nabla \cdot (|\nabla u|^{p-2} \nabla u)$ is the p-Laplace operator for $p > 1$.

As highlighted by the already mentioned symmetry result of Gidas, Ni and Nirenberg, positive solutions of semilinear boundary value problems are much better understood than sign changing ones. This is unsatisfactory since in general (1.2) admits many nodal solutions, and they appear in highly interesting problems. In particular, nodal solutions play a key role in describing segregation phenomena in elliptic and parabolic systems with a competitive coupling [29, 37, 38]. Moreover, nodal solutions of (1.2) with $f(x, u) = \sinh(u)$ arise in the study of counter-rotating vortices in planar Euler flows [9, 10]. A further example is given by extremal functions in Poincaré-Sobolev type inequalities [51] which we will discuss in detail in Sect. 6.1 below. As summarized in the recent survey [11], so far mainly questions concerning the existence and multiplicity of nodal solutions have been studied. The challenge in studying qualitative properties of nodal solutions is that many techniques available for positive solutions—e.g. the moving plane method—do not work anymore.

In the present survey we focus on symmetry results obtained with methods relying on hyperplane reflections and related transformations. Most of these results apply to positive and to nodal solutions. The main ingredients of the underlying methods are geometric characterizations of symmetries and rather well known tools from partial differential equations like the maximum principle (and its variants), the unique continuation principle and regularity theory for solutions of (degenerate) elliptic equations. We will also make use of a topological tool: the Borsuk-Ulam Theorem (see Theorem 6.21 below). It seems that—in the context of symmetry problems for (1.2)—this theorem was first used by Pacella and the author [72] and more recently by Mariş [67].

One purpose of the present survey is to stress the common concepts of a large part of the recent 'hyperplane reflection methods' introduced and elaborated in

[13, 22–24, 52, 64, 65, 67, 72, 77, 84–86]. The classical moving plane method—which is also based on hyperplane reflections—does not play a prominent role in the present survey, and we refer the reader to [18] for a concise account on this method.

For some results discussed here we will give rather detailed proofs, while for others we refer to an extended version of this survey, see [89]. This is done since we also consider generalizations and variants of recent results. Moreover, some proofs given in the literature are not self contained and do not seem to be well known, whereas a quite elementary presentation is possible in the framework of this survey.

The article is organized as follows. In Sect. 2 we review different notions of symmetry and discuss possible characterizations via hyperplanes and half spaces. For the sake of consistency Sect. 3 is devoted to polarization, a simple rearrangement relative to half spaces which is extremely useful in the context of symmetry problems. In particular, following and extending work of Brock [23], we will characterize notions of *local symmetry* via polarization. Moreover, we review results of van Schaftingen [84–86] on universal approximations of symmetrizations by polarization. In Sect. 6 we apply the tools of the preceding sections to constrained minimization problems which admit minimizers solving equations of type (1.2) and (1.3). We give special attention to problems where the corresponding minimizers change sign and therefore classical methods do not work. In this section we also sketch an interesting recent approach of Mariş [67]—partly based on ideas of Lopes [64]—which makes use of hyperplane reflections in a different way than the methods presented before. Finally, in Sect. 7 we review results obtained in joint work with Gladiali and Pacella [52, 72] on solutions of (1.2) with Morse index bounds.

We emphasize that the present survey is solely devoted to symmetry of solutions in radial domains via reflection methods and related transformations. For a discussion of symmetry, rearrangements and symmetrization in a broader context including domain-dependent problems, we refer the reader to [18, 48, 54, 56, 58–60, 69] and the references therein. Even within the present restricted framework, the survey is far from complete. We therefore briefly mention some recent developments which we had to leave out. In [83, 84], van Schaftingen proved the existence of symmetric critical points located on the minimax level corresponding to Liusternik-Schnirelman or linking type characterization. In the context of elliptic systems, reflection methods were applied e.g. in [23, 64, 65, 67], whereas Lopes and Mariş also considered integro-differential equations [66]. Finally, in joint work with Gazzola, Berchio and Ferrero [15, 47] we studied symmetry via polarization for higher order semilinear boundary value problems.

1.1 General Notation

Throughout the paper, $B_r(x)$ denotes the open ball centered at $x \in \mathbb{R}^N$. Moreover, S_r denotes the r-sphere centered at zero, and in the special case of the unit sphere we write $\mathcal{S}$ in place of S_1.

Let $\mathcal{P}$ denote the set of all *affine* hyperplanes in $\mathbb{R}^N$ and $\mathcal{P}_0$ the set of all hyperplanes, i.e. the set of all $T \in \mathcal{P}$ containing the origin. For $T \in \mathcal{P}$ we denote by $R_T : \mathbb{R}^N \to \mathbb{R}^N$ the reflection at T.

We also consider the set $\mathcal{H}$ of all *open* affine half spaces in $\mathbb{R}^N$ and $\mathcal{H}_0$ the subset of all $H \in \mathcal{H}$ with $0 \in \partial H$. For given $H \in \mathcal{H}$ we write $R_H : \mathbb{R}^N \to \mathbb{R}^N$ instead of $R_{\partial H}$

for the reflection at the hyperplane ∂H, and we let $\widehat{H} \in \mathcal{H}$ denote the complementary half space, i.e. $\widehat{H} = R_H(H) = \mathbb{R}^N \setminus \overline{H}$. We also set

$$\mathcal{H}(p) := \{H \in \mathcal{H} : p \in H\} \quad \text{and} \quad \mathcal{H}_0(p) := \{H \in \mathcal{H}_0 : p \in H\} \quad \text{for } p \in \mathbb{R}^N.$$

Given a nonzero vector $e \in \mathbb{R}^N \setminus \{0\}$, we write $T(e) = e^\perp$ for the perpendicular hyperplane and $H(e) = \{x \cdot e > 0\} \in \mathcal{H}_0$ for the half space of vectors which form an acute angle with e. Here and in the following, $\cdot$ denotes the Euclidean inner product in $\mathbb{R}^N$. In this situation we also write R_e in place of $R_{T(e)}$ for the reflection at $T(e)$. Moreover, we write $\Omega(e) = H(e) \cap \Omega$ for subsets $\Omega \subset \mathbb{R}^N$.

Next we consider functions $u : \Omega \to \mathbb{R}$, where $\Omega \subset \mathbb{R}^N$. We write $u^+ := \max\{u, 0\}$ and $u^- := \min\{u, 0\}$ for the positive and negative part of u, so that $u = u^+ + u^-$. For $c \in \mathbb{R}$, $u_c := \{x \in \Omega : u(x) \geq c\}$ denotes the *c-superlevel set* of u. We call a hyperplane $T \in \mathcal{P}$ a *symmetry hyperplane for u* if $\Omega = R_T(\Omega)$ and $u \equiv u \circ R_T$. Moreover, we call a half space $H \in \mathcal{H}$ *dominant for u* if $u(x) \geq u(R_H x)$ for all $x \in \Omega \cap H$ with $R_H x \in \Omega$, and we call it *subordinate for u* if $u(x) \leq u(R_H x)$ for all $x \in \Omega \cap H$ with $R_H x \in \Omega$. If u is defined on $\Omega = \mathbb{R}^N$, we write $z * u : \mathbb{R}^N \to \mathbb{R}$ for the translation of u with respect to $z \in \mathbb{R}^N$, i.e., $[z * u](x) = u(x - z)$.

We use standard notation for function spaces such that Lebesgue and Sobolev spaces. We will mainly be working with the first order Sobolev spaces $W^{1,p}(\Omega)$, $W^{1,p}_{loc}(\Omega)$ and $W^{1,p}_0(\Omega)$, see e.g. [50].

In the remainder of this survey, unless stated otherwise, we will always assume that $\Omega \subset \mathbb{R}^N$ *is a radial domain, so either* Ω *is* $\mathbb{R}^N$, *a ball, an annulus, or the exterior of a ball centered at the origin.*

2 Symmetry via Hyperplanes, Half Spaces and Reflections

In this section we discuss different types of symmetry and their characterizations using hyperplanes and reflections.

2.1 Radial Symmetry

A function $u : \Omega \to \mathbb{R}$ is called radial (or radially symmetric) if $u(x) = u(y)$ for every $x, y \in \Omega$ with $|x| = |y|$, $x \neq y$. Since for any such points there exists $T \in \mathcal{P}_0$ with $R_T x = y$, we find that

$$u : \Omega \to \mathbb{R} \text{ is radial if and only if every } T \in \mathcal{P}_0 \text{ is a symmetry hyperplane for } u. \tag{2.1}$$

Next we consider a subspace $V \subset \mathbb{R}^N$, and we call a function $u : \Omega \to \mathbb{R}$ *radial with respect to* V if $u(x)$ only depends on the orthogonal projection of x onto V and the distance of x to V, which means that $u(x) = u(y)$ for every $x, y \in \mathbb{R}^N$ which have the same distance to V and satisfy $x - y \in V^\perp$. Similarly as in (2.1) we then have:

$$\begin{aligned}&u \text{ is radial with respect to } V \text{ if and only if}\\ &\text{every hyperplane } T \text{ containing } V \text{ is a symmetry hyperplane for } u.\end{aligned} \tag{2.2}$$

Let us briefly consider some special cases. In case $V = \{0\}$ we recover the usual notion of radial symmetry. If $V = \mathbb{R}p$ for some $p \in \mathbb{R}^N \setminus \{0\}$, we also say that u is axially symmetric with respect to the axis $\mathbb{R}p$. If V is a hyperplane, then radial symmetry with respect to V is just reflection symmetry with respect to V, while radial symmetry with respect to $V = \mathbb{R}^N$ is an empty condition. In case V has codimension at least two in $\mathbb{R}^N$, we can express radial symmetry with respect to V in terms of normal derivatives at hyperplanes. This fact is used in [67], where a proof of the following basic observation is given.

Lemma 2.1 *For $u \in C^1(\Omega)$ and a subspace $V \subset \mathbb{R}^N$ with* $\dim V \leq N-2$ *the following are equivalent.*

(i) *u is radially symmetric with respect to V.*
(ii) *For every hyperplane $T \in \mathcal{P}$ containing V we have $u_\nu \equiv 0$ on T, where u_ν denotes the normal derivative of u at T.*

2.2 Schwarz Symmetry

A function $u : \Omega \to \mathbb{R}$ is said to be *Schwarz symmetric* if for every $c \in \mathbb{R}$ the superlevel set u_c of u is equal to Ω or the intersection of Ω with a ball centered at zero. In other words, Schwarz symmetric functions are radial and nonincreasing in the radial variable. Note that this notion is slightly more general than in the literature since it also includes functions defined in an annulus or the exterior of a ball. Schwarz symmetry can be characterized easily via half spaces. For simplicity we restrict our attention to continuous functions.

Lemma 2.2 *A continuous function $u : \Omega \to \mathbb{R}$ is Schwarz symmetric if and only if every half space $H \in \mathcal{H}(0)$ is dominant for u.*

Proof We note that for every pair of points $x, y \in \Omega$ with $|x| < |y|$ there exists precisely one half space $H \in \mathcal{H}(0)$ containing x and such that $R_H x = y$. Henve every $H \in \mathcal{H}(0)$ is dominant for u if and only if $u(x) \geq u(y)$ for every $x, y \in \Omega$ with $|x| < |y|$. By continuity of u, this is true if and only if $u(x) \geq u(y)$ for every $x, y \in \Omega$ with $|x| \leq |y|$, and the latter property is obviously equivalent to the Schwarz symmetry of u. □

For functions defined in the entire space, it is sometimes useful to extend the notion of Schwarz symmetry as follows. We call a function $u : \mathbb{R}^N \to \mathbb{R}$ *Schwarz symmetric up to translation* if the translated function $z * u : \mathbb{R}^N \to \mathbb{R}$ is Schwarz symmetric for some $z \in \mathbb{R}^N$. We have the following useful characterization.

Proposition 2.3 *Let $u : \mathbb{R}^N \to \mathbb{R}$ be continuous and such that* $\lim_{|x|\to\infty} u(x) \in \mathbb{R} \cup \{\pm\infty\}$ *exists. Then the following statements are equivalent*:

(i) *u or $-u$ is Schwarz symmetric up to translation.*
(ii) *Every half space $H \in \mathcal{H}$ is dominant or subordinate for u.*

The fact that (i) implies (ii) is an immediate consequence of Lemma 2.2, but the other implication is not so obvious, see [89] for a proof. Here we need the extra assumption on the existence of the limit. Indeed, if $g : \mathbb{R} \to \mathbb{R}$ is monotone but not constant, then the function $u : \mathbb{R}^n \to \mathbb{R}$, $u(x) = g(x_1)$ satisfies (ii) but is not Schwarz symmetric up to translation.

The result can be generalized slightly; one can replace the assumption on the continuity of u in Proposition 2.3 by (upper or lower) semicontinuity. This however is crucial; consider the function $u : \mathbb{R} \to \mathbb{R}$ defined by $u \equiv 1$ on $(-1, 1]$ and $u \equiv 0$ elsewhere. Then u has property (ii) of Proposition 2.3 but is not Schwarz symmetric up to translation.

2.3 Foliated Schwarz Symmetry

Let $N \geq 2$ in this section, and let $p \in \mathcal{S}$ be a unit vector. A function $u : \Omega \to \mathbb{R}$ will be called *foliated Schwarz symmetric with respect to* p if, for every $r > 0$ and $c \in \mathbb{R}$, the restricted superlevel set $\{x \in S_r : u(x) \geq c\}$ is equal to S_r or a geodesic ball in the sphere S_r centered at rp. Hence u is axially symmetric with respect to the axis $\mathbb{R}\, p$ and nonincreasing in the polar angle $\theta = \arccos[\frac{x}{|x|} \cdot p]$. We simply call u *foliated Schwarz symmetric* if u has this property for some unit vector p. The name 'foliated Schwarz symmetric' was introduced in [77] and refers to the foliation of Ω by spheres with the same center. Alternatively, one could use the notion 'cap symmetric' to stress the relationship with spherical cap symmetrization, see Sect. 5 below. We have the following elementary characterization of foliated Schwarz symmetry.

Proposition 2.4 *Let* $u : \Omega \to \mathbb{R}$ *be a continuous function.*

(i) u *is foliated Schwarz symmetric with respect to* $p \in \mathcal{S}$ *if and only if every half space* $H \in \mathcal{H}_0(p)$ *is dominant for* u.
(ii) u *is foliated Schwarz symmetric if and only if every half space* $H \in \mathcal{H}_0$ *is dominant or subordinate for* u.

Part (i) is proved similarly as Lemma 2.2, using now the fact that for every two points $x, y \in \Omega$ with $|x| = |y|$ and $|x - p| < |y - p|$ there exists precisely on half space $H \in \mathcal{H}_0(p)$ with $R_H x = y$. Part (ii) is stated in [22, Lemma 4.2] under slightly stronger assumptions on u. Since the proof given there is not completely self contained, we also give a proof in [89].

As a byproduct of Propositions 2.3 and 2.4(ii), we have the following.

Corollary 2.5 *Let* $u : \mathbb{R}^N \to \mathbb{R}$ *be a continuous function such that* $\lim_{|x| \to \infty} u(x) \in \mathbb{R} \cup \{\pm\infty\}$ *exists. Suppose moreover that for every* $z \in \mathbb{R}^N$ *the translated function* $z * u : \mathbb{R}^N \to \mathbb{R}$ *is foliated Schwarz symmetric. Then* u *or* $-u$ *is Schwarz symmetric up to translation.*

Proof By assumption and Proposition 2.4(ii), for every $z \in \mathbb{R}^N$, every $H \in \mathcal{H}_0$ is dominant or subordinate for the translated function $z * u$. This yields that every $H \in \mathcal{H}$ is dominant or subordinate for u. Hence Proposition 2.3 implies that u or $-u$ is Schwarz symmetric up to translation. □

As in Proposition 2.3, the assumption concerning the existence of the limit for $|x| \to \infty$ is crucial in Corollary 2.5, as can be seen by considering $u(x) = g(x_1)$, where g is a monotone but non-constant function.

3 Polarization

In this section we discuss polarization, a simple rearrangement which appeared already more than 50 years ago as a set transformation in a paper of Wolontis [91] on a planar capacity problem. In a similar context it was used by Ahlfors [2]. Baernstein and Taylor [7] introduced polarization for functions, and in [6] Baernstein applied it to derive general integral inequalities associated with different types of symmetrizations. Moreover, Dubinin [42–44] and Solynin [78] studied capacity problems in higher dimensions with the help of polarization. In their seminal paper [24], Brock and Solynin applied polarization to variational problems related to nonlinear partial differential equations, and from then on it has been used extensively in this context, see e.g. [13, 15, 22, 23, 47, 77, 83–85] and the references therein.

Given an affine half space $H \in \mathcal{H}$, the *polarization* $u_H : \mathbb{R}^N \to \mathbb{R}$ of a function $u : \mathbb{R}^N \to \mathbb{R}$ with respect to H is defined by

$$u_H(x) = \begin{cases} \max\{u(x), u(R_H x)\}, & x \in H, \\ \min\{u(x), u(R_H x)\}, & x \in \mathbb{R}^N \setminus H. \end{cases}$$

This rearrangement is clearly related to the notions of dominant and subordinate half spaces introduced in Sect. 2. More precisely, the half space H is dominant for u if and only if u coincides with u_H, and this is true if and only if the complementary half space $\widehat{H}$ is subordinate for u. One goal of this survey is to illustrate the usefulness of polarization in the analysis of symmetries of solutions to variational problems associated with integral functionals. We will consider problems of this type in Sect. 6 below. For this we need the following invariance properties of polarization.

Lemma 3.1 *Let $u : \mathbb{R}^N \to \mathbb{R}$ be a measurable function and $H \in \mathcal{H}$.*

(i) *If $F : \mathbb{R} \to \mathbb{R}$ is a continuous function such that $\int_{\mathbb{R}^N} |F(u)|\, dx < \infty$, then $\int_{\mathbb{R}^N} F(u_H)\, dx = \int_{\mathbb{R}^N} F(u)\, dx$.*

(ii) *If $u \in W^{1,1}_{loc}(\mathbb{R}^N)$, then $u_H \in W^{1,1}_{loc}(\mathbb{R}^N)$. If in addition $G : \mathbb{R} \times [0, \infty) \to \mathbb{R}$ is a continuous function such that $\int_{\mathbb{R}^N} |G(u, |\nabla u|)|\, dx < \infty$, then $\int_{\mathbb{R}^N} G(u_H, |\nabla u_H|)\, dx = \int_{\mathbb{R}^N} G(u, |\nabla u|)\, dx$.*

We note that (i) is a general consequence of Cavalieri's principle and the equimeasurability of rearrangements, but in the case of polarization it can also be proved by a simple change of variables. It is a nice feature of polarization that the proof of (ii) is similarly easy, although the integrands depend on the gradient.

Proof of Lemma 3.1(ii) If $u \in W^{1,1}_{loc}(\mathbb{R}^N)$, then the restriction of u_H to H is in $W^{1,1}_{loc}(H)$ since it is the maximum of two $W^{1,1}_{loc}$-functions. Similarly, the restriction of u_H to $\widehat{H}$ belongs to $W^{1,1}_{loc}(\widehat{H})$. Moreover, both restrictions have a common trace on ∂H, so it follows that $u_H \in W^{1,1}_{loc}(\mathbb{R}^N)$. Now let $M \subset \mathbb{R}^N$ denote the coincidence set of u and u_H. Then $\nabla u_H = \nabla u$ almost everywhere on M, whereas $u_H = u \circ R_H$ and $\nabla u_H = R_T \circ \nabla u \circ R_H$ almost everywhere on $\mathbb{R}^N \setminus M$, where T is the hyperplane in $\mathcal{P}_0$ parallel to ∂H. Since the set M is symmetric with respect to R_H, we conclude that

$$\int_{\mathbb{R}^N} G(u_H, |\nabla u_H|)\,dx = \int_M G(u, |\nabla u|)\,dx + \int_{\mathbb{R}^N \setminus M} G(u \circ R_H, |\nabla u \circ R_H|)\,dx$$
$$= \int_{\mathbb{R}^N} G(u, |\nabla u|)\,dx. \qquad \square$$

A rather immediate consequence of Lemma 3.1 is the following.

Corollary 3.2 *Let* $1 \le p < \infty$, $u \in W^{1,p}(\mathbb{R}^N)$ *and* $H \in \mathcal{H}$. *Then* $u_H \in W^{1,p}(\mathbb{R}^N)$, *and*

$$\|\nabla u_H\|_{L^p(\mathbb{R}^N)} = \|\nabla u\|_{L^p(\mathbb{R}^N)}, \qquad \|\nabla u_H^{\pm}\|_{L^p(\mathbb{R}^N)} = \|\nabla u^{\pm}\|_{L^p(\mathbb{R}^N)}. \tag{3.1}$$

Proof Applying Proposition 3.1 with $F(t) = |t|^p$ and $G(t, s) = |s|^p$, we infer that $u_H \in W^{1,p}(\mathbb{R}^N)$ and the first equality in (3.1) holds. The other equalities follow from the first one applied to $u^{\pm}$, since by definition of polarization we have $(u_H)^{\pm} = (u^{\pm})_H$ pointwise on $\mathbb{R}^N$. Here it is essential that we use the definition $u^- = \min\{u, 0\}$. $\square$

Remark 3.3 (i) (Extension to functions defined on radial subdomains) If $\Omega \subset \mathbb{R}^N$ is a radial subdomain and $H \in \mathcal{H}_0$, then Ω is symmetric with respect to R_H. Hence we may define the polarization of a function $u : \Omega \to \mathbb{R}$ with respect to H simply by

$$u_H(x) = \begin{cases} \max\{u(x), u(R_H x)\}, & x \in \Omega \cap H, \\ \min\{u(x), u(R_H x)\}, & x \in \Omega \setminus H. \end{cases}$$

It is easy to see that analogues of Lemma 3.1 and Corollary 3.2 hold in this setting. Moreover, if Ω is a ball in $\mathbb{R}^N$ and $H \in \mathcal{H}$ is an affine half space, then we can also define $u_H : \Omega \to \mathbb{R}$ as the polarization of the trivial extension of u to $\mathbb{R}^N$ restricted to Ω. However, in this case analogues of Lemma 3.1(ii) and Corollary 3.2 only hold for half spaces in $\mathcal{H}(0)$ and *nonnegative functions* $u : \Omega \to \mathbb{R}$ satisfying Dirichlet boundary conditions in weak sense. More precisely, if $H \in \mathcal{H}(0)$, then Lemma 3.1 resp. Corollary 3.2 hold for nonnegative functions $u \in W_0^{1,1}(\Omega)$, $u \in W_0^{1,p}(\Omega)$, respectively, and then the polarized functions are also in $W_0^{1,1}(\Omega)$ resp. $W_0^{1,p}(\Omega)$.

(ii) The invariance properties stated in Lemma 3.1 also hold for integral functionals with x-dependence provided that the integrands are symmetric with respect to the reflection R_H. Moreover, integrands satisfying pointwise reflection inequalities lead to inequalities between the functional values for u and u_H, see e.g. [83, Proposition 2.19]. In this survey, we restrict our attention to integrands which do not depend (explicitly) on the space variable. Moreover, we merely remark that polarization also gives rise to integral inequalities of convolution type, see e.g. [6].

(iii) It is obvious from the definition that the existence of derivatives in the strong sense are in general *not* preserved by polarization. In particular, if $u \in C^1(\mathbb{R}^N)$, then u_H does not need to belong to $C^1(\mathbb{R}^N)$. In fact, if $u \in C^1(\mathbb{R}^N)$ is a function such that $u_H \in C^1(\mathbb{R}^N)$ for *every* half space $H \in \mathcal{H}$, then u has some form of local symmetry which will be analyzed in Sect. 4. Similarly, higher order weak derivatives are also not preserved by polarization. In particular, if $u \in W^{m,2}(\mathbb{R}^N)$ and $m \geq 2$, then u_H does not need to be in $W^{m,2}(\mathbb{R}^N)$. So at first glance it is surprising that in [15] polarization has also been applied to derive foliated Schwarz symmetry of a class of solutions to higher order Dirichlet problems of the type

$$(-\Delta)^m u = f(|x|, u) \quad \text{in } B_1(0), \qquad u = \frac{\partial u}{\partial r} = \cdots = \frac{\partial^{m-1} u}{\partial r^{m-1}} = 0 \quad \text{on } \partial B_1(0). \tag{3.2}$$

This is possible since, instead of working in a subspace of $W^{m,2}(\mathbb{R}^N)$, one can transform (3.2) into an integral equation posed in $L^p(\mathbb{R}^N)$ for some $p > 1$, using the Dirichlet-Greenfunction of the polyharmonic operator $(-\Delta)^m$. We refer the reader to [15] for details.

4 Local Symmetry via Polarization

In this section we restrict our attention to C^1-functions. Following and extending work of Brock [23], we discuss notions of *local symmetry*. We will see that these notions, although somewhat strange at first glance, appear naturally in constrained minimization problems (see e.g. Example 6.6 below). Moreover, as has been observed by Brock [23] in the case of local Schwarz symmetry, they can be characterized in a simple way via polarization, see Proposition 4.3 below.

Definition 4.1 Let $u \in C^1(\mathbb{R}^N)$.

(i) u is called *locally Schwarz symmetric* if, for every $c \in \mathbb{R}$, the superlevel set u_c is $\mathbb{R}^N$ or a ball in $\mathbb{R}^N$, and $|\nabla u|$ is constant on ∂u_c.
(ii) u is called *locally foliated Schwarz symmetric* if, for every $c \in \mathbb{R}$ and $r > 0$, the restricted superlevel set $\{x \in S_r : u(x) \geq c\}$ is a geodesic ball in S_r and $|\nabla_{S_r} u|$ is constant on the relative boundary of this set in S_r.

Here and in the following, $\nabla_{S_r} u$ denotes the projection of ∇u onto the tangent bundle of S_r, i.e.

$$\nabla_{S_r} u(x) = \nabla u(x) - \frac{\nabla u(x) \cdot x}{r^2} x \quad \text{for } x \in S_r.$$

The definition of local Schwarz symmetry is extracted from Brock [23, p. 232], although it is not stated explicitly there. It is stronger than the notion of local symmetry given in other papers of Brock (see [20, 21]), which includes functions with superlevel sets given as disjoint unions of balls. Concerning the shape of locally symmetric functions, the following can be shown.

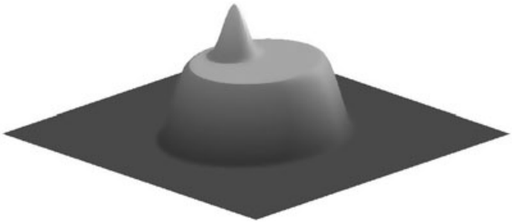

Fig. 2 A locally Schwarz symmetric function which is not radially symmetric

Proposition 4.2 *Let* $u \in C^1(\mathbb{R}^N)$.

(i) *If* $N \geq 2$, *u is locally Schwarz symmetric and A is a connected component of the set* $\{x \in \mathbb{R}^N : \nabla u(x) \neq 0\}$, *then A is radially symmetric with respect to some* $z \in \mathbb{R}^N$, *and the function* $u|_A : A \to \mathbb{R}$ *is radially symmetric and strictly decreasing in the distance to z.*

(ii) *If* $N \geq 3$, *u is locally foliated Schwarz symmetric and A is a connected component of the set* $\{x \in S_r : \nabla_{S_r} u(x) \neq 0\}$ *for some* $r > 0$, *then A is axially symmetric with respect to the axis* $\mathbb{R}p$ *for some unit vector* $p \in \mathbb{R}^N$, *and the function* $u|_A : A \to \mathbb{R}$ *is axially symmetric w.r.t.* $\mathbb{R}\,p$ *and strictly decreasing in the polar angle* $\theta = \arccos[\frac{x}{|x|} \cdot p]$.

The proof of Part (i) is sketched in [23, Corollary 1]. Part (ii) is proved along the same lines, see [89] for details. The following simple criterion for local symmetry in terms of polarization is the main result of this section. We point out that Part (i) is essentially due to Brock [23], although it is not stated in the same form in [23].

Proposition 4.3 *Let* $u \in C^1(\mathbb{R}^N)$.

(i) *If* $u_H \in C^1(\mathbb{R}^N)$ *for every* $H \in \mathcal{H}$, *then every nonempty superlevel set* u_c *is either* $\mathbb{R}^N$, *a ball, a hyperplane or the exterior of a ball, and* $|\nabla u|$ *is constant on* ∂u_c.
If furthermore $c_\infty = \lim_{|x|\to\infty} u(x) \in \mathbb{R} \cup \{-\infty\}$ *exists and* $u \geq c_\infty$ *on* $\mathbb{R}^N$, *then u is locally Schwarz symmetric.*

(ii) *If* $u_H \in C^1(\mathbb{R}^N)$ *for every* $H \in \mathcal{H}_0$, *then u is locally foliated Schwarz symmetric.*

The proof of this characterization relies on the following geometric lemma. Here we recall that for $v \in \mathbb{R}^N \setminus \{0\}$ we write $R_v : \mathbb{R}^N \to \mathbb{R}^N$ for the reflection at the hyperplane $T(v) = v^\perp$, i.e.,

$$R_v\, x = x - 2\frac{v \cdot x}{|v|^2} v \quad \text{for } x \in \mathbb{R}^N.$$

Lemma 4.4

(i) (*See* [23, *Lemma R*]) *Let* $U \subset \mathbb{R}^N$ *be a nonempty open set with* C^1*-boundary* Γ, *and let* $\nu(x)$ *denote the exterior normal to U at* $x \in \Gamma$. *If*

$$\nu(y) = R_{y-z}\, \nu(z) \quad \textit{for all } \; y, z \in \Gamma,\ y \neq z, \tag{4.1}$$

then U is either a half space, a ball or the exterior of a ball in $\mathbb{R}^N$.

(ii) *Let* $S \subset \mathbb{R}^N$ *be a sphere* (*not necessarily centered at the origin*) *and* $U \subset S$ *be a nonempty open set with* C^1*-boundary* Γ *satisfying* (4.1), *where* $\nu(x) \in T_x S$ *denotes the exterior normal to* U *at* $x \in \Gamma$. *Then* U *is a geodesic ball in* S.

Proof (i) and (ii) can be proved simultaneously. The proof is similar to the one of (i) given by Brock in [23, Lemma R], the modifications being a matter of taste. If the normal ν is constant on Γ, then U is a half space in case (i) and a hemisphere in case (ii). Suppose now that $\nu(y_1) \neq \nu(y_2)$ for some points $y_1, y_2 \in \Gamma$. By (4.1) we have $\nu(y_1) \cdot (y_2 - y_1) = \nu(y_2) \cdot (y_1 - y_2) \neq 0$. Hence the lines $t \mapsto y_1 + t\nu(y_1)$ and $t \mapsto y_2 + t\nu(y_2)$ intersect precisely in one point given by

$$w = y_1 + \frac{|y_1 - y_2|^2}{2\nu(y_1) \cdot (y_2 - y_1)} \nu(y_1) = y_2 + \frac{|y_1 - y_2|^2}{2\nu(y_2) \cdot (y_1 - y_2)} \nu(y_2) \in \mathbb{R}^N,$$

and $|y_1 - w| = |y_2 - w|$. We may assume that $w = 0$, so that $|y_1| = |y_2| =: r$. Now consider arbitrary $x \in \Gamma \setminus \{y_1, y_2\}$; it then suffices to show $|x| = r$. Since $y_i = r\nu(y_i)$ for $i = 1, 2$ or $y_i = -r\nu(y_i)$ for $i = 1, 2$, equation (4.1) implies that $R_{y_i - x}\, y_i = r\nu(x)$ for $i = 1, 2$ or $R_{y_i - x}\, y_i = -r\nu(x)$ for $i = 1, 2$. Hence the values

$$R_{y_i - x}\, y_i - x = \frac{|y_i - x|^2 - 2y_i \cdot (y_i - x)}{|y_i - x|^2}(y_i - x) = \frac{|x|^2 - r^2}{|y_i - x|^2}(y_i - x), \quad i = 1, 2,$$

coincide. If $|x| \neq r$, then $\frac{y_1 - x}{|y_1 - x|^2} = \frac{y_2 - x}{|y_2 - x|^2}$ and therefore $y_1 = y_2$, contrary to the choice of y_1 and y_2. Hence $|x| = r$, as claimed. □

We may now complete the

Proof of Proposition 4.3 (i) Fix $x, y \in \mathbb{R}^N$ with $x \neq y$ and $u(x) = u(y)$. Let $H \in \mathcal{H}$ be such that $x \in H$ and $R_H x = y$. Moreover, let $v = u \circ R_H$, so that in a neighborhood of x the function u_H is the maximum of u and v. In this situation it is well known that one-sided directional derivatives of u_H at x satisfy

$$\lim_{\substack{t \to 0 \\ t > 0}} \frac{u_H(x + td) - u_H(x)}{t} = \max\{\nabla u(x) \cdot d, \nabla v(x) \cdot d\} \quad \text{for } d \in \mathbb{R}^N.$$

Since by assumption u_H is differentiable at x, this implies that $\nabla u(x) = \nabla v(x)$, while by definition $\nabla v(x) = R_{x-y}\, \nabla u(y)$. Hence we have

$$\nabla u(x) = R_{x-y}\, \nabla u(y) \quad \text{for every } x, y \in \mathbb{R}^N \text{ with } x \neq y \text{ and } u(x) = u(y). \tag{4.2}$$

Now, for arbitrary $c \in \mathbb{R}$, (4.2) implies that

$$|\nabla u(x)| = |\nabla u(y)| \quad \text{for } x, y \in \partial u_c \text{ with } x \neq y, \tag{4.3}$$

so that $|\nabla u|$ is constant on ∂u_c. From this we easily deduce that the regular values of u are dense in $\mathbb{R}$ (note that this does not follow from Sard's Lemma since we only assume that u is a C^1-function). Now if c is a regular value of u, then (4.2) and (4.3) yield

$$\nu(x) = R_{x-y}\,\nu(y) \quad \text{for } x, y \in \partial u_c \text{ with } x \neq y,$$

where $\nu(x) = -\frac{\nabla u(x)}{|\nabla u(x)|}$ is the exterior normal to u_c at $x \in \partial u_c$. Hence the asserted shape of u_c is a consequence of Lemma 4.4(i).

If c is a singular value of u, then $u_c = \bigcap_{i=1}^{\infty} u_{c_i}$ for an increasing sequence $(c_i)_i$ of regular values converging to c, and therefore we also get the asserted shape.

Finally, if $u(x) \to c_\infty$ as $|x| \to \infty$ and $u \geq c_\infty$ on $\mathbb{R}^N$, then u_c is a ball for $c > c_\infty$ and equal to $\mathbb{R}^N$ for $c \leq c_\infty$. Hence u is locally Schwarz symmetric.

(ii) Let $r > 0$. If $x, y \in S_r$ and $x \neq y$, then there exists $H \in \mathcal{H}_0$ such that $x \in H$ and $R_H x = y$. Hence, by the same argument as in (i),

$$\nabla u(x) = R_{x-y}\,\nabla u(y) \quad \text{for all } x, y \in S_r \text{ with } x \neq y \text{ and } u(x) = u(y). \tag{4.4}$$

From this it is easy to see that also

$$\nabla_{S_r} u(x) = R_{x-y}\,\nabla_{S_r} u(y) \quad \text{for all } x, y \in S_r \text{ with } x \neq y \text{ and } u(x) = u(y), \tag{4.5}$$

so that $|\nabla_{S_r} u|$ is constant on the relative boundary of the set $\{x \in S_r : u(x) \geq c\}$ for every $c \in \mathbb{R}$, and $\nu(x) = R_{x-y}\,\nu(y)$ for different points x, y on this boundary if c is a regular value of $u|_{S_r}$. We can therefore proceed as in (i)—now using Lemma 4.4(ii) —to show that all sets $\{x \in S_r : u(x) \geq c\}$, $c \in \mathbb{R}$ are geodesic balls in S_r. □

Remark 4.5 (Extension to functions defined on radial subdomains)

(i) If $\Omega \subset \mathbb{R}^N$ is a radial subdomain, the notion of local foliated Schwarz symmetry can be extended in an obvious way to functions in $C^1(\overline{\Omega})$ by only considering spheres $S_r \subset \overline{\Omega}$ in Definition 4.1(ii). Clearly Proposition 4.3(ii) extends to this situation: *if $u_H \in C^1(\overline{\Omega})$ for every $H \in \mathcal{H}_0$, then u is locally foliated Schwarz symmetric.*

(ii) Let $\Omega \subset \mathbb{R}^N$ be a ball centered at zero. Then $u \in C^1(\overline{\Omega})$ is called *locally Schwarz symmetric* if u_c is a ball in $\overline{\Omega}$ for every $c \in \mathbb{R}$ and $|\nabla u|$ is constant on ∂u_c. We then have the following variant of Proposition 4.3:

If $u \in C^1(\overline{\Omega})$ is a nonnegative function such that $u = 0$ on $\partial\Omega$ and $u_H \in C^1(\mathbb{R}^N)$ for every $H \in \mathcal{H}$, then u is locally Schwarz symmetric.

5 Symmetrization via Polarization

So far we have characterized different symmetry properties of functions with the help of hyperplane reflections and polarization. In this context, it also seems appropriate to review some results on the approximation of symmetrizations by iterated polarization. These results are very useful to analyze the behavior of integral functionals under symmetrization, see e.g. Theorem 5.1 below. Let $u : \mathbb{R}^N \to \mathbb{R}$ be a Lebesgue measurable function. If the superlevel set u_c has finite measure for sufficiently large $c \in \mathbb{R}$, then the *Schwarz symmetrization* $u_* : \mathbb{R}^N \to \mathbb{R}$ of u is defined as the unique upper semicontinuous Schwarz symmetric function such that the superlevel sets of u and u_* have the same measure for every c. More precisely, we define

$$u_*(y) := \sup\{c \in \mathbb{R} : |u_c| \geq |B_r(0)|\} \quad \text{for } y \in \mathbb{R}^N \text{ with } r = |y|,$$

where $|\cdot|$ denotes Lebesgue measure. Similarly, the *spherical cap symmetrization* $u_\sharp$ of u is defined as the function which is axially symmetric with respect to the axis $\mathbb{R}e_N$ and such that the superlevel sets of the restrictions of u and $u_\sharp$ to S_r have the same $N-1$-dimensional Lebesgue measure for every $r > 0$. Strictly speaking, this definition is only valid for those $r > 0$ such that the restriction of u to the sphere S_r is measurable, i.e., for almost every $r > 0$. Hence $u_\sharp$ is only defined up to sets of measure zero. By Cavalieri's principle, we have the implications

$$u \in L^p(\mathbb{R}^N), \quad u \geq 0 \quad \Longrightarrow \quad u_* \in L^p(\mathbb{R}^N) \quad \text{and} \quad \int_{\mathbb{R}^N} u_*^p \, dx = \int_{\mathbb{R}^N} u^p \, dx, \tag{5.1}$$

$$u \in L^p(\mathbb{R}^N) \quad \Longrightarrow \quad u_\sharp \in L^p(\mathbb{R}^N) \quad \text{and} \quad \int_{\mathbb{R}^N} |u_\sharp|^p \, dx = \int_{\mathbb{R}^N} |u|^p \, dx, \tag{5.2}$$

for $1 \leq p < \infty$. We also have the following inequalities for Dirichlet type integrals.

Theorem 5.1 *Let* $1 < p < \infty$, *and let* $u \in W^{1,p}(\mathbb{R}^N)$. *Then*:

(i) $u_\sharp \in W^{1,p}(\mathbb{R}^N)$, *and* $\|\nabla u_\sharp\|_{L^p(\mathbb{R}^N)} \leq \|\nabla u\|_{L^p(\mathbb{R}^N)}$.
(ii) *If* $u \geq 0$, *then* $u_* \in W^{1,p}(\mathbb{R}^N)$ *and* $\|\nabla u_*\|_{L^p(\mathbb{R}^N)} \leq \|\nabla u\|_{L^p(\mathbb{R}^N)}$.

Part (ii) was proved for $p = 2$ in [74] and then for general $p \geq 1$ in [5, 82] with the help of the coarea formula. Part (i) is due to Kawohl [56, Corollary 2.35] for $N = 2$ (see also [73] for the special case $N = 2$ and $p = 2$) and to Smets and Willem [77] for general N. Theorem 5.1(i) and (ii) are examples of *Polya-Szegö inequalities*. The name *Polya-Szegö inequality* or *Polya-Szegö principle* usually stands for a statement of the form that a gradient-depending functional is nonincreasing under a certain rearrangement. Polya-Szegö inequalities for more general rearrangements can be found e.g. in [56, 85]. The strategy of the proofs in [77, 85] is to approximate symmetrization by polarization. In the case of Schwarz symmetrization, this strategy has already been introduced by Brock and Solynin [24]. The following general result on symmetrization via iterated polarization is due to van Schaftingen, see [85, 86].

Theorem 5.2 *Let* $1 \leq p < \infty$.

(i) *There exists a sequence* $(H_n)_n \subset \mathcal{H}(0)$ *of affine half spaces such that, for any nonnegative function* $u \in L^p(\mathbb{R}^N)$, *the sequence* $(u_n)_n \subset L^p(\mathbb{R}^N)$, $n \in \mathbb{N}$ *defined by*

$$u_1 := u \quad \text{and} \quad u_{n+1} := [u_n]_{H_n} \quad \text{for } n \in \mathbb{N}, \tag{5.3}$$

converges strongly to u_* *in* $L^p(\mathbb{R}^N)$.

(ii) *There exists a sequence* $(H_n)_n \subset \mathcal{H}_0(e_N)$ *such that, for any function* $u \in L^p(\mathbb{R}^N)$, *the sequence* $(u_n)_n \subset L^p(\mathbb{R}^N)$ *defined by* (5.3) *converges strongly to* $u_\sharp$ *in* $L^p(\mathbb{R}^N)$.

Using this result and arguing as in [24, 77], we complete the proof of Theorem 5.1 as follows: Let $u \in W^{1,p}(\mathbb{R}^N)$, and let $(u_n)_n$ be the sequence given in part (ii) of Theorem 5.2. By Lemma 3.1 and Corollary 3.2, we have

$$\|u_n\|_{L^p(\mathbb{R}^N)} = \|u\|_{L^p(\mathbb{R}^N)} \quad \text{and} \quad \|\nabla u_n\|_{L^p(\mathbb{R}^N)} = \|\nabla u\|_{L^p(\mathbb{R}^N)} \quad \text{for every } n \in \mathbb{N},$$

hence $(u_n)_n \subset W^{1,p}(\mathbb{R}^N)$ is a bounded sequence. Since $u_n \to u_\sharp$ in $L^p(\mathbb{R}^N)$ by Theorem 5.2(ii), it follows that, for a subsequence, $u_n \rightharpoonup u_\sharp$ weakly in $W^{1,p}(\mathbb{R}^N)$. Hence $\|\nabla u_\sharp\|_{L^p(\mathbb{R}^N)} \le \|\nabla u\|_{L^p(\mathbb{R}^N)}$ by weak lower semicontinuity of the function $u \mapsto \|\nabla u\|_{L^p(\mathbb{R}^N)}$.

Similarly, assuming $u \ge 0$ and using Theorem 5.2(i), we obtain $u_* \in W^{1,p}(\mathbb{R}^N)$ and $\|\nabla u_*\|_{L^p(\mathbb{R}^N)} \le \|\nabla u\|_{L^p(\mathbb{R}^N)}$.

We do not include a proof of Theorem 5.2 here, but we comment on the construction of the 'symmetrizing sequences' of half spaces. As observed in [86], the sequences $(H_n)_n \subset \mathcal{H}(0)$ resp. $(H_n)_n \subset \mathcal{H}_0(e_N)$ in Theorem 5.2 can be taken of the form

$$(H_n)_n = (H_1', H_2', H_1', H_2', H_3', H_1', H_2', H_3', H_4', \ldots),$$

where $(H_n')_n \subset \mathcal{H}(0)$ resp. $(H_n')_n \subset \mathcal{H}_0(e_N)$ is an arbitrary dense sequence of half spaces in $\mathcal{H}(0)$, $\mathcal{H}_0(e_N)$, respectively. Here density refers to standard metrics on the sets $\mathcal{H}(0)$ and $\mathcal{H}_0(e_N)$, see [86]. In [84] it was proved that also random sequences $(H_n)_n \subset \mathcal{H}(0)$ resp. $(H_n)_n \subset \mathcal{H}_0(e_N)$ are admissible in Theorem 5.2. Prior to these results, weaker versions of Theorem 5.2 had been established in [24, 77] where the sequence of half spaces was constructed such that it depends on the initial function u. It is tempting to guess that every dense sequence $(H_n)_n \subset \mathcal{H}(0)$ resp. $(H_n)_n \subset \mathcal{H}_0(e_N)$ is admissible in Theorem 5.2. This guess is wrong, as shown by the following counterexample.

Example 5.3 Let $v : \mathbb{R}^N \to \mathbb{R}$ be a nontrivial Schwarz symmetric continuous function with compact support, i.e., v is radial and decreasing in the radial variable, $v(0) > 0$ and $v \equiv 0$ outside a sufficiently large ball centered at zero. We also consider translated function $u = p_0 * v$, where p_0 is an arbitrary unit vector. Finally, we let $(q_n)_n$ be a strictly decreasing sequence of real numbers such that $q_0 = 1$ and $\lim_{n\to\infty} q_n > 0$. Next, let $(H_n) \subset \mathcal{H}(0)$ be an arbitrary dense subset. We define half spaces $I_n \in \mathcal{H}(0)$ such that iterated polarization of u with respect to the mixed sequence

$$(I_1, H_1, I_2, H_2, I_3, H_3, I_4, H_4, \ldots) \tag{5.4}$$

does not converge to u_*, although this sequence is still dense in $\mathcal{H}(0)$. Note that every H_n can be written as

$$H_n = \{x : x \cdot p_n > -\lambda_n\} \quad \text{with unit vectors } p_n \in \mathbb{R}^N \text{ and } \lambda_n > 0,\ n \in \mathbb{N}. \tag{5.5}$$

Since for any $n \in \mathbb{N}$ we have $|q_n p_n| < |q_{n-1} p_{n-1}|$, there exists precisely one half space $I_n \in \mathcal{H}(0)$ containing the point $q_n p_n$ and such that the reflection at ∂I_n maps $q_n p_n$ onto $q_{n-1} p_{n-1}$. Hence, defining $u_n \in C(\mathbb{R}^N)$ inductively by $u_0 := u$ and $u_n := [u_{n-1}]_{I_n}$ (polarization with respect to I_n) for $n \in \mathbb{N}$, it is easy to see that $u_n = [q_n p_n] * v$ for every $n \in \mathbb{N}$, and therefore $[u_n]_{H_n} = u_n$ for every $n \in \mathbb{N}$ by (5.5). Consequently, the sequence $(u_n)_n$ coincides with the one obtained from u_0 by iterated polarization with respect to the mixed sequence in (5.4). However, since $\lim_{n\to\infty} q_n > 0$, it is clear that u_n does not tend to $u_* = v$ in $L^p(\mathbb{R}^N)$ for any $p \geq 1$.

Theorem 5.1 also extends to $p = 1$ with a somewhat different proof, see [24, p. 1781] for details. In applications to variational problems, it is important to know under which conditions equality holds in the Polya-Szegö type inequalities. For the Schwarz symmetrization we have the following classical result of Brothers and Ziemer [25].

Theorem 5.4 *Let* $1 < p < \infty$. *If* $u \in W^{1,p}(\mathbb{R}^N)$ *is nonnegative with* $\|\nabla u_*\|_p = \|\nabla u\|_p$ *and such that the set* $\{x \in \mathbb{R}^N \ : u(x) > 0,\ \nabla u(x) = 0\}$ *has zero Lebesgue measure, then u is Schwarz symmetric up to translation.*

This result does not extend to the case $p = 1$, as pointed out in [25]. Moreover, it has no analogue for spherical cap symmetrization of functions $u \in W^{1,p}(\mathbb{R}^N)$, as the following example shows.

Example 5.5 Consider an arbitrary C^1-function $\tilde{u} : [0, \infty) \to \mathbb{R}$ such that

$$\tilde{u}(0) = \tilde{u}'(0) = 0, \quad \tilde{u} > 0 \quad \text{in } (0, 1), \qquad \tilde{u}(1) = 0 \quad \text{and} \quad \tilde{u} < 0 \quad \text{in } (1, \infty),$$

and suppose also that u decays to zero exponentially as $|x| \to \infty$. Moreover, let $u \in C^1(\mathbb{R}^N)$ be defined by $u(x) = \tilde{u}(|x|)\cos\theta$ for $x \neq 0$, where $\theta = \arccos[\frac{x}{|x|} \cdot e_N]$ is the polar angle from the x_N-axis. Then the zero set of ∇u has measure zero, and it is easy to see that $u_\sharp \equiv u$ in $B_1(0)$ and $u_\sharp \equiv -u$ in $\mathbb{R}^N \setminus B_1(0)$. Hence $\|\nabla u_\sharp\|_p = \|\nabla u\|_p$, but u does not coincide with $u_\sharp$ up to rotation.

Since the construction of this example solely relies on the fact that spherical cap symmetrization is a rearrangement along submanifolds of lower dimension, it also appears in Steiner symmetrization and monotone rearrangement in cylindrical domains. In the latter context it has been noted in [16]. Under additional assumptions on the underlying domain, a sharp characterization of the equality case in Steiner symmetrization has been obtained by Cianchi and Fusco [27], extending earlier work of Kawohl [56].

Remark 5.6 (Extension to functions defined on radial subdomains)

(i) If $\Omega \subset \mathbb{R}^N$ is a radial subdomain, then the definition of spherical cap symmetrization extends to measurable functions $u : \Omega \to \mathbb{R}$ by only considering spheres S_r contained in Ω, and Theorem 5.1(i) holds within this setting.

(ii) If Ω is a ball in $\mathbb{R}^N$ centered at zero, then also the definition of Schwarz symmetrization extends to measurable functions $u : \Omega \to \mathbb{R}$. However, in this setting Theorem 5.1(ii) only holds for nonnegative functions in $W_0^{1,p}(\Omega)$.

6 A Class of Constrained Minimization Problems

In this section we apply the concepts of the preceding sections to constrained minimization problems. We consider a rather simple setting which shows the potential and the limits of different types of arguments. Let, as before, $\Omega \subset \mathbb{R}^N$ be a radial domain. We fix $1 < p < \infty$ and let, as usual, $p^* := \frac{Np}{N-p}$ denote the critical Sobolev exponent if $N > p$. Moreover, we consider continuous functions $f_0, \dots, f_k : \mathbb{R} \to \mathbb{R}$ satisfying the following growth condition.

(H0) There exists a constant $c_0 > 0$ such that, for $i = 0, \dots, k$:
If $N > p$, then

- $|f_i(t)| \le c_0(1 + |t|^{p^*-1})$ for $t \in \mathbb{R}$ if Ω is bounded;
- $|f_i(t)| \le c_0(|t|^{p-1} + |t|^{p^*-1})$ for $t \in \mathbb{R}$ if Ω is unbounded.

If $N = p$, then, for some $s > p$,

- $|f_i(t)| \le c_0(1 + |t|^{s-1})$ for $t \in \mathbb{R}$ if Ω is bounded;
- $|f_i(t)| \le c_0(|t|^{p-1} + |t|^{s-1})$ if Ω is unbounded.

If $N < p$ and Ω is unbounded, then $|f_i(t)| \le c_0|t|^{p-1}$ in a neighborhood of $t = 0$.

We now put $F_i(t) = \int_0^t f_i(\tau)\, d\tau$ for $i = 0, \dots, k$ and consider the functional

$$E : \mathcal{W} \to \mathbb{R}, \quad E(u) = \frac{1}{p}\int_\Omega |\nabla u|^p\, dx - \int_\Omega F_0(u)\, dx.$$

Here and in the following, $\mathcal{W}$ stands for either one of the Sobolev spaces $W^{1,p}(\Omega)$ and $W_0^{1,p}(\Omega)$. By standard arguments in critical point theory, (H0) implies that E is a well-defined C^1-functional. We consider the following constrained minimization problem.

(MP) Minimize E subject to the constraints $\int_\Omega F_i(u)\, dx = \gamma_i$, $i = 1, \dots, k$, where the values $\gamma_1, \dots, \gamma_k \in \mathbb{R}$ are given.

We will not be concerned with the existence of minimizers, which can be proved in various cases by arguments based on compactness or concentration compactness, see e.g. [80, 90]. Instead we discuss the symmetries of minimizers, and we start with the following observation.

Theorem 6.1 *Suppose that* (MP) *admits a minimizer* u. *Then*:

(i) *The spherical cap symmetrization* $u_\sharp$ *of* u *is also a minimizer of* (MP).
(ii) *If* $u \ge 0$ *and either* $\Omega = \mathbb{R}^N$ *or* Ω *is a ball and* $\mathcal{W} = W_0^{1,p}(\Omega)$, *then the Schwarz symmetrization* u_* *of* u *is also a minimizer of* (MP).

Proof (i) By Cavalieri's principle, it is easy to see that $\int_\Omega F_i(u_\sharp)\,dx = \int_\Omega F_i(u)\,dx$ for $i = 0,\dots,k$. Moreover, by Theorem 5.1(i) and Remark 5.6(i), we have $\int_\Omega |\nabla u_\sharp|^p\,dx \le \int_\Omega |\nabla u|^p\,dx$, hence $E(u_\sharp) \le E(u)$ and $u_\sharp$ must also be a minimizer of (MP). The proof of (ii) is similar, using now Theorem 5.1(ii) and Remark 5.6(ii). □

Theorem 6.1 is important for the (numerical or theoretical) calculation of the energy minimum corresponding to the minimization problem (MP), since it ensures that minimizing among symmetric functions yields the global minimum. On the other hand, Theorem 6.1 does not rule out the existence of non-symmetric minimizers. In order to analyze the shape of all minimizers, we need the following regularity assumption.

(H1) Every minimizer u of (MP) is a C^1-function on Ω.

This property is known in many cases. In particular, it is ensured by the following weak nondegeneracy condition:

(H2) If $\alpha_1,\dots,\alpha_k \in \mathbb{R}$ are such that $\sum_{i=1}^k \alpha_i f_i \equiv 0$ on a non-empty open interval in $\mathbb{R}$, then $\alpha_1,\dots,\alpha_k = 0$.

To see that (H2) implies (H1), let u be a minimizer of (MP). We may assume that u is non-constant in Ω. Then a neighborhood of u in the closed set

$$M := \left\{ u \in \mathcal{W} : \int_\Omega F_i(u)\,dx = \gamma_i \text{ for } i = 1,\dots,k \right\} \tag{6.1}$$

is a C^1-submanifold of $\mathcal{W}$ of codimension k, and by the Lagrange-Multiplier rule (see e.g. [92, Theorem 43.D]), u is a weak solution of

$$-\Delta_p u = f_0(u) + \sum_{i=1}^k \alpha_i f_i(u) \quad \text{in } \Omega, \tag{6.2}$$

for some $\alpha_1,\dots,\alpha_k \in \mathbb{R}$. Moreover, u satisfies Dirichlet boundary conditions on $\partial\Omega$ in case $\mathcal{W} = W_0^{1,p}(\Omega)$ and Neumann boundary conditions in case $\mathcal{W} = W^{1,p}(\Omega)$. Combining the growth assumption (H0) with the regularity results in [53, Sect. 1] and [61], we infer that $u \in C^1(\overline{\Omega})$ (in fact, ∇u is even locally Hölder continuous but we do not need this here). Hence (H2) implies (H1).

If (H1) is assumed, the shape of minimizers is described by the following theorem which—apart from part (i)—is a combination of (variants of) results by Brock [23] and Mariş [67].

Theorem 6.2 *Suppose that* (H1) *holds, and let u be a minimizer of* (MP). *Then*:

(i) *u is locally foliated Schwarz symmetric.*
(ii) *If $k \le N-2$, then there exists a subspace $V \subset \mathbb{R}^N$ with* $\dim V = k$ *such that u is radial with respect to V.*
(iii) *If $\Omega = \mathbb{R}^N$, then u^+ and $-u^-$ are locally Schwarz symmetric, and $\lim_{|x|\to\infty} u(x) = 0$. If in addition* (H2) *holds, then u or $-u$ is locally Schwarz symmetric, and either $u \equiv 0$ or u is nonzero everywhere in $\mathbb{R}^N$.*

(iv) *If* Ω *is a ball*, $\mathcal{W} = W_0^{1,p}(\Omega)$ *and* u *is nonnegative*, *then* u *is locally Schwarz symmetric*.

Proof (i) Let $H \in \mathcal{H}_0$. Since $u \in M$, also $u_H \in M$ by Lemma 3.1 and Remark 3.3(i), and $E(u_H) = E(u)$ by Corollary 3.2. Hence u_H is also a solution of (MP), so that $u_H \in C^1(\Omega)$ by (H2). By Proposition 4.3 and Remark 4.5 we conclude that u is locally foliated Schwarz symmetric.

(ii) is a special case of a recent result of Mariş [67]. It relies on a rather different reflection method. We will sketch the method and the proof of (ii) in Sect. 6.3 below.

(iii) As in (i) we show that $u_H \in C^1(\Omega)$ for every $H \in \mathcal{H}$. Taking into account that $u \in L^p(\mathbb{R}^N)$, Proposition 4.3 then implies that the superlevel set u_c of u is a ball for $c > 0$ and the exterior of a ball for $c < 0$, and that $|\nabla u|$ is constant on ∂u_c. Hence u^+ and $-u^-$ are locally Schwarz symmetric, and $u(x) \to 0$ as $x \to \infty$. Now suppose that in addition (H2) holds, and suppose by contradiction that u changes sign in $\mathbb{R}^N$. By the local Schwarz symmetry of u^+ and u^-, it is easy to see that there exists a half space H such that the set $\{u > 0\}$ is contained in H and $\{u < 0\}$ is contained in $\widehat{H}$. Without loss we assume that $H = \{x_1 > 0\}$, and we consider the function $v \in W^{1,p}(\mathbb{R}^N)$ defined by

$$v(x) = \begin{cases} u(x_1 - 1, x_2, \dots, x_n) & \text{if } x_1 \geq 1; \\ 0 & \text{if } 0 < x_1 < 1; \\ u(x) & \text{if } x_1 \leq 0. \end{cases}$$

It is clear that v is also a minimizer of (MP). By (H2) and the subsequent remarks, it is therefore a solution of (6.2). However, the restriction of v to H is nonnegative, so the strong maximum principle of Vazquez [88] implies that either $v \equiv 0$ or $v > 0$ in H, which is a contradiction. We point out that the strong maximum principle is applicable since $f_0(t), \dots, f_k(t)$ are of order $|t|^{p-1}$ as $t \to 0$ by assumption (H0). So we conclude that u does not change sign, which implies that either u or $-u$ is (nonnegative and) locally Schwarz symmetric. Finally, applying again the strong maximum principle, we infer that $u \equiv 0$ or u is nonzero everywhere in $\mathbb{R}^N$.

(iv) Taking into account the definition of u_H given in Remark 3.3(i), we can again show that $u_H \in C^1(\Omega)$ for every $H \in \mathcal{H}$. Since u is nonnegative, we therefore conclude by Remark 4.5 that u is locally Schwarz symmetric. □

Remark 6.3

(i) In the special case $N \geq 3$ and $k = 1$, it follows from Theorem 6.2 that, under assumption (H1), any minimizer u of (MP) is locally foliated Schwarz symmetric and axially symmetric with respect to an axis $\mathbb{R}\, p$. Note however that in general axial symmetry and local foliated Schwarz symmetry do not imply foliated Schwarz symmetry, as shown by Example 5.5.

(ii) In case $\Omega = \mathbb{R}^N$, $k \leq N - 1$ and $\gamma_i \neq 0$ for at least one of the constraints in (MP), Mariş showed in [67, Theorem 2] that there exists a subspace $V \subset \mathbb{R}^N$ with $\dim V = k - 1$ and $z \in \mathbb{R}^N$ such that the translated function $z * u$ is radial with respect to V. In the special case $k = 1$, we then conclude by Theorem 6.2(iii) that u or $-u$ is Schwarz symmetric up to translation.

(iii) The proof of Theorem 6.2(iii) shows that, as a consequence of (H0) and the strong maximum principle, u cannot have plateaus at level zero if Ω is unbounded. Here by plateaus we mean open regions where ∇u is zero. Plateaus may appear at levels different from zero, as shown by examples in [21, 76]. Moreover, in case Ω is bounded, our growth assumption (H0) does not exclude plateaus at level zero.
(iv) With the exception of the latter statement in part (iii), Theorem 6.2 can be extended to quite general constrained minimization problems where the constraints also depend on $|\nabla u|$. This is due to the fact that Lemma 3.1 is the only tool in the proof which depends on the assumptions on the energy functional and the constraints. However, it is more difficult to ensure assumption (H1) in this context. For an example with constraints depending on $|\nabla u|$, see Sect. 6.2.
(v) In [21], Brock showed that any positive solution u of the Dirichlet problem for the equation $-\Delta_p u = f(u)$ in a ball with continuous f has some form of local symmetry. However, as remarked already in Sect. 4, this symmetry is weaker than local Schwarz symmetry since superlevel sets of u may be disjoint unions of balls. See [21] for explicit examples.

In the semilinear case $p = 2$, we get a better symmetry result for minimizers of (MP) under an additional Lipschitz condition. The following result and its proof is inspired by Brock [22, Sect. 4].

Theorem 6.4 *Let $u \in \mathcal{W}$ be a minimizer of* (MP), *and suppose that in addition to* (H2) *the following is satisfied*:

(H3) $p = 2$, *and* $f_0, \dots, f_k$ *are locally Lipschitz continuous.*

Then:

(i) *u is foliated Schwarz symmetric.*
(ii) *If Ω is a ball, $\mathcal{W} = W_0^{1,p}(\Omega)$ and $u > 0$ in Ω, then u is Schwarz symmetric.*
(iii) *If $\Omega = \mathbb{R}^N$, then u or $-u$ is Schwarz symmetric up to translation.*

We remark that, by standard elliptic regularity, (H3) implies that minimizers of (MP) are classical C^2-solutions of (6.2).

Proof (i) Let $H \in \mathcal{H}_0$. By Proposition 2.4(ii), it suffices to show that

$$H \text{ is dominant or subordinate for } u. \tag{6.3}$$

As noted in the proof of Theorem 6.2(i), $v := u_H$ is also a solution of (MP). We may assume that u is non-constant in Ω, so that v is non-constant as well. Hence, as noted above, (H3) implies that u and v are classical solutions of

$$-\Delta u = f_0(u) + \sum_{i=1}^{k} \alpha_i f_i(u), \quad -\Delta v = f_0(u) + \sum_{i=1}^{k} \beta_i f_i(v) \quad \text{in } \Omega, \tag{6.4}$$

for some $\alpha_i, \beta_i \in \mathbb{R}$, $i = 1, \dots, k$. We suppose by contradiction that (6.3) is false, which means that $u \not\equiv v$ and $u \circ R_H \not\equiv v$. Then there exists a nonempty open subset

$U \subset \Omega$ such that $u \equiv v$ on U and u is non-constant on U. By (6.4) we then have $\sum_{i=1}^{k}(\alpha_i - \beta_i) f_i(u(x)) = 0$ for all $x \in U$, hence $\alpha_i = \beta_i$ for $i = 1, \dots, k$ by assumption (H2). As a consequence, u, v are solutions of $-\Delta u = h(u)$ and $-\Delta v = h(v)$ in Ω with $h(t) = f_0(t) + \sum_{i=1}^{k} \alpha_i f_i(t)$. Since h is locally Lipschitz continuous by (H3), $w := u - v$ satisfies a linear equation of the form

$$-\Delta w = V(x) w \quad \text{in } \Omega$$

with a locally bounded function V. Moreover, since $w \equiv 0$ in U, the unique continuation principle (see e.g. [55]) implies that $w \equiv 0$ in Ω and therefore $u \equiv v$ in Ω, contrary to what we have assumed. Hence (6.3) is true, and the assertion follows by Proposition 2.4(ii).

(ii) By Lemma 2.2, it suffices to show that every half space $H \in \mathcal{H}(0)$ is dominant for u. If $\Omega \subset H$, then H is dominant for u by definition. If $\Omega \not\subset H$, it is easy to see that there exists a nonempty open subset $U \subset \widehat{H} \cap \Omega$ close to the boundary such that $u < u \circ R_H$ on U and u is non-constant on U. Indeed, this follows since u is continuous, $u > 0$ in Ω and $u = 0$ on $\partial\Omega$. Hence u coincides with u_H on U, where the polarization u_H of u is now defined according to Remark 3.3(i). As in (i) we now deduce that $u \equiv u_H$ on Ω, which means that H is dominant for u, as required.

(iii) By Theorem 6.2(iii), $u(x)$ tends to zero as $|x| \to \infty$. Hence u is Schwarz symmetric up to translation by (i) and Corollary 2.5. Alternatively, instead of using (i) one could show directly that every $H \in \mathcal{H}$ is dominant or subordinate for u; then the Schwarz symmetry follows from Proposition 2.3. □

Remark 6.5

(i) Part (ii) of Theorem 6.4 also follows from the symmetry result of Gidas, Ni and Nirenberg [49] discussed in the introduction. It also holds for $1 < p < 2$ if $f_0, \dots, f_k$ are locally Lipschitz continuous, since in [32, 33] the moving plane method has been extended to this case.

(ii) Results on Schwarz symmetry of positive solutions to the Dirichlet problem $-\Delta_p u = f(u)$ in Ω, $u = 0$ on $\partial\Omega$ have also been obtained for $p > 2$ in case Ω is a ball and f is a *positive* function, see [20, 34]. Moreover, in [34] also Steiner symmetric domains were considered.

(iii) Even in the simplest case where $k = 1$ and (H3) holds, minimizers of (MP) are in general not radial when $\mathcal{W} = W^{1,2}(\Omega)$, which corresponds to Neumann boundary conditions. An example for this has been considered by Ni and Takagi [70] in the case where Ω is a ball. Moreover, Esteban [45] considered an example in the case where Ω is the exterior of a ball. In the case of an annulus, examples for nonradial minimizers corresponding to one constraint can be given both for $\mathcal{W} = W^{1,2}(\Omega)$ and $\mathcal{W} = W_0^{1,2}(\Omega)$, see e.g. [28] and [19, p. 453].

(iv) Even in the semilinear case $p = 2$, the Lipschitz assumption is crucial in all parts of Theorem 6.4. For non-Lipschitz data, the minimization problem (MP) may admit solutions with plateaus, and then symmetries can be broken easily by varying the location of these plateaus. We illustrate this by Example 6.6 below which is a slight modification of an example given by Mariş [67].

The appearance of plateaus is somewhat related to the nonuniqueness of solutions of initial value problems for ordinary differential equations with non-Lipschitz data. The simplest example in this context seems to be the equation $\dot{x} = \sqrt[3]{x^2}$ which allows nontrivial solutions vanishing identically on arbitrarily large intervals.

Example 6.6 (A minimizer subject to two constraints without axial symmetry) Following [67, p. 326], we let $\alpha \in (0, 1)$, and let $j \in C(\mathbb{R}) \cap C^1(0, \infty)$ be a function satisfying

- $j \equiv 0$ on $(-\infty, 0]$, and $j(s) = s^\alpha$ for $0 < s \le 1$.
- The function $J : \mathbb{R} \to \mathbb{R}$, $J(s) = \int_0^s j(\tau)\, d\tau$ has compact support.

We consider the Dirichlet integral $E : W_0^{1,2}(\Omega) \to \mathbb{R}$, $E(u) = \int_\Omega |\nabla u|^2\, dx$ and the following minimization problem:

$$(M_\Omega) \quad \text{minimize } E \text{ under the constraints } \int_\Omega J(u)\, dx = \int_\Omega J(-u)\, dx = 1.$$

Note that this minimization problem is a special case of (MP) only if Ω is bounded, since otherwise (H0) is not satisfied. Nevertheless, it has been shown in [67, Example 7] that $(M_{\mathbb{R}^N})$ admits minimizers of the form $u_{z,w} = z * u_0 - w * u_0$, where u_0 is a nonnegative Schwarz symmetric function with compact support and $z, w \in \mathbb{R}^N$ are chosen such that $z * u_0$ and $w * u_0$ have disjoint supports. Fix $z, w \in \mathbb{R}^N \setminus \{0\}$, $z \ne -w$ with this property and $R > 0$ such that $\operatorname{supp} u_{z,w} \subset B_R$, where $B_R := B_R(0)$. Since $W_0^{1,2}(B_R) \subset W_0^{1,2}(\mathbb{R}^N)$ by trivial extension, we find that $u_{z,w} \in W_0^{1,2}(B_R)$ is a minimizer of (M_{B_R}) which is not axially symmetric, hence in particular not foliated Schwarz symmetric. Note however that $u_{z,w}$ is locally foliated Schwarz symmetric.

We finally remark that the nonnegativity assumption in Theorem 6.2(iv) and Theorem 6.4(ii) can be removed if $f_0, \ldots, f_k$ are odd functions, hence $F_0, \ldots, F_k$ are even. Note that in this case minimizers of (MP) come in pairs $\{\pm u\}$.

Theorem 6.7 *Suppose* (H1) *holds*, Ω *is a ball*, $\mathcal{W} = W_0^{1,p}(\Omega)$ *and that the functions* $f_0, \ldots, f_k$ *are odd. Let* u *be a minimizer of* (MP). *Then* u *or* $-u$ *is locally Schwarz symmetric. If in addition* (H2) *and* (H3) *hold, then* u *or* $-u$ *is Schwarz symmetric.*

Proof By Theorems 6.2 and 6.4, it suffices to show that u does not change sign. Since $F_0, \ldots, F_k$ are even, it is easy to see that also $|u|$ is a minimizer of (MP), so $|u|$ is locally Schwarz symmetric by Theorem 6.2(iv). Hence the set $\{|u| > 0\} = \bigcup_{n\in\mathbb{N}}\{|u| \ge \frac{1}{n}\}$ is a countable union of nested balls and therefore connected. Since by continuity u does not change sign in this set, it does not change sign at all. □

In the following two sections, we consider a special class of examples for the minimization problem (MP) and a related constrained minimization problem.

6.1 Extremal Functions in Poincaré-Sobolev-type Inequalities

Assume that Ω is bounded, and consider the family of Poincaré-Sobolev-type inequalities

$$\left(\int_\Omega |u-u_\Omega|^q\,dx\right)^{\frac{p}{q}} \le C(p,q,\Omega)\int_\Omega |\nabla u|^p\,dx, \quad u\in W^{1,p}(\Omega), \tag{6.5}$$

where $u_\Omega = \frac{1}{|\Omega|}\int_\Omega u\,dx$ is the average of u on Ω and

$$1\le q\le p^* \quad \text{if } N>p, \qquad 1<q<\infty \quad \text{if } N\le p.$$

This family of inequalities can be derived by combining Poincaré inequalities with Sobolev embeddings. However, this derivation neither yields optimal constants $C(p,q,\Omega)$, nor does it say whether equality can be achieved and how extremal functions look like. It is easy to see that $C(p,q,\Omega)$ is the inverse of the number

$$L_{p,q}(\Omega)=\inf\left\{\int_\Omega |\nabla u|^p\,dx : u\in W^{1,p}(\Omega),\ \int_\Omega u\,dx=0,\ \int_\Omega |u|^q\,dx=1\right\}. \tag{6.6}$$

In the following we assume that Ω is a bounded radial domain; then the minimization problem corresponding to (6.6) is a special case of (MP). Concerning the existence of minimizers, we have the following result. It follows by a standard compactness argument for subcritical q and is due to Demyanov and Nazarov in the critical case, see [41, Theorem 7.3].

Theorem 6.8 *There exists $\beta>0$ such that the infimum in* (6.6) *is attained under each of the following assumptions.*

(i) $p\ge N$.
(ii) $p<N$ *and* $1\le q<p^*$.
(iii) $p<\frac{N+1}{2}+\beta$ *and* $q=p^*$.

From Theorem 6.1 we therefore deduce

Corollary 6.9 *Under the assumptions of Theorem* 6.8, *there exists a foliated Schwarz symmetric minimizer of* (6.6).

Concerning the shape of *arbitrary* minimizers, Theorems 6.2 and 6.4 provide the following information. Note here that assumptions (H0)–(H2) are satisfied, and that (H3) is satisfied if $p=2$.

Theorem 6.10 *Let u be an extremal function for* (6.5), *i.e., a minimizer of* (6.6) *up to subtraction of a constant. Then*:

(i) *u is locally foliated Schwarz symmetric.*
(ii) *If $N\ge 4$, then there exists a two-dimensional subspace $V\subset\mathbb{R}^N$ such that u is radial with respect to V.*
(iii) *If $p=2$, then u is foliated Schwarz symmetric.*

In [51] we considered the superlinear case $p = 2$ and $q \geq 2$, and we proved the following geometric properties of minimizers in addition to the foliated Schwarz symmetry.

Theorem 6.11 *Let $p = 2$, $q \geq 2$, and let u be a minimizer of (6.6) which is foliated Schwarz symmetric with respect to some unit vector $\mathfrak{p}$, i.e., $u = u(r, \theta)$ with $r = |x|$ and $\theta = \arccos(\frac{x}{|x|} \cdot \mathfrak{p})$. Then*

(i) *u is strictly decreasing in $\theta \in (0, \pi)$.*
(ii) *if q is sufficiently close to 2, then u is odd with respect to the reflection at the hyperplane $T(\mathfrak{p})$.*

If Ω is the unit ball, then we have in addition:

(iii) *$\partial_{\mathfrak{p}} u > 0$ on $\overline{\Omega} \setminus \{\pm\mathfrak{p}\}$. If τ is another unit vector in $\mathbb{R}^N$ orthogonal to $\mathfrak{p}$, then $\partial_\tau u$ has precisely four nodal domains. Here $\partial_{\mathfrak{p}}$ and ∂_τ denote the directional derivatives in direction $\mathfrak{p}$ and τ, respectively.*
(iv) *if $N = 2$, the function u is not antisymmetric with respect to the reflection at $H(\mathfrak{p})$ when q is sufficiently large.*

We point out that if Ω is a ball, properties (i) and (ii) imply that u takes its maximum and minimum precisely at two antipodal points $\{\pm\mathfrak{p}\}$ on the boundary of Ω, and u has precisely two nodal domains. Moreover, in the case where u is odd with respect to the hyperplane $T(\mathfrak{p})$, the four nodal domains of $\partial_\tau u$ considered in (ii) are precisely the four quadrants in Ω cut off by the hyperplanes $T(\mathfrak{p})$ and $T(\tau)$. This holds in particular for q close to 2.

The methods underlying the proof of Theorem 6.11 are quite different from the focus of the present survey, so we only add brief comments and refer the reader to [51] for details. The most difficult parts of the proof are the strict inequality in (i) and property (ii), see [51, Sect. 5]. For both parts we need to carefully study the boundary values of the directional derivatives $\partial_{\mathfrak{p}} u$ and $\partial_\tau u$ for τ perpendicular to e. In a first step, we show that $\partial_{\mathfrak{p}} u$ is positive on $\partial\Omega \setminus \{\pm\mathfrak{p}\}$ and on the hyperplane $T(\mathfrak{p})$ defined above. In a second step, we show that $\partial_{\mathfrak{p}} u$ can have at most two nodal domains. It then follows that $\partial_{\mathfrak{p}} u$ must be positive in one of the half balls cut off by the hyperplane $T(\mathfrak{p})$. With this information, we then can conclude the proof of (ii) by a moving plane argument. This is one of few examples where the moving plane method is applied to a problem with Neumann boundary conditions.

We finally note that in case $p = q = 2$, minimizers of (6.6) are precisely the eigenfunctions of the Neumann-Laplacian on Ω corresponding to the first nonzero eigenvalue. These eigenfunctions are of the form $u(r, \theta) = g(r)\cos\theta$, and properties (i)–(iii) can be verified easily.

6.2 Least Energy Nodal Solutions

We consider the Dirichlet problem

$$\begin{cases} -\Delta u = |u|^{p-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{6.7}$$

in a smooth bounded domain Ω, where $p > 2$ is subcritical, i.e. $p < 2^* = \frac{2N}{N-2}$ if $N \geq 3$. Solutions of (6.7) are critical points of the functional

$$E : W_0^{1,2}(\Omega) \to \mathbb{R}, \quad E(u) = \frac{1}{2}\int_\Omega |\nabla u|^2\,dx - \frac{1}{p}\int_\Omega |u|^p\,dx.$$

Multiplication of (6.7) with $u^\pm$ and partial integration shows that all nodal solutions belong to the set

$$\mathcal{C} = \left\{ u \in W_0^{1,2}(\Omega) : u^+, u^- \neq 0 : \int_\Omega |\nabla u^\pm|^2\,dx = \int_\Omega |u^\pm|^p\,dx \right\}.$$

In order to find nodal (i.e., sign changing) solutions with least possible energy, one can try to minimize E on $\mathcal{C}$. We note however that $\mathcal{C}$ is not a C^1-submanifold of $W_0^{1,2}(\Omega)$. Nevertheless, the following result was proved by Castro-Cossio-Neuberger [26].

Theorem 6.12 *The value $c := \inf_{u\in\mathcal{C}} E(u)$ is positive and attained by some $u \in \mathcal{C}$. Moreover, every minimizer $u \in \mathcal{C}$ of $E|_\mathcal{C}$ is a classical sign changing solution of* (6.7) *which has Morse index two with respect to E and precisely two nodal domains.*

We recall that the Morse index of a solution u of (6.7) is the number of negative Dirichlet eigenvalues of the linearized operator $-\Delta + p|u|^{p-2}$ in Ω. A more general definition of the Morse index will be given in Sect. 7 below. Concerning the symmetry of least energy nodal solutions, i.e., of minimizers $u \in \mathcal{C}$ of $E|_\mathcal{C}$, we prove the following in [13, Sect. 3].

Theorem 6.13 *If Ω is a bounded radial domain in $\mathbb{R}^N$, $N \geq 2$, then every least energy nodal solution of* (6.7) *is foliated Schwarz symmetric.*

Remark 6.14

(i) The existence and foliated Schwarz symmetry of least energy nodal solutions has been established in a more general setting than considered here, see [12, 13, 26, 63, 87] for details.
(ii) Aftalion and Pacella [1] showed subsequently that every least energy nodal solution u of (6.7) is nonradial. Furthermore, the nodal set $\{x \in \Omega : u(x) = 0\}$ of u touches the boundary of Ω, see [1, Theorem 1.3]. If $N \geq 2$ and $p \geq 3$, these statements are also true for any nodal solution of (6.7) with Morse index less than or equal to N, see Remark 7.8 below.

Strictly speaking, Theorem 6.13 does not fit in the abstract framework of the minimization problem (MP), since the constraint contains gradient terms and $\mathcal{C}$ is not a C^1-manifold. However, the proof is a nice and easy application of the abstract tools developed so far combined with Theorem 6.12.

Proof of Theorem 6.13 Let $r > 0$ be such that $S_r \subset \Omega$, and let $\mathfrak{p} \in \mathcal{S}$ be a point such that the restriction of u to S_r attains its maximum at $r\mathfrak{p}$. We fix a half space $H \in$

$\mathcal{H}_0(\mathfrak{p})$ and show that it is dominant for u. Indeed, as a consequence of Lemma 3.1 and Corollary 3.2, the values of the integrals $\int_\Omega |u^\pm|^p\,dx$ and $\int_\Omega |\nabla u^\pm|^2\,dx$ remain unchanged when passing from u to u_H. Hence u_H is also a minimizer of $\Phi|_{\mathcal{C}}$, and therefore a solution of (6.7) by Theorem 6.12. Since the function $w = u_H - u$ is nonnegative in $H \cap \Omega$ and therefore also $-\Delta w = |u_H|^{p-2}u_H - |u|^{p-2}u \geq 0$, we conclude by the strong maximum principle that either $w \equiv 0$ in $H \cap \Omega$ or $w > 0$ in $H \cap \Omega$. However, the latter is impossible since $w(r\mathfrak{p}) = 0$ by the choice of $\mathfrak{p}$. Consequently we have $w \equiv 0$, i.e., $u \equiv u_H$ on $H \cap \Omega$. Hence the half space H is dominant for u, as claimed. By Proposition 2.4(i), we now conclude that u is foliated Schwarz symmetric with respect to $\mathfrak{p}$. □

6.3 Radial Symmetry with Respect to Subspaces via the Method of Mariş

As before, let $\Omega \subset \mathbb{R}^n$ be a radial domain. Here we review an approach of Mariş [67]—partly based on earlier ideas by Lopes [64, 65]—which yields radial symmetry of solutions to the minimization problem (MP) with respect to subspaces of $\mathbb{R}^N$. The following transformation is the basic ingredient in this approach. For a function $u : \Omega \to \mathbb{R}$ and $H \in \mathcal{H}_0$, let $u^H : \Omega \to \mathbb{R}$ be the even extension of $u|_H$, i.e.,

$$u^H(x) = \begin{cases} u(x), & x \in H \cap \Omega, \\ u(R_H x), & x \in \Omega \setminus H. \end{cases}$$

Although we use a similar notation as for polarization, there should be no danger of confusion since in this section we shall not use polarization at all. Note that the transformation $u \mapsto u^H$ preserves continuity and maps $W^{1,p}(\Omega)$ into $W^{1,p}(\Omega)$ for $1 \leq p \leq \infty$, but it is not a rearrangement. Hence we cannot expect invariance of integral functionals as stated for polarization in Lemma 3.1. On the other hand, given a solution u of the minimization problem (MP), one can ask which half spaces in $\mathcal{H}$ have the property that also u^H satisfies the constraints in (MP). Following Mariş [67], this gives rise to the following definition.

Definition 6.15 Let $u \in M$, where M is defined in (6.1). We say that a half space $H \in \mathcal{H}_0$ splits the constraints in two for u if one of the following equivalent conditions hold:

(i) $u^H \in M$.
(ii) $u^{\widehat{H}} \in M$.
(iii) $\int_{\Omega\cap H} F_i(u(x))\,dx = \int_{\Omega\cap\widehat{H}} F_i(u(x))\,dx$ for $i = 1, \dots, k$.

Similarly, we say that a hyperplane $T \in \mathcal{P}_0$ splits the constraints in two if one (and then both) of the half spaces separated by T satisfy the conditions (i)–(iii).

The equivalence of conditions (i)–(iii) follows by a simple change of variable. We note the following crucial observation.

Lemma 6.16 *Let $u \in M$ be a minimizer of* (MP), *and suppose that some half space $H \in \mathcal{H}_0$ splits the constraints in two for u. Then*:

(i) u^H *and* $u^{\widehat{H}}$ *are also minimizers of* (MP).
(ii) *If* (H1) *holds, then the normal derivative* u_ν *vanishes on* ∂H.

Proof (i) By definition, it is easy to check that $E(u^H) + E(u^{\widehat{H}}) = 2E(u)$. On the other hand, since both u^H and $u^{\widehat{H}}$ satisfy the constraints, we have $E(u^H) \geq E(u)$ and $E(u^{\widehat{H}}) \geq E(u)$. Hence in both cases equality holds, and the assertion follows.

(ii) By (i), u^H is a minimizer of (MP) and therefore a C^1-function by (H1). This immediately implies that $u_\nu \equiv 0$ on ∂H. □

We now sketch the proof of Theorem 6.2(ii), essentially following the argument of Mariş [67]. We will recall the statement for convenience.

Theorem 6.17 *Suppose that the number k of constraints in* (MP) *is less than or equal to $N-2$ and that* (H1) *holds. Then every minimizer u of* (MP) *is radial with respect to some k-dimensional subspace $V \subset \mathbb{R}^N$.*

In fact, Mariş proves a more general result in [67] related to elliptic systems, cf. Remark 6.23 below. We also warn the reader that the order of arguments differs a little bit from [67] in our sketch. We will derive the result as a consequence of the following three lemmas.

Lemma 6.18 *Suppose that $k \leq N-2$, and that u is a minimizer of* (MP). *Then u admits a symmetry hyperplane.*

Lemma 6.19 *Let $u \in M$, and let $V \subset \mathbb{R}^N$ be a subspace with* $\dim V \geq k+1$. *Then there exists a unit vector $e \in V$ such that the hyperplane $T(e)$ splits the constraints in two for u.*

Lemma 6.20 *Let u be a minimizer of* (MP), *and suppose that u is radial with respect to some subspace $V \subset \mathbb{R}^N$ with* $\dim V \leq N-1$. *Furthermore, let $e \in V$ be a unit vector such that the hyperplane $T(e)$ splits the constraint in two for u. Then u is radial with respect to the subspace $V \cap T(e)$.*

Using these lemmas, we can easily complete the proof of Theorem 6.17 as follows. Inductively, we prove that for $j = k, \dots, N-1$, u is radial with respect to a j-dimensional subspace of $\mathbb{R}^N$. The case $j = N-1$ follows directly from Lemma 6.18. Now suppose that u is radial with respect to some subspace V, where $k+1 \leq \dim V \leq N-1$. By Lemma 6.19, there exists $e \in V$ such that $T(e)$ splits the constraints in two for u. By Lemma 6.20, u is therefore radially symmetric with respect to $V \cap T(e)$, which is a subspace of dimension $\dim V - 1$. Hence the claim follows by induction.

Next we give a proof of Lemmas 6.19 and 6.20. After that we sketch the proof of Lemma 6.18, where we will use Lemmas 6.19 and 6.20. This will be the most difficult part of the argument. Lemma 6.19 is a nice and easy application of the following well known topological result.

Theorem 6.21 (Borsuk-Ulam Theorem, see e.g. [79, Theorem 9, p. 266]) *If n, m are integers with $n \geq m \geq 1$, then any continuous map $f : S^n \to \mathbb{R}^m$ admits a point $x \in S^n$ with $f(x) = f(-x)$. Here S^n denotes the unit sphere in $\mathbb{R}^{n+1}$.*

Proof of Lemma 6.19 Let $\mathcal{S}_V$ denote the unit sphere in V, and let $f : \mathcal{S}_V \to \mathbb{R}^k$ be defined by

$$f(e) = \left(\int_{H(e)\cap\Omega} G_1(u(x))\,dx, \ldots, \int_{H(e)\cap\Omega} G_k(u(x))\,dx \right).$$

Then f is continuous, and since $\mathcal{S}_V$ is a sphere of dimension larger than or equal to k, Theorem 6.21 yields $e \in \mathcal{S}_V$ such that $f(e) = f(-e)$, i.e.,

$$\int_{H(e)\cap\Omega} F_i(u(x))\,dx = \int_{H(-e)\cap\Omega} F_i(u(x))\,dx \quad \text{for } i = 1, \ldots, k.$$

Since $H(-e) = \widehat{H}(e)$, we thus conclude that $T(e)$ splits the constraint in two for u. □

Next we give the

Proof of Lemma 6.20 Set $H = H(e)$. Since the function u^H is symmetric with respect to reflection at $T(e)$ and radial with respect to V, we have

$$u^H(x + y) = u^H(x - y) \quad \text{for every } x \in V \cap T(e),\ y \in [V \cap T(e)]^{\perp}.$$

Therefore, by a simple change of variable, every hyperplane T containing $V \cap T(e)$ splits the constraint in two for u^H. Hence Lemma 6.16(ii) and Lemma 2.1 imply that u^H is radial with respect to $V \cap T(e)$. By the same argument, $u^{\widehat{H}}$ is also radial with respect to $V \cap T(e)$. Since u^H and $u^{\widehat{H}}$ coincide with u on $T(e)$, it easily follows that u is also symmetric with respect to $V \cap T(e)$. □

We conclude this section by sketching the proof of Lemma 6.18. As a consequence of Lemma 6.19, there exist a hyperplane $T = T(e)$ which splits the constraints in two for u. Hence $\check{u} := u^{H(e)}$ and $\hat{u} := u^{\hat{H}(e)}$ are minimizers of (MP) by Lemma 6.16. Since $k + 1 \leq N - 1 = \dim T(e)$ by assumption, Lemma 6.19 implies that there exists unit vectors $\check{e}, \hat{e}$ perpendicular to e such that $T(\check{e})$ splits the constraints in two for $\check{u}$ and $T(\hat{e})$ splits the constraints in two for $\hat{u}$. From Lemma 6.20 we therefore infer that

$$\check{u} \text{ is radial with respect to } T(e) \cap T(\check{e}), \text{ and } \hat{u} \text{ is radial with respect to } T(e) \cap T(\hat{e}). \tag{6.8}$$

If $\check{e}$ and $\hat{e}$ coincide up to sign, then u is also radial with respect to $T(e) \cap T(\check{e})$, and in particular $T(e)$ is a symmetry hyperplane for u. Hence we may assume that $\check{e} \neq \pm\hat{e}$, and we note that the restriction $\tilde{u}$ of u to $T(e)$ is symmetric both with respect to the reflection at $T(\check{e})$ and the reflection at $T(\hat{e})$. By a very subtle case distinction concerning the value of the angle θ spanned by e and $\check{e}$, Mariş manages to show that one of the following alternatives is true:

(i) u is invariant under the transformation $x + y \mapsto -x + y$ with $x \in \operatorname{span}\{\check{e}, \hat{e}\}$ and $y \in \{\check{e}, \hat{e}\}^{\perp}$.
(ii) There is a unit vector $\tilde{e} \in \operatorname{span}\{\check{e}, \hat{e}\}$ such that u is invariant under the transformation $x + y \mapsto -x + y$ with $x \in \operatorname{span}\{e, \tilde{e}\}$ and $y \in \{e, \tilde{e}\}^{\perp}$.

It is beyond the scope of this survey to present this part of the argument in detail, so we refer the reader to [67]. Given one of the properties (i) and (ii), it is easy to conclude the proof. If for instance (ii) holds, then a simple change of variable shows that every hyperplane T containing $\{e, \tilde{e}\}^{\perp}$ splits the constraints in two for u, and again by Lemmas 2.1 and 6.16(ii) we conclude that u is radially symmetric with respect to $\{e, \tilde{e}\}^{\perp}$. In particular, $T(e)$ is a symmetry hyperplane for u. If (i) holds, the same argument shows that u is radially symmetric with respect to $\{\check{e}, \hat{e}\}^{\perp}$. Combining this information with (6.8), one easily concludes that u is radial with respect to $\{e, \check{e}, \hat{e}\}^{\perp}$, so again $T(e)$ is a symmetry hyperplane for u. This completes the proof of Lemma 6.18.

As remarked in [67], the proof of Lemma 6.18 is very easy under the additional assumptions (H2) and (H3). Indeed, then we have the following stronger result observed already by Lopes in a more general setting, see [64, Theorem II.5].

Theorem 6.22 *Suppose that* (H0), (H2) *and* (H3) *are satisfied, and let u be a minimizer of* (MP). *Moreover, let $T \in \mathcal{P}_0$ be a hyperplane which splits the constraints in two for u. Then T is a symmetry hyperplane for u.*

Proof We may assume that u is not constant. Let $H \in \mathcal{H}_0$ such that $\partial H = T$, so that $v := u^H$ is also a minimizer of (MP) which coincides with u on H. Up to replacing H by $\widehat{H}$, we may assume that u is not constant in $H \cap \Omega$. We may therefore proceed precisely as in the proof of Proposition 6.4(i)—using (H2) and the unique continuation principle—to show that $u \equiv v$ in Ω. This implies that T is a symmetry hyperplane for u. □

Remark 6.23 In [64], Lopes used a variant of Theorem 6.22 to prove axial symmetry of solutions of minimization problems subject to one constraint. One of the major advantages of the approach of Lopes and Mariş is the fact that it carries over—without any restriction—to constrained minimization problems related to systems, i.e., problems for vector valued functions $u : \Omega \to \mathbb{R}^m$. For details, we refer the reader to [64, 65, 67] where this more general framework is considered. In contrast, arguments based on polarization do not carry over to problems related to systems, unless additional cooperativity conditions are assumed as in [23].

7 Solutions with Morse Index Bounds

This section is devoted to a further connection—studied in [52, 72]—between symmetry of solutions of semilinear elliptic equations and their variational properties. We consider solutions of

$$-\Delta u = f(|x|, u) \quad \text{in } \Omega \tag{7.1}$$

where, as before, Ω is a (bounded or unbounded) radial subdomain of $\mathbb{R}^N$, $N \geq 2$ and $f : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ is locally a $C^{1,\alpha}$-function. We complement (7.1) with Dirichlet boundary conditions

$$u = 0 \quad \text{on } \partial\Omega, \tag{7.2}$$

which are empty conditions if $\Omega = \mathbb{R}^N$. The aim of this section is to derive symmetry properties of solutions of (7.1), (7.2) from Morse index bounds. To define the Morse index of a solution u to (7.1), (7.2), we introduce the quadratic form

$$Q_u(\varphi, \psi) = \int_\Omega \left[\nabla\varphi\nabla\psi - V_u(x)\varphi\psi\right] dx \quad \text{for } \psi, \varphi \in C_c^1(\Omega), \tag{7.3}$$

corresponding to a solution u of (7.1), (7.2). Here $V_u(x) = f'(|x|, u(x))$, and f' stands for the derivative of f with respect to u. Moreover, $C_c^1(\Omega)$ denotes the space of all C^1-functions $\Omega \to \mathbb{R}$ with compact support in Ω.

Definition 7.1 We say that a C^2-solution of (7.1) and (7.2)

- is stable if $Q_u(\psi, \psi) \geq 0$ for all $\psi \in C_c^1(\Omega)$;
- has Morse index equal to $K \in \mathbb{N} \cup \{\infty\}$ if K is the maximal dimension of a subspace X of $C_c^1(\Omega)$ such that

$$Q_u(\psi, \psi) < 0 \quad \text{for all } \psi \in X \setminus \{0\}.$$

These definitions of stability and Morse index are standard in the context of semilinear elliptic PDE in (possibly) unbounded domains. In bounded domains Ω, the Morse index as defined above is just the number of negative Dirichlet eigenvalues of the operator $-\Delta + V_u(x)$ in Ω. Recently there has been a growing interest in stable and finite Morse index solutions of elliptic equations in unbounded domains, see e.g. [35, 36, 39, 40, 46]. As noted already by Bahri and Lions in [8], these solutions appear as possible obstructions for a priori bounds in related boundary value problems where variational principles allow to control the Morse index. Another motivation comes from boundary value problems with small diffusion as discussed in [35, 36], where an asymptotic description of the shape of solutions with finite Morse index is of interest. We first note a symmetry result for stable solutions of (7.1), (7.2).

Theorem 7.2 ([52]) *Let u be a stable solution of* (7.1) *and* (7.2) *such that* $|\nabla u| \in L^2(\Omega)$. *Then u is radial.*

In case $\Omega = \mathbb{R}^N$ and $f = f(u)$ does not depend on x, every stable solution of (7.1) and (7.2) is constant, as follows by an application of Theorem 7.2 to all translations of u. This latter result has already been obtained by Dancer [36] under weaker integrability assumptions on $|\nabla u|$. To see that some kind of decay or integrability assumption for $|\nabla u|$ is needed, let us consider the case $\Omega = \mathbb{R}^N$ and the Allen-Cahn nonlinearity $f(u) = u - u^3$. In this case (7.1) is given by $-\Delta u + u^3 = u$, which admits the stable but nonradial solution $u(x) = \tanh(\frac{x_1}{\sqrt{2}})$. We note that the stability of this solution can be derived from the fact that $u_{x_1} := \frac{\partial u}{\partial x_1}$ is a positive solution of $-\Delta u_{x_1} - V_u(x)u_{x_1} = 0$ in $\mathbb{R}^N$.

We quickly sketch the proof of Theorem 7.2 in the case of a bounded radial domain Ω. In this case the assumption $|\nabla u| \in L^2(\Omega)$ is automatically satisfied and Theorem 7.2 is a well known observation, see e.g. [4, 57, 58, 69] and the references therein. For arbitrarily chosen orthonormal vectors $e_1, e_2 \in \mathcal{S}$, we introduce cylinder coordinates (r, η, y) defined by $x = r[\cos\eta e_1 + \sin\eta e_2] + y$ with $y \in \{e_1, e_2\}^\perp$. It then suffices to show that the angular derivative $u_\eta = \frac{\partial u}{\partial \eta}$ satisfies

$$u_\eta \equiv 0. \tag{7.4}$$

Differentiating the equation $-\Delta u = f(|x|, u)$ and the boundary conditions with respect to η, we infer that u_η solves

$$\begin{cases} -\Delta u_\eta - V_u(x)u_\eta = 0 & \text{in } \Omega, \\ u_\eta = 0 & \text{on } \partial\Omega. \end{cases} \tag{7.5}$$

If the underlying domain Ω is bounded, the first Dirichlet eigenvalue of $-\Delta - V_u(x)$ in Ω is nonnegative by the stability of u. Moreover, it is well known that this eigenvalue is simple with a positive eigenfunction. Combining this information with (7.5), we readily infer that u_η may not change sign. Since u is periodic in η, this implies $u_\eta \equiv 0$.

In the case where Ω is unbounded, the proof of (7.4) is more subtle, and we refer the reader to [52] for details. We merely remark that a crucial role is played by the Cauchy-Schwarz-inequality

$$Q_u(\psi, \rho)^2 \le Q_u(\psi, \psi)Q_u(\rho, \rho) \quad \text{for all } \psi, \rho \in C_c^1(\Omega) \tag{7.6}$$

which is an immediate consequence of the stability of u.

Next we discuss symmetry properties of higher Morse index solutions of (7.1). Note that even for solutions with Morse index one we cannot—in general—expect radial symmetry, since the examples mentioned in Remark 6.5(i) correspond to solutions having Morse index one. Extending earlier work in [72] to unbounded domains, the following result has been obtained by Gladiali, Pacella and the author.

Theorem 7.3 ([52]) *Suppose that $f(|x|, s)$ or $f'(|x|, s)$ is convex in s for every $x \in \Omega$. Then every solution u of (7.1) and (7.2) with $|\nabla u| \in L^2(\Omega)$ and Morse index $j \le N$ is foliated Schwarz symmetric.*

Remark 7.4

(i) The special case $j = 1$, f convex had been considered earlier in [68, 71].

(ii) The bound N in Theorem 7.3 is optimal in general. In case Ω is a disc in $\mathbb{R}^2$, this can be seen by considering $f(|x|, u) = \lambda u$, where λ is the fourth Dirichlet eigenvalue of the Laplacian in Ω (counted with multiplicity). It is known that there exists a corresponding Dirichlet eigenfunction of the form $u(x) = J(r)\cos 2\theta$ with $r = |x|$ and $\theta = \arccos \frac{x_1}{|x|}$, where J is a rescaled positive Bessel function. This function u has Morse index three and is not foliated Schwarz symmetric.

In the case when $\Omega = \mathbb{R}^N$ and f does not depend on $|x|$, we deduce the following result from Theorem 7.3 and Corollary 2.5.

Theorem 7.5 *Assume that* $\Omega = \mathbb{R}^N$, *that* $f = f(u)$ *does not depend on* x *and that* f *or* f' *is convex. Moreover, suppose that* u *is a solution of* (7.1) *with Morse index* $j \le N$, $|\nabla u| \in L^2(\mathbb{R}^N)$ *and such that* u *has a limit as* $|x| \to \infty$. *Then* u *or* $-u$ *is Schwarz symmetric up to translation.*

Proof By translation invariance, for every $z \in \mathbb{R}^N$ the translated function $z * u$ is also a solution of (7.1) with Morse index $j \le N$ and $|\nabla u| \in L^2(\mathbb{R}^N)$, hence it is foliated Schwarz symmetric by Theorem 7.3. Therefore Corollary 2.5 implies that u or $-u$ is Schwarz symmetric up to translation. □

An immediate corollary of Theorem 7.5 is a nonexistence result for sign changing solutions.

Corollary 7.6 *Under the assumptions of Theorem* 7.5, (7.1) *does not admit sign changing solutions* u *with Morse index* $j \le N$ *and such that*

$$|\nabla u| \in L^2(\mathbb{R}^N) \quad \text{and} \quad u(x) \to 0 \quad \text{as } |x| \to \infty.$$

In the case $\Omega = \mathbb{R}^N \setminus B$, we have the following stronger nonexistence result for solutions which may or may not change sign. In contrast to Corollary 7.6, the proof of this result is quite involved and requires completely different techniques. We refer the reader to [52, Sect. 4] for details.

Theorem 7.7 *Assume that* $\Omega = \mathbb{R}^N \setminus B$ *and* $f = f(u)$ *does not depend on* x *and that either* f *is convex or* f' *is convex. Then there are no solutions* u *of* (7.1) *and* (7.2) *with*

$$|\nabla u| \in L^2(\Omega), \qquad u(x) \to 0 \quad \text{as } |x| \to \infty$$

and Morse index $j \le N$.

Remark 7.8 Suppose that Ω is bounded and radial, $f = f(u)$ does not depend on $|x|$ and f or f' is convex. Moreover, let u be a sign changing solution of (7.1), (7.2) with Morse index less than or equal to N. Then u is nonradial, and the nodal set $\{x \in \Omega : u(x) = 0\}$ of u touches the boundary of Ω. The first statement was proved by Aftalion and Pacella in [1], and the second statement follows by combining the arguments in [1] with further stability information in half domains which are derived in the course of the proof of Theorem 7.3. See [72, Theorem 1.2] for details in the case where f' is convex. As a consequence, we infer that least energy nodal solutions of (6.7) have these properties if $N \ge 2$ and $p \ge 3$.

In the remainder of this section, we sketch some ideas used in the proof of Theorem 7.3. We restrict our attention to *bounded* radial domains Ω from now on, referring the reader to [52] for the case of unbounded Ω where major additional difficulties have to be circumvented. To simplify notation, we assume that $f = f(u)$ does not depend on x, but the arguments are the same for nonlinearities depending on $|x|$. Then the argument consists in three main steps. In the first step, we reduce the foliated Schwarz symmetry of a solution u of (7.1), (7.2) to the existence of a symmetry hyperplane for which u is stable in the corresponding half domains. In the second step, we further reduce the foliated Schwarz symmetry to nonnegativity of an auxiliary operator—depending on u—in some half domain. In both steps, we neither use the convexity assumptions on the nonlinearity nor the Morse index bound on u. These assumptions are used in the last step, where the Borsuk-Ulam Theorem is applied to find a direction such that the corresponding half domain has the property required in Step 2.

For the remainder of this section, we fix a solution u of (7.1), (7.2). Moreover, for a unit vector $e \in \mathcal{S}$ we denote by $\lambda_1(e, V_u)$ the first Dirichlet eigenvalue of the linearized operator $-\Delta + V_u$ in the half domain $\Omega(e)$, cf. Sect. 1.1.

Proposition 7.9 *Suppose Ω is bounded, and suppose that there exists $e \in \mathcal{S}$ such that $T(e)$ is a symmetry hyperplane for u and*

$$\lambda_1(e, V_u) \geq 0. \tag{7.7}$$

Then u is foliated Schwarz symmetric.

Proof After a rotation, we may assume that $e = e_2 = (0, 1, \ldots, 0)$, hence $T(e) = \{x_2 = 0\}$. By Proposition 2.4(ii), it suffices to show that every half space $H \in \mathcal{H}_0$ is dominant or subordinate for u. So we consider an arbitrary unit vector $e' \in S$ different from $\pm e$ and the corresponding half space $H := H(e') \in \mathcal{H}_0$. After another orthogonal transformation which leaves e_2 and $H(e_2)$ invariant, we may assume that $e' = (\cos\eta_0, \sin\eta_0, 0, \ldots, 0)$ for some $\eta_0 \in (-\frac{\pi}{2}, \frac{\pi}{2})$. Now we choose new coordinates, replacing x_1, x_2 by polar coordinates r, η with $x_1 = r\cos\eta$, $x_2 = r\sin\eta$, and leaving $\tilde{x} := (x_3, \ldots, x_N)$ unchanged. The angular derivative $u_\eta = \frac{\partial u}{\partial \eta}$ then satisfies

$$\begin{cases} -\Delta u_\eta - V_u(x)u_\eta = 0 & \text{in } \Omega(e_2), \\ u_\eta = 0 & \text{on } \partial\Omega(e_2), \end{cases} \tag{7.8}$$

where the boundary condition comes from the symmetry of u with respect to $H(e_2)$. In the new coordinates, this symmetry can be expressed in form

$$u(r\cos(-\eta), r\sin(-\eta), \tilde{x}) = u(r\cos\eta, -r\sin\eta, \tilde{x}) = u(r\cos\eta, r\sin\eta, \tilde{x}), \tag{7.9}$$

which implies that u_η is odd in the η-variable. Moreover, similarly as in the proof of Theorem 7.2, the assumption (7.7) implies that u_η does not change sign in $\Omega(e_2)$. We therefore may distinguish the following cases.

(i) $u_\eta \leq 0$ in $\Omega(e_2)$ and $u_\eta \geq 0$ in $\Omega(-e_2)$.
(ii) $u_\eta \geq 0$ in $\Omega(e_2)$ and $u_\eta \leq 0$ in $\Omega(-e_2)$.

An elementary calculation shows that the half space $H(e')$ is dominant for u in case (i) and subordinate for u in case (ii). For details, see [72]. □

The next task is to find a direction $e \in \mathcal{S}$ satisfying the assumptions of Proposition 7.9. For this we consider the difference $w_e : \Omega \to \mathbb{R}$ between u and its reflection at the hyperplane $T(e)$, i.e., $w_e(x) = u(x) - u(R_e x)$. Note that the restriction of w_e to the half domain $\Omega(e)$ solves the linear problem

$$\begin{cases} -\Delta w_e - V_e(x) w_e = 0 & \text{in } \Omega(e), \\ w_e = 0 & \text{on } \partial\Omega(e), \end{cases} \tag{7.10}$$

where

$$V_e(x) = \begin{cases} \frac{1}{w_e(x)}[f(u(x)) - f(u(R_e x))], & w_e(x) \neq 0; \\ f'(u(x)), & w_e(x) = 0. \end{cases}$$

We write $\lambda_1(e, V_e)$ for the first Dirichlet eigenvalue of $-\Delta - V_e$ in the half domain $\Omega(e)$, and we claim the following.

Proposition 7.10 *Suppose that there exists a direction $e \in \mathcal{S}$ such that $\lambda_1(e, V_e) \geq 0$. Then the assumptions of Proposition 7.9 are satisfied, and hence u is foliated Schwarz symmetric.*

Proof If $w_e \equiv 0$, then $T(e)$ is a symmetry hyperplane for u; hence $V_e = V_u$, and therefore $\lambda_1(e, V_u) = \lambda_1(e, V_e) \geq 0$ by assumption. Hence we may assume that $w_e \not\equiv 0$. Then, by (7.10), the restriction of w_e to $\Omega(e)$ is a Dirichlet eigenfunction of $-\Delta - V_e$ in $\Omega(e)$ corresponding to the eigenvalue zero, so by assumption we have $\lambda_1(e, V_e) = 0$. Thus w_e does not change sign. Replacing e by $-e$ if necessary, we find that $w_e > 0$ in $\Omega(e)$ by the strong maximum principle. Hence the set $\mathcal{A} := \{\tilde{e} \in \mathcal{S} : w_{\tilde{e}} > 0 \text{ in } \Omega(\tilde{e})\}$ is nonempty. We claim the following about $\mathcal{A}$:

$$\mathcal{A} \subset \mathcal{S} \text{ is open, } \partial\mathcal{A} \text{ is nonempty, and } \lambda_1(\tilde{e}, V_{\tilde{e}}) = 0 \text{ for every } \tilde{e} \in \mathcal{A}. \tag{7.11}$$

Here $\partial\mathcal{A}$ denotes the relative boundary of $\mathcal{A}$ in $\mathcal{S}$. For the moment, we take these properties for granted and conclude the argument. Let $e' \in \partial\mathcal{A}$; then $w_{e'} \geq 0$ and $\lambda_1(e', V_{e'}) = 0$ by the continuity of u and (7.11). Since $w_{e'}$ solves (7.10) with e' in place of e, the strong maximum principle implies that either $w_{e'} > 0$ or $w_{e'} \equiv 0$ in $\Omega(e')$. However, the former case is excluded since $\mathcal{A}$ is open and therefore $e' \notin \mathcal{A}$. We thus conclude $w_{e'} \equiv 0$, hence $T(e')$ is a symmetry hyperplane for u and $V_{e'} = V_u$. This again yields $\lambda_1(e', V_u) = \lambda_1(e', V_{e'}) = 0$, as required.

It thus remains to show (7.11). Since $\mathcal{A}$ does not contain antipodal points by definition, it is clear that $\partial\mathcal{A}$ is nonempty. Moreover, if $\tilde{e} \in \mathcal{A}$, then, by (7.10), $w_{\tilde{e}}$ is a *positive* Dirichlet eigenfunction of the operator $-\Delta - V_{\tilde{e}}$ in $\Omega(\tilde{e})$ corresponding to the eigenvalue zero, hence $\lambda_1(\tilde{e}, V_{\tilde{e}}) = 0$. To prove that $\mathcal{A}$ is open in S, we use an argument in the spirit of the moving plane method. Let $\mu > 0$ be chosen sufficiently small such that, for any subdomain $M \subset \Omega$ with $|M| < \mu$ and any $e \in S$, the operator $-\Delta - V_e(x)$ fulfills the strong maximum principle in M, which means that any nontrivial C^2-function satisfying $-\Delta w = V_e(x) w$ in M and $w \geq 0$ on ∂M is strictly positive in M. The existence of such a number $\mu > 0$ is a consequence of the *maximum principle in thin domains*, see [17, Proposition 1.1]. Next, let $\tilde{e} \in \mathcal{A}$, and choose a compact set $K \subset \Omega(\tilde{e})$ such that $|\Omega(\tilde{e}) \setminus K| < \mu$. Since u is continuous and positive in the compact set K, there exists a neighborhood $N \subset \mathcal{S}$ of $\tilde{e}$ such that for every $\hat{e} \in N$ we have $K \subset \Omega(\hat{e})$, $|\Omega(\hat{e}) \setminus K| < \mu$ and $w_{\hat{e}} > 0$ in K. By the choice of μ, this implies that $w_{\hat{e}} > 0$ in $\Omega(\hat{e})$ for every $\hat{e} \in N$, so that N is contained in $\mathcal{A}$. This shows that $\mathcal{A}$ is open and finishes the proof. □

By Proposition 7.10, we only need to find a direction $e \in \mathcal{S}$ such that $\lambda_1(e, V_e)$ is nonnegative. Due to the implicit dependence of the potential V_e on u and f, it seems difficult to estimate $\lambda_1(e, V_e)$ for general nonlinearities f. At this point our convexity assumptions on f resp. f' in Theorem 7.3 enter. Exemplarily we will only consider the case where

$$f' \textit{ is convex in } u.$$

In this case we introduce, for every direction $e \in S$ the even part V_{es} of the potential $V_u(x) = f'(u(x))$ relative to the reflection at the hyperplane $H(e)$, i.e., $V_{es}(x) = \frac{1}{2}[f'(u(x)) + f'(u(R_e x))]$. We also denote by $\lambda_1(e, V_{es})$ the first Dirichlet eigenvalue of the operator $-\Delta - V_{es}$ in the half domain $\Omega(e)$. Since f' is convex in u, we find that $V_e \le V_{es}$ in Ω, which immediately implies that

$$\lambda_1(e, V_e) \ge \lambda_1(e, V_{es}). \tag{7.12}$$

This inequality is crucial since V_{es} is much closer related to the linearized potential V_u, therefore we can hope to use the Morse index bound for u to derive estimates for the eigenvalue $\lambda_1(e, V_{es})$ for some $e \in \mathcal{S}$. Here again the Borsuk-Ulam theorem enters. We first consider the case where the Morse index of u is less or equal to $N-1$.

Proposition 7.11 *Suppose that u has Morse index $j \le N-1$. Then there exists $e \in \mathcal{S}$ such that $\lambda_1(e, V_{es}) \ge 0$. Hence, as a consequence of (7.12) and Proposition 7.10, u is foliated Schwarz symmetric.*

Proof By assumption, the linearized operator $-\Delta - V_u(x)$ has precisely j negative Dirichlet eigenvalues in Ω (counted with multiplicity), and we choose L^2-orthonormal eigenfunctions $\varphi_1, \varphi_2, \dots, \varphi_j$ corresponding to these eigenfunctions. It then follows that

$$Q_u(\psi, \psi) \ge 0 \quad \text{for every } \psi \in W_0^{1,2}(\Omega) \text{ which is } L^2\text{-orthogonal to } \varphi_1, \dots, \varphi_j. \tag{7.13}$$

For $e \in \mathcal{S}$, we let $\psi_e \in W_0^{1,2}(\Omega)$ denote the odd extension of the unique positive L^2-normalized Dirichlet eigenfunction of $-\Delta - V_{es}$ in the half domain $\Omega(e)$ corresponding to $\lambda_1(e, V_{es})$. We then consider the odd and continuous map

$$h : \mathcal{S} \to \mathbb{R}^j, \quad h(e) = \left[\int_{\Omega \cap B_R} \psi_e(x)\varphi_1(x)dx, \dots, \int_{\Omega \cap B_R} \psi_e(x)\varphi_j(x)dx \right].$$

Since $j \le N-1$, h must have a zero $e \in \mathcal{S}$ by the Borsuk Ulam Theorem (see Theorem 6.21). Then ψ_e is L^2-orthogonal to $\varphi_1, \dots, \varphi_j$, so $Q_u(\psi_e, \psi_e)$ is nonnegative by (7.13). On the other hand, since ψ_e is odd with respect to the reflection at $T(e)$, we have

$$Q_u(\psi_e,\psi_e) = \int_\Omega \Big[|\nabla \psi_e|^2 - V_u(x)\psi_e^2\Big]\,dx$$
$$= \int_\Omega \Big[|\nabla \psi_e|^2 - V_{es}(x)\psi_e^2\Big]\,dx = 2\lambda_1(e, V_{es}).$$

Hence $\lambda_1(\Omega(e), V_{es})$ is nonnegative, as claimed. □

It remains to consider the case where the Morse index j of u equals N. This case is essentially more complicated, and it is beyond the scope of this survey to go into details here. We merely remark that in this case we are not able to find $e \in \mathcal{S}$ such that $\lambda_1(e, V_{es}) \geq 0$. Instead, we use the even potential V_{es} in another way to find $e \in \mathcal{S}$ such that $\lambda_1(e, V_e)$ is nonnegative, see [52, 72]. So also in this case Proposition 7.10 yields the foliated Schwarz symmetry of u.

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ÜBERSICHTSARTIKEL

Nonlinear Aspects of Calderón-Zygmund Theory

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Abstract Calderón-Zygmund theory is classically a linear fact and amounts to get sharp integrability and differentiability properties of solutions of linear equations depending on those of the given data. A typical question is for instance: Given the Poisson equation $-\triangle u = \mu$, in which Lebesgue space do Du or D^2u lie if we assume that $\mu \in L^\gamma$ for some $\gamma \geq 1$? Questions of this type have been traditionally answered using the theory of singular integrals and using Harmonic Analysis methods, which perfectly fit in the case of linear equations. The related results lie at the core of nowadays analysis of partial differential equations (pde) as they often provide the first regularity information after which further qualitative properties of solutions can be established. In the last years there has anyway been an ever growing number of results concerning nonlinear equations: put together, they start shaping what we may call a nonlinear Calderón-Zygmund theory. This means a theory which reproduces for non-linear equations the results and phenomena known for linear ones, without necessarily appealing to linear techniques and tools. The approaches are in this case suited to the special equations under consideration. Yet, although bypassing general Harmonic Analysis tools, in some way they preserve the general spirit of some the basic Harmonic Analysis ideas, applying them directly at a pde level. This is a report on some of the main results available in this context.

Keywords Calderón-Zygmund theory · Regularity · Quasilinear equations

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1 Classics

The readers of the Jahresbericht, especially those with some interest in pdes, most likely know the basics of so called Calderón-Zygmund theory—from now on also abbreviated as CZ theory. This is, at least in its basic version, a by now classical topic in the analysis of partial differential equations, which is concerned with determining, possibly in a sharp way, the integrability and differentiability properties of solutions to elliptic and parabolic equations in terms of the regularity of the given data. A brief introduction for the beginner is here outlined in Sect. 1.1 below, that we invite to read as a first approach to the subject. The point is—and we restrict to the elliptic case to fix the ideas—that when dealing with an elliptic equation of the type

$$-\text{div}\, a(x, Du) = \mu \tag{1.1}$$

thanks to the regularization properties of the left hand side operator, the solution u inherits the integrability or differentiability properties of the right hand side datum μ. In a first stage—"the linear age"—such a theory has been widely developed in the case the equations considered were linear, and the approaches used were largely relying on linearity via explicit representation formulas and/or linear interpolation methods. On the other hand, starting by the pioneering work of Tadeusz Iwaniec [52], over the last years a series of nonlinear results for possibly degenerate operators of the type in (1.1) has been accumulating, up to the stage that allows us to start talking of a nonlinear Calderón-Zygmund theory. In this survey we will try to summarize some of such results.

1.1 Basics

The most classical instance of CZ theory occurs when considering the Poisson equation

$$-\triangle u = \mu, \tag{1.2}$$

which for simplicity we shall initially consider in the whole $\mathbb{R}^n$ for $n > 2$. Here μ is again for simplicity assumed to be smooth and compactly supported, while u is the unique solution which decays to zero at infinity. The point we will emphasize in this introductory section is the possibility of getting a priori estimates, from which regularity results for more general data μ eventually follow via approximation procedures.

The classical approach—going back to Calderón & Zygmund [29, 30]—to the integrability properties of solutions to (1.2) goes via a representation formula involving the so called fundamental solution

$$u(x) = \int G(x, y)\, d\mu(y) \tag{1.3}$$

where $G(\cdot)$ is the Green's function

$$G(x, y) \approx \begin{cases} |x - y|^{2-n} & \text{if } n \geq 3 \\ \log\, |x - y| & \text{if } n = 2, \end{cases} \tag{1.4}$$

while here we recall that for simplicity we concentrate on the case $n > 2$. The symbol $\approx$ denotes a relation of proportionality via a fixed constant whose value is in principle not relevant for our purposes. The representation formula in (1.3) allows to derive all the relevant integrability properties of u and its derivatives in terms of those of the right hand side datum. Let us recall the strategy of the proof.

Definition 1.1 Let $\beta \in [0, n)$; the linear operator defined by

$$I_\beta(\mu)(x) := \int_{\mathbb{R}^n} \frac{d\mu(y)}{|x-y|^{n-\beta}}, \tag{1.5}$$

is called the β-Riesz potential of μ, where μ is a Borel measure defined on $\mathbb{R}^n$.

By (1.3) we gain the following inequalities:

$$|u(x)| \leq |I_2(\mu)(x)| \quad \text{and} \quad |Du(x)| \leq I_1(|\mu|)(x), \tag{1.6}$$

with the second one that has been actually obtained differentiating (1.3). By mean of the previous inequalities and of the following regularizing property of the Riesz potential [83]

$$I_\beta \colon L^\gamma \to L^{\frac{n\gamma}{n-\beta\gamma}}, \quad \gamma > 1,\ \beta\gamma < n, \tag{1.7}$$

we immediately infer the a priori estimate

$$\|Du\|_{L^{\frac{n\gamma}{n-\gamma}}} \leq c\|\mu\|_{L^\gamma}, \tag{1.8}$$

which holds whenever $\gamma < n$. Eventually, estimates like (1.6) and (1.8) extend to that case when $\mu \in L^\gamma$ by approximation arguments. In the case $\gamma = 1$ the previous inequalities clearly fail—think for instance to the case $-\Delta u = \delta$ (Dirac measure charging the origin) where the solution is indeed the Green's function in (1.4)—and in order to get an optimal analog we need to recall the notion of Marcinkiewicz spaces, often called weak Lebesgue spaces.

Definition 1.2 Let $t \geq 1$ and let $\Omega \subseteq \mathbb{R}^n$ be an open subset; a measurable map $w : \Omega \to \mathbb{R}^k$ belongs to $\mathcal{M}^t(\Omega, \mathbb{R}^k) \equiv \mathcal{M}^t(\Omega)$ iff

$$\sup_{\lambda \geq 0} \lambda^t |\{x \in A \ : \ |w| > \lambda\}| =: \|w\|^t_{\mathcal{M}^t(\Omega)} < \infty. \tag{1.9}$$

These are the right spaces to analyze the case $\gamma = 1$—and eventually, especially, the case when μ is a measure—and indeed we have (see for instance [5, 83])

$$I_\beta \colon L^1 \to \mathcal{M}^{\frac{n}{n-\beta}}, \tag{1.10}$$

and therefore

$$\|Du\|_{\mathcal{M}^{\frac{n}{n-1}}} \leq c\|\mu\|_{L^1}. \tag{1.11}$$

Again, the latter inequality immediately extends to the case where μ is a measure by approximation arguments. The importance of the space $\mathcal{M}^t$ also lies in the fact that it serves to describe in a sharp way the integrability of solutions involving measure data problems—and therefore L^1: typical solutions in such cases are given by the potential-like functions as $|x|^{-n/t}$; note that

$$|x|^{-n/t} \in \mathcal{M}^t(B(0,1)) \setminus L^t(B(0,1)) \tag{1.12}$$

for every $t \geq 1$. In general the following inclusions hold

$$L^t \subsetneqq \mathcal{M}^t \subsetneqq L^{t-\varepsilon} \quad \text{for every } \varepsilon > 0. \tag{1.13}$$

As for the first relation in (1.13), observe that

$$|\{|w| > \lambda\}| = \int_{\{|w|>\lambda\}} dx \leq \int_{\{|w|>\lambda\}} \frac{|w|^t}{\lambda^t} dx \leq \frac{\|w\|_{L^t}^t}{\lambda^t} \tag{1.14}$$

so that $\|w\|_{\mathcal{M}^t} \leq \|w\|_{L^t}$ holds, and in fact the estimation in (1.14) motivates the definition of Marcinkiewicz spaces.

The analysis of the integrability properties second derivatives requires a further differentiation of (1.3), indeed differentiating (1.3) twice we arrive at a new representation formula of solutions to (1.2):

$$D^2u(x) \approx \int K(x-y)\, d\mu(y) \tag{1.15}$$

where now $K(\cdot)$ is a so called Calderón-Zygmund kernel, that is

$$\|K\|_{L^2} + \|\hat{K}\|_{L^\infty} \leq B, \tag{1.16}$$

where $\hat{K}(\cdot)$ denotes the Fourier transform of $K(\cdot)$, and moreover the following (so called Hörmander) cancellation condition holds:

$$\int_{|x|\geq 2|y|} |K(x-y) - K(x)|\, dx \leq B \quad \text{for every } y \in \mathbb{R}^n; \tag{1.17}$$

here B denotes a certain finite constant. At this point the standard *CZ theory of singular integrals* comes into the play: the linear operator $\mu \mapsto CZ(\mu)$ defined by

$$CZ(\mu)(x) := \int K(x-y)\, d\mu(y), \tag{1.18}$$

is bounded from L^γ to L^γ, for every $\gamma \in (1,\infty)$. The outcome is the following a priori estimate for second order derivatives

$$\|D^2u\|_{L^\gamma} \leq c\|\mu\|_{L^\gamma} \quad \text{whenever } 1 < \gamma < \infty, \text{ where } c \equiv c(B). \tag{1.19}$$

Once again well-known counterexamples show that (1.19) fails for $\gamma = 1$.

The basic difference between this case and the one related to the estimates in (1.8), is that the convolution kernel $K(\cdot)$ appearing in (1.15) *is not integrable*, and the possibility of an estimate as (1.19) is linked to the *cancellation properties* expressed by (1.17). This is a sort of paradigm in modern analysis: when *size properties*—i.e. integrability—do not suffice then one should *look for additional cancellations*.

Similar results hold for equations with a right hand side in divergence form of type

$$\triangle u = \operatorname{div} F. \tag{1.20}$$

In this case similar methods yield the a priori estimate

$$\|Du\|_{L^\gamma} \le c\|F\|_{L^\gamma} \quad \text{for every } 1 < \gamma < \infty, \text{ where } c \equiv c(\gamma). \tag{1.21}$$

Finally, it is worth to remark here that the approach via fundamental solutions and singular integrals extends also to non-Euclidean settings; for this we refer to the nice survey [61]. We also refer to the monograph [46] for properties of fundamental solutions to higher order problems.

1.2 More on the Borderline Case $\gamma = 1$

When the right hand side of the equation belongs to L^1 or it is a measure one cannot go beyond Marcinkiewicz regularity (1.11). There are anyway intermediate cases allowing to improve such an information to full integrability. A classical one is linked to so called Hardy spaces, whose functions enjoy some subtle cancellation properties. We shall not deal with this aspect since such spaces seem to be in this context too much linked to linear problems. Here the emphasis is on nonlinear ones and we look for conditions that keep on working in nonlinear cases. The space $L\log L(\Omega)$ instead fits our purposes; with $\Omega \subseteq \mathbb{R}^n$ being a bounded domain, $L\log L(\Omega)$ is defined as the set of those functions w satisfying

$$\int_\Omega |w|\log(\mathrm{e}+|w|)\,dx < \infty.$$

This space, a particularly important instance of what are called Orlicz spaces, becomes a Banach space when equipped with the following Luxemburg norm:

$$\|w\|_{L\log L(\Omega)} := \inf\left\{\lambda > 0 \;:\; \int_\Omega \left|\frac{w}{\lambda}\right| \log\left(\mathrm{e}+\left|\frac{w}{\lambda}\right|\right)\,dx \le 1\right\} < \infty. \tag{1.22}$$

An obvious consequence of the definition above is the inclusion $L\log L(\Omega) \subsetneqq L^1(\Omega)$. As a matter of fact the space $L\log L$ is sent into L^1 by singular integrals operators $\|CZ(\mu)\|_{L^1} \lesssim \|\mu\|_{L\log L}$. As a consequence we have limiting L^1-estimates in (1.19) and (1.20): these turn to

$$\|Du\|_{L^{\frac{n}{n-1}}} + \|D^2u\|_{L^1} \lesssim \|\mu\|_{L\log L} \tag{1.23}$$

and

$$\|Du\|_{L^1} \lesssim \|F\|_{L\log L} \tag{1.24}$$

respectively, while (1.10) turns to $I_\beta\colon L\log L \to L^{\frac{n}{n-\beta}}$. Ultimately, for the Poisson equation (1.2) it holds that

$$\mu \in L\log L \quad \Longrightarrow \quad Du \in L^{\frac{n}{n-1}}. \tag{1.25}$$

See for instance [5] and references therein.

2 Nonlinearities

The results in the previous section are concerned with linear equations, and, although explicit representation formulas as (1.3) are not always an unavoidable tool—for instance interpolation techniques may be employed as well—all the classical approaches to CZ theory found till the beginning of the eighties strongly rely on the linearity of the problems considered. In this section we shall report on the first nonlinear results of Calderón-Zygmund type, mainly referring to possibly degenerate quasilinear equations of the type (1.1) with p-growth. When $p \neq 2$ the chief model case is given by the p-Laplacian operator

$$u \to \operatorname{div}(|Du|^{p-2}Du) = \triangle_p u. \tag{2.1}$$

In the rest of the paper $\Omega \subset \mathbb{R}^n$ will denote a bounded, Lipschitz regular domain, and $n \geq 2$; by $B(x,R) \subset \mathbb{R}^n$ we denote the open ball with radius $R > 0$, centered at x, i.e.

$$B(x,R) := \{y \in \mathbb{R}^n \,:\, |x-y| < R\}.$$

When the center will not be relevant we shall simply denote $B_R \equiv B(x,R)$. In a similar way, we shall denote by Q_R the general Euclidean hypercube with sidelength equal to $2R$, and sides parallel to the coordinate axes. In the rest of the paper we shall denote by c, δ, ε etc. general positive constants; we shall emphasize its functional dependence on the relevant parameters by displaying them in parentheses; for example, to indicate a dependence of c on the real parameters n, p, ν, L we shall write $c \equiv c(n,p,\nu,L)$. Finally, according to a standard notation, given a set $A \subset \mathbb{R}^n$ with positive measure and a map $v \in L^1(A,\mathbb{R}^k)$, we shall denote by

$$(v)_A := \fint_A v(x)\,dx$$

its integral average over the set A.

2.1 The Notion of Solution

The general setting we are going to examine concerns nonlinear equations and systems which in the most general form look like

$$-\operatorname{div} a(x,Du) = H \quad \text{in } \Omega, \tag{2.2}$$

where $a\colon \Omega \times \mathbb{R}^{Nn} \to \mathbb{R}^{Nn}$ is a Carátheodory vector field—and therefore a priori only measurable with respect to x—satisfying the following strong p-monotonicity and growth assumptions:

$$\begin{cases} \nu(s^2+|z_1|^2+|z_2|^2)^{\frac{p-2}{2}}|z_2-z_1|^2 \leq \langle a(x,z_2)-a(x,z_1), z_2-z_1\rangle \\ |a(x,z)| \leq L(s^2+|z|^2)^{\frac{p-1}{2}} \end{cases} \tag{2.3}$$

whenever $x \in \Omega$, $z, z_1, z_2 \in \mathbb{R}^n$ where $0 < \nu \leq L$. Here we take $p > 1$, $s \geq 0$. When $N = 1$ (2.2) reduces to an equation and we are in the scalar case. On the right hand side of (2.2) we initially assume that $H \in \mathcal{D}'(\Omega, \mathbb{R}^N)$. As we shall see below, assumptions (2.3) are nearly minimal in order to obtain low order regularity results, such as for instance integrability and continuity results for solutions. When instead looking for higher regularity results—for instance higher integrability estimates on Du—we shall need additional regularity on the vector field and we shall for instance consider the following:

$$\begin{cases} |a(x,z)| + (s^2+|z|^2)^{\frac{1}{2}}|a_z(x,z)| \leq L(s^2+|z|^2)^{\frac{p-1}{2}} \\ \nu(s^2+|z|^2)^{\frac{p-2}{2}}|\lambda|^2 \leq \langle a_z(x,z)\lambda, \lambda\rangle \\ |a(x,z)-a(x_0,z)| \leq L_1\omega\left(|x-x_0|\right)(s^2+|z|^2)^{\frac{p-1}{2}}, \end{cases} \tag{2.4}$$

whenever $x, x_0 \in \Omega$, $z, \lambda \in \mathbb{R}^{Nn}$ where $0 < \nu \leq L$ and $s, L_1 \geq 0$. The symbol a_z denotes the partial derivative of $a(\cdot)$, and a_z is again to be assumed Carátheodory regular, while $\omega(\cdot)$ is *a modulus of continuity*, i.e. a non-decreasing function such that

$$\lim_{R\to 0} \omega(R) = 0.$$

The previous condition essentially serves to prescribe that the dependence of the vector field $a(\cdot)$ upon the "coefficients" x is continuous, while the parameter $s \geq 0$ is used to distinguish the case of degenerate ellipticity ($s = 0$) from the nondegenerate one ($s > 0$).

We remark that both assumptions (2.3) and (2.4) are satisfied by the model case given by

$$-\operatorname{div}\left[\gamma(x)(s^2+|Du|^2)^{\frac{p-2}{2}}Du\right] = H$$

with $0 < c_1 \leq \gamma(x) \leq c_2 < \infty$, where $\gamma(\cdot)$ is a measurable function in the case assumptions (2.3) are considered, and is continuous otherwise; in this case the choice of ν and L depend on n, p, c_1, c_2. By taking $\gamma(\cdot) \equiv 1$ and $s = 0$ we obtain in the left hand side the p-Laplacian operator in (2.1).

Remark 2.1 We emphasize that a crucial difference between assumptions (2.3) and (2.4) is that in the first case the partial map $x \mapsto a(x,z)$ is just measurable, while in the second is assumed to be continuous.

Remark 2.2 Some of the results we are going to report upon below can be obtained under assumptions slightly more general than those considered in (2.3)–(2.4). On the other hand for the sake of simplicity we confine ourselves to present a sample of basic facts, rather than aiming at the largest possible generality.

To introduce the notion of weak solution we say that a map $u \in W^{1,1}(\Omega, \mathbb{R}^N)$ is a weak solution to (2.2) iff u is such that $a(x, Du) \in L^1(\Omega, \mathbb{R}^N)$ and satisfies

$$\int_\Omega \langle a(x, Du), D\varphi \rangle \, dx = \langle H, \varphi \rangle \quad \text{for every } \varphi \in C_c^\infty(\Omega, \mathbb{R}^N). \tag{2.5}$$

This definition turns out to be too general, as it will become clear very soon. Therefore in the following we shall mainly distinguish two situations:

- The first is when $H \in W^{-1,p'}$, that is the dual of the natural Sobolev space $W_0^{1,p}(\Omega, \mathbb{R}^N)$.

  In this case standard monotonicity methods apply [66], allowing to find—for instance when solving a Dirichlet problems—a so called **energy solution**, that is a solution belonging to the natural energy space associated to the problem: $u \in W^{1,p}(\Omega, \mathbb{R}^N)$. This is actually the standard situation and solutions are unique in their Dirichlet class provided strict monotonicity properties, as for instance $(2.3)_1$, are assumed. For this reason when considering this situation we shall mainly consider local solutions to (2.5), without specifying the boundary datum.
- The second is when $H \notin W^{-1,p'}$, and it is more delicate.

  Indeed in this situation we shall assume that H is in the most general case a Borel measure μ with finite total mass, while the notion of solution must be specified more carefully since specific phenomena appear. Solutions that do not lie in the natural space $W^{1,p}$ must be also considered, indeed called **very weak solutions**.

Remark 2.3 (Abundance of very weak solutions) In general, very weak solutions may also exist beside usual energy solutions, even for simple linear homogeneous equations of the type

$$\text{div}\,(A(x)Du) = 0. \tag{2.6}$$

Indeed, as shown by a classical counterexample of Serrin [81], for a proper choice of the strongly elliptic and bounded, measurable matrix $A(x)$, (2.6) admits at least two distributional solutions solving the same Dirichlet problem on a smooth domain. One of them belongs to the natural energy space $W^{1,2}$, and it is therefore an energy solution; the other one does not belong to $W^{1,2}$, and for this reason in a time where the concept of very weak solution was not familiar yet, was conceived as a pathological solution. This situation immediately poses the problem of *uniqueness of distributional solutions*. More generally, when restricting to the case of a measure right hand side, the problem arises to find a function class where unique solvability is possible.

In the following, when dealing with very weak solutions—that in our case will always happen when the right hand side datum H does not belong to the dual of $W_0^{1,p}$—although we shall mainly deal with local regularity results, for the sake of exposition we shall restrict ourselves to the case of homogeneous Dirichlet problems of the type

$$\begin{cases} -\mathrm{div}\, a(x, Du) = \mu & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.7}$$

with μ being in the most general case a (signed) Borel measure with finite total mass $|\mu|(\Omega) < \infty$. Non-homogeneous boundary data can be dealt with by standard reductions, and will not be treated here. Finally, without loss of generality, we shall assume in what follows that μ is defined on the whole $\mathbb{R}^n$, by eventually letting $\mu(\mathbb{R}^n \setminus \Omega) = 0$. This situation leads to consider special very weak solutions, called SOLA, which are described in the next section.

2.2 SOLA

Under assumptions (2.3) with $2 - 1/n < p \le n$ and $N = 1$ (scalar case) a distributional solution to (2.7) can be obtained by regularization methods as shown in [17, 18], and this generates a notion of solution called SOLA (Solution Obtained by Limits of Approximations)—in general *they are only very weak solutions*. Let us outline the strategy, which is on the other hand very natural. One considers solutions $u_k \in W_0^{1,p}(\Omega)$ to the regularized Dirichlet problems

$$\begin{cases} -\mathrm{div}\, a(x, Du_k) = \mu_k & \text{in } \Omega \\ u_k = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.8}$$

where $\mu_k \in C^\infty$ can be canonically obtained by smoothing μ via convolution with a sequence of smooth standard mollifiers $\{\phi_k\}$. Actually any smooth sequence $\{\mu_k\} \subset C^\infty$ such that μ_k weakly converges to μ in the sense of measures can be used. The arguments in [17] lead to establish that there exists $u \in W_0^{1,p-1}(\Omega)$ such that, up to a not relabeled subsequence, $u_k \to u$ in $W^{1,p-1}(\Omega)$ and (2.7) is solved by u in the sense of (2.5). The fundamentals of this approach have been given by Boccardo & Gallouët; we record this fact in the following theorem, which also gives the best possible regularity in terms of Sobolev spaces:

Theorem 2.1 [14, 17, 18, 32] *Under the assumptions* (2.3) *with* $2 - 1/n < p \le n$, *there exists a SOLA* $u \in W_0^{1,p-1}(\Omega)$ *to* (2.7). *Moreover, it holds that*

$$u \in W_0^{1,q}(\Omega) \quad \textit{for every } q < \frac{n(p-1)}{n-1}. \tag{2.9}$$

Finally, there exists a unique SOLA when $\mu \in L^1(\Omega)$ *or* $p = 2$.

The result in (2.9) is nearly optimal and for this reason SOLA are in general only very weak solutions; compare with Remark 2.6 below also for the lower bound $2 - 1/n < p$. The uniqueness in the case $\mu \in L^1$ is described for instance in [14, 32] and means that *by considering a different approximating sequence* $\{\bar{\mu}_k\}$ *converging to* μ *in* $L^1(\Omega)$*, we still get the same limiting solution* u. The uniqueness of SOLA when μ is nothing more than a general measure is on the other hand an open problem.

Remark 2.4 (The vectorial case) SOLA to (2.7) always exist by Theorem 2.1 in the scalar case $N = 1$ under assumptions (2.3). Existence in the vectorial case $N > 1$ is known only in certain special situations, more precisely when an assumption of the type $a(x, Du) = g(x, |Du|)Du$ is considered. The model case of the p-Laplacian system is therefore covered and we refer for details to the papers [37, 39]; see also Theorem 2.7 below.

Remark 2.5 (The superconformal case) We finally notice that the problem in (2.7) involves a priori only very weak solutions when $p \leq n$, otherwise usual energy solutions can be found as well. Indeed, as a consequence of Sobolev embedding theorem, when $p > n$ then $\mu \in W^{-1,p'}$ and we are in the realm of usual energy solutions, where SOLA coincide with the usual energy solutions.

Remark 2.6 (Again on the notion of solutions) The solutions of Theorem 2.1 for the measure data case actually satisfy

$$|Du|^{p-1} \in \mathcal{M}^{\frac{n}{n-1}}(\Omega) \tag{2.10}$$

which is an optimal result. Indeed, for $p \leq n$ the problem

$$\begin{cases} -\triangle_p u = \delta & \text{in } B_1 \\ u = 0 & \text{on } \partial B_1, \end{cases} \tag{2.11}$$

involving the Dirac measure charging the origin on the right hand side, has only one SOLA, given by what is conventionally called the nonlinear Green's function

$$G_p(x) \approx \begin{cases} (|x|^{\frac{p-n}{p-1}} - 1) & \text{if } 1 < p \neq n \\ \log\, |x| & \text{if } p = n. \end{cases} \tag{2.12}$$

This is the unique solution *amongst those obtainable via approximation with non-negative smooth functions* μ_k—compare [75, Sect. 4.4]. The result in (2.10) now tells us that $Du \in L^1$ iff $p > 2 - 1/n$, and we see that $G_p \notin W^{1,1}$ when $p \leq 2 - 1/n$. Indeed, the notion of solution must be further changed when $p \leq 2 - 1/n$. Moreover, some of the results below take a different form also in the case $2 - 1/n < p < 2$. For these reasons, in order to keep the presentation at a reasonably non-technical level we shall always assume

$$p \geq 2$$

when dealing with solutions to (2.7). Many of the results we are going to present indeed extend readily to the case $p > 2 - 1/n$ and for this we refer to [13, 31, 38].

Summarizing, *in the rest of the paper*, when considering a right hand side which does not belong to the dual space $W^{-1,p'}$, we shall always consider a problem of the type (2.7). In this case u will always be considered as a solution obtained by approximation—that is a SOLA—according to the scheme described in the lines above. In the case the right hand side belongs to the dual, we shall consider traditional energy solutions.

2.3 Weak and Very Weak Solutions

In the following, and for the ease of the reader, we shall distinguish the case when we shall deal with energy solutions from the one when considering very weak solutions in specific situations.

Let us consider the equation

$$-\text{div } a(x, Du) = \mu. \tag{2.13}$$

In the case $\mu \in L^\gamma$ then by using Sobolev embedding theorem it follows that

$$\gamma \geq \frac{np}{np - n + p} = (p^*)' > 1 \quad \Longrightarrow \quad \mu \in W^{-1,p'} \quad \text{when } p \leq n \tag{2.14}$$

and for this reason we shall appeal to $(p^*)'$ as the duality exponent.

Here p^* denotes the usual Sobolev embedding exponent given by $np/(n-p)$ when $p < n$ and $(p^*)'$ is its conjugate. We shall abuse the notation in so far that when $p = n$ we are implicitly setting $p^* := \infty$ and $(p^*)' = 1$.

In the case $p > n$ clearly $\mu \in W^{-1,p'}$ for every Borel measure with finite mass, see also Remark 2.5. Therefore when considering (2.13) and the condition

$$\begin{cases} 1 < \gamma < (p^*)' & \text{and } p \leq n \\ \text{or} & \\ \mu \in L^1, \ \mu \text{ is a measure} & \text{for } p \leq n \end{cases} \tag{2.15}$$

we shall be dealing with SOLA. For reasons explained in Remark 2.7 such cases will be the significant ones for us in the situation (2.13).

For equations of the type

$$\text{div } a(x, Du) = \text{div}(|F|^{p-2}F) \tag{2.16}$$

i.e. with the right hand side in divergence form, we have that the right hand side itself belongs to the dual $W^{-1,p'}$ iff $F \in L^q$ for $q \geq p$. This is essentially the only case when a priori estimates for solutions to (2.16) are available; see also Remark 2.8 below. Note that there is no loss of generality in writing the right hand side in (2.16) in the form displayed. Indeed, any equation of the form

$$\text{div } a(x, Du) = \text{div } G \tag{2.17}$$

can be rewritten in the form in (2.16) by a change of variable; obviously in this situation the right hand side belongs to the dual space $W^{-1,p'}$ iff $G \in L^q$ with $q \geq p/(p-1)$.

Remark 2.7 (Reduction to diverge form) Comparing cases (2.13) and (2.16) can be indeed useful: in the case converse to the one in (2.15) we can reduce problems of the type (2.13) to those of the type (2.17) and therefore in turn to those of the type (2.16), and then apply the results known for this case when the right hand side belongs to the dual. In case of (2.16) with $G \equiv |F|^{p-2}F$ this happens for instance when $G \in L^q$ and $q \geq p/(p-1)$. Indeed, such a reduction can be done as follows: with μ being given, up to an inessential additive constant vector, we can locally solve the equation the equation $\operatorname{div} G = \mu$. At this stage observe that when $p > n$, whenever $\gamma \geq 1$ this gives that $G \in L^q$ for every $q < n/(n-1)$ and therefore $G \in L^{p/(p-1)}$. When $p < n$ and $\gamma \geq (p^*)'$ or when $p = n$ and $\gamma > 1$ we have again that $G \in L^{p/(p-1)}$. Therefore, all in all the situation is as follows: when dealing with a right hand side of the type in (2.16) a satisfying theory is available only when the right hand side is in the dual of $W_0^{1,p}(\Omega)$, that is $F \in L^p$. For the case *below the duality exponent* results are instead available only for the case (2.13), and in this case we consider the situation (2.15); see also [76].

2.4 The Duality Range

This section is devoted to the case of solutions belonging to the natural Sobolev space $W^{1,p}$, and therefore mainly to the case when the right hand side belongs to the dual space $W^{-1,p'}$. The starting point here is the following natural p-Laplacian analog of (1.20):

$$\operatorname{div}\,(|Du|^{p-2}Du) = \operatorname{div}\,(|F|^{p-2}F) \quad \text{for } p > 1. \tag{2.18}$$

The first, fundamental result, has been obtained by Tadeusz Iwaniec, and *marks the beginning* of what might be called nonlinear Calderón-Zygmund theory.

Theorem 2.2 [52] *Let $u \in W^{1,p}(\mathbb{R}^n)$ be a weak solution to (2.18) in $\mathbb{R}^n$. Then*

$$F \in L^{\gamma}(\mathbb{R}^n, \mathbb{R}^n) \quad \Longrightarrow \quad Du \in L^{\gamma}(\mathbb{R}^n, \mathbb{R}^n) \quad \textit{for every } \gamma \geq p. \tag{2.19}$$

Remark 2.8 An interesting, and certainly difficult open problem, stems from a comparison between the result in the previous theorem and the one described in (1.21). Indeed, when considering the solutions of the related Dirichlet problem

$$\begin{cases} \operatorname{div}\,(|Du|^{p-2}Du) = \operatorname{div}\,(|F|^{p-2}F) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.20}$$

one could ask for solvability and a priori estimates in L^{γ} as long as $\gamma > p-1$, with a related estimate of the type

$$\int_{\Omega} |Du|^{\gamma}\, dx \leq c \int_{\Omega} |F|^{\gamma}\, dx \tag{2.21}$$

which is usually referred to as an estimate below the natural growth exponent. Unfortunately, such a result still remains a conjecture [54]. The only progress available is due to Iwaniec & Sbordone [54] (see also [53]) who proved the solvability of (2.20) with an estimate of the type (2.21) for $\gamma > p - \varepsilon$; c, ε are essentially universal constants depending on n, N and $\gamma_1 < \gamma_2$ as long as $p \in [\gamma_1, \gamma_2]$. This result, although still far from covering the full range $\gamma > p - 1$, is highly non-trivial, and involves the use of delicate rigidity properties of the Hodge decomposition.

The local version of Theorem 2.2, involving more general equations, is

Theorem 2.3 *Let $u \in W^{1,p}(\Omega)$ be a weak solution to the equation* (2.16) *in Ω under the assumptions* (2.4), *with $p > 1$. Then*

$$F \in L^{\gamma}_{\mathrm{loc}}(\Omega, \mathbb{R}^n) \quad \Longrightarrow \quad Du \in L^{\gamma}_{\mathrm{loc}}(\Omega, \mathbb{R}^n) \quad \textit{for every } \gamma \geq p. \tag{2.22}$$

Moreover, there exists a constant $c \equiv c(n, p, \gamma)$ such that for every ball $B_R \subseteq \Omega$ with radius $R > 0$ it holds that

$$\left(\fint_{B_{R/2}} |Du|^{\gamma}\, dx \right)^{1/\gamma} \leq c \left(\fint_{B_R} |Du|^{p}\, dx \right)^{1/p} + c \left(\fint_{B_R} |F|^{\gamma}\, dx \right)^{1/\gamma}. \tag{2.23}$$

A proof can be adapted from [3, 60]; we remark that in [3] a different approach to Theorem 2.3 is proposed, by using ideas from [28].

Remark 2.9 It would be interesting to extend Theorem 2.3 to anisotropic operators of the type considered in [70]. At the moment the only version known is the one regarding systems with so called $p(x)$-growth of the type

$$\operatorname{div}(|Du|^{p(x)-2} Du) = \operatorname{div}(|F|^{p(x)-2} F) \tag{2.24}$$

therefore involving a variable growth exponent function $p(\cdot)$. In this setting a result of the type (2.22) has been proved in [3], under suitable continuity condition of the function $p(x)$. A higher order problem version has been later obtained in [48]. Operators of the type in (2.24) appear in several models; see for instance [1, 2, 79] and references therein.

One may of course wonder whether or not results of the type of Theorem 2.3 hold for general systems—i.e. u takes values in $\mathbb{R}^N$, $N > 1$ and $a \colon \Omega \times \mathbb{R}^{Nn} \to \mathbb{R}^{Nn}$ under the same assumptions; the answer is obviously no. Indeed, already in the case of homogeneous systems $\operatorname{div} a(Du) = 0$, we have that minimizers may be unbounded [84], while a result like Theorem 2.3 would immediately imply their boundedness. On the other hand it is known that for certain special elliptic systems, satisfying an additional quasidiagonal structure of the type

$$a(Du) = g(|Du|)Du, \quad g(|Du|) \approx |Du|^{p-2} \tag{2.25}$$

solutions are as regular as the corresponding scalar equations, this being a fundamental result of Uhlenbeck and Ural'tseva [87, 88]. It is therefore natural to look for a generalization of Theorem 2.3 in the case (2.25), and this has been done by DiBenedetto & Manfredi [36] in the case of the p-Laplacian system (2.19); in the same paper limiting BMO estimates for the gradient are derived as well. See also [3] for more general vectorial cases. A further interesting extension of Theorem 2.3, valid for equations with possibly discontinuous coefficients of VMO type, has been obtained by Kinnunen & Zhou [60]; see also [25] for a global result on rough domains. This means that coefficients, instead of being continuous, are only assumed to show a controlled type of discontinuity, described in an integral way. More precisely, whenever v is an integrable map, one defines

$$\omega_v(R) := \sup_{B_\varrho,\, \varrho \leq R} \fint_{B_\varrho} |v - (v)_{B_\varrho}|\, dx, \quad (v)_{B_\varrho} := \fint_{B_\varrho} v\, dx,$$

where the supremum is take over all possible balls in the domain. Then one says that v is VMO-regular iff

$$\lim_{R \to 0} \omega_v(R) = 0. \tag{2.26}$$

Notice that in the case v is continuous $\omega_v(\cdot)$ is dominated by the usual modulus of continuity of v defined by

$$\tilde{\omega}_v(R) := \sup_{x, y \in \Omega, |x-y| \leq R} |v(x) - v(y)|,$$

and in this sense VMO-regularity generalizes (uniform) continuity.

Finally, results in the same spirit of Theorem 2.3 have been obtained in [21] for related obstacle problems: here it is the integrability of the obstacle function which governs the one of solutions.

Now, while on one hand for general elliptic systems there is no hope to get a full CZ theory in the style of Theorem 2.3, on another one something still remains. More precisely, Theorem 2.3 still holds when γ is not too large and indeed we have the following:

Theorem 2.4 [63] *Let $u \in W^{1,p}(\Omega, \mathbb{R}^N)$ be a weak solution to the system (2.16) in Ω under the assumptions (2.4) with $p > 1$. Then there exists $\delta \equiv \delta(n, N, p, L/\nu) > 0$ such that (2.22) and (2.23) hold provided*

$$p \leq \gamma < p + \frac{2p}{n-2} + \delta \quad \textit{when } n > 2, \tag{2.27}$$

while no upper bound is prescribed on γ in the two-dimensional case $n = 2$.

It is worth remarking here that the results as the previous one play a crucial role in deriving certain improved bounds on the Hausdorff dimension of the singular sets of vectorial problems [62, 63, 71–73] and in the analysis of their boundary regularity [40, 64, 65].

We close this section by mentioning a few parabolic counterparts of the above results, since in the evolutionary case there are several substantial differences coming up. The model case here is again the evolutionary p-Laplacian equation/system:

$$u_t - \operatorname{div}\, (|Du|^{p-2} Du) = \operatorname{div}\, (|F|^{p-2} F). \tag{2.28}$$

All the parabolic problems in the following, starting with (2.28), will be considered in the cylindrical domain $\Omega_T := \Omega \times (0, T)$, where, as usual, $\Omega \subset \mathbb{R}^n$ is a bounded Lipschitz domain and $T > 0$.

The extension of the elliptic nonlinear CZ theory in the sense of Theorem 2.3 to the parabolic case has remained an open problem since [52] until it has been settled, both in the vectorial and in the scalar case, in [4]. The reason is that the proof of the elliptic results strongly relies on the use of Harmonic Analysis tools such as maximal and sharp maximal operators, an approach which cannot at all be carried over to the case of (2.28) for $p \neq 2$. This is deeply linked to the fact that the homogeneous system

$$u_t - \text{div}\,(|Du|^{p-2}Du) = 0 \tag{2.29}$$

locally follows an *intrinsic geometry* dictated by the solution itself. This is essentially DiBenedetto's approach to the regularity of parabolic problems [35] that we are going to briefly outline. The cylinders on which the system (2.29) enjoys good a priori estimates—we consider $p \geq 2$ for simplicity—are of the type

$$\tilde{Q} = Q_{z_0}(\lambda^{2-p}R^2, R) \equiv B_R(x_0) \times (t_0 - \lambda^{2-p}R^2, t_0 + \lambda^{2-p}R^2),$$

where $z_0 \equiv (x_0, t_0) \in \mathbb{R}^{n+1}$ and the main point is that λ is required to satisfy a relation of the type

$$\fint_{Q_{z_0}(\lambda^{2-p}R^2, R)} |Du|^p \approx \lambda^p. \tag{2.30}$$

The last line says that $Q_{z_0}(\lambda^{2-p}R^2, R)$ is defined in an intrinsic way, and therefore in the terminology of [35] is called an *intrinsic cylinder*. It is actually the main core of DiBenedetto's ideas to show that such cylinders can be constructed and used to prove regularity: on such a cylinder, roughly speaking, equations as (2.29) re-homogenize and behave like the heat equation up to an extent which is sufficient to prove the desired regularity. Now the point is very simple: since the cylinders as $\tilde{Q}$ depend on the size of the solution itself, it is not possible to associate to them, and therefore to the operator (2.29), a universal family of cylinders—that is independent of the solution considered. In turn this rules out the possibility of using parabolic type maximal operators but for the case $p = 2$.

In the paper [4] this obstruction has been overcome by introducing a completely new technique which *bypasses the use of maximal operators*, providing the first *Harmonic Analysis-free, purely pde proof* of nonlinear CZ estimates. This technique in turn allows to give a new, Harmonic Analysis-free approach to all the previous elliptic results, see for instance [24]. Indeed in [89] even a new, elementary and interpolation-free proof of the classical Calderón-Zygmund theorem on the L^p-boundedness of singular integrals has been given adapting the techniques of [4].

Theorem 2.5 [4] *Let $u \in C(0, T, L^2(\Omega, \mathbb{R}^N)) \cap L^p(0, T, W^{1,p}(\Omega, \mathbb{R}^N))$ be a weak solution to the parabolic system* (2.28), *where Ω is a bounded domain in $\mathbb{R}^n$, and*

$$p > \frac{2n}{n+2}. \tag{2.31}$$

Then

$$F \in L^{\gamma}_{\mathrm{loc}}(\Omega_T, \mathbb{R}^{Nn}) \implies Du \in L^{\gamma}_{\mathrm{loc}}(\Omega_T, \mathbb{R}^{Nn}) \quad \textit{for every } \gamma \geq p. \tag{2.32}$$

Here Du denotes the spatial gradient of u.

For reasons which are by now well-understood in the theory of degenerate parabolic problems, lower bounds as (2.31) are essential to get results as the previous one; we refer to [35] for further discussion and counterexamples.

The result of Theorem 2.5 readily extends to all parabolic equations of the type

$$u_t - \mathrm{div}\, a(x, t, Du) = \mathrm{div}\, (|F|^{p-2}F),$$

with the vector field $a\colon \Omega \times (0, T) \times \mathbb{R}^n \to \mathbb{R}^n$ being such that the partial map $(x, z) \mapsto a(x, \cdot, z)$ is satisfying assumptions as (2.4). It is worth to be mentioned that in order to prove a result as in (2.32) no continuity condition is required with respect to the time variable: the partial map $t \mapsto a(x, t, z)$ can be assumed to be just measurable and for this we refer to [44], where some extensions of the results of [4] are also proposed. Further non-trivial extensions of Theorem 2.5 are contained in [21], where parabolic variational inequalities are considered, and again in [44, 80], where parabolic versions of Theorem 2.4 are included.

As already in the elliptic case, one could again wonder whether it is possible to extend the result of (2.32) below the natural growth exponent and taking $\gamma > p - 1$. A progress in this direction is in the important papers [58, 59], where the case $\gamma \in (p - \varepsilon, p + \varepsilon)$ of (2.32) is examined for a small $\varepsilon > 0$. Higher order versions of the results can be found in [19, 20].

2.5 The Subdual Range

In this section we deal with problems as in (2.7), where we assume that the right hand side *does not* need to belong to the dual space $W^{-1,p'}$. In particular, when considering (2.13) we shall assume condition (2.15) and we therefore shall treat the case of gradient estimates below the duality exponent [76], with special emphasis on the case when μ is nothing more that a Borel measure with finite mass. Recall that the case $\gamma \geq (p^*)'$ with $p \leq n$ and $\gamma > 1$ can be dealt with via Theorem 2.3, as pointed out in Remark 2.7. We shall deal with SOLA, and keeping Remarks 2.5–2.6 in mind, we shall restrict to the case $2 \leq p \leq n$ in order to simplify the exposition.

The fundamentals of the CZ theory for the subdual case have been established by Boccardo & Gallouët and it holds the following:

Theorem 2.6 [17, 18] *Under the assumptions* (2.3) *with $2 \leq p \leq n$, every SOLA $u \in W^{1,p-1}_0(\Omega)$ to* (2.7) *is such that*

$$|Du|^{p-1} \in \mathcal{M}^{\frac{n}{n-1}}(\Omega) \quad \textit{when } \mu \textit{ is a Borel measure with finite mass}, \tag{2.33}$$

$$|Du|^{p-1} \in L^{\frac{n}{n-1}}(\Omega) \quad \textit{when } \mu \in L\log L(\Omega), \tag{2.34}$$

and

$$|Du|^{p-1} \in L^{\frac{n\gamma}{n-\gamma}}(\Omega) \quad \text{when } \mu \in L^{\gamma}(\Omega),\ 1 < \gamma \leq (p^*)' \text{ and } p < n.$$

The delicate, conformal case $p = n$—when $Du \in \mathcal{M}^n$—has been actually established in [39] (it was indeed left as an open problem in [13]), while another approach to this case, together with explicit local estimates, has been proposed in [74], see Theorem 2.8 below. Related estimates were obtained in certain situations in the works [51, 68], where fine properties of solutions are also studied in the context of what is usually called nonlinear potential theory.

A theory for a class of elliptic systems satisfying for instance assumption (2.25) is also available. For instance it holds the following:

Theorem 2.7 [37, 39] *Let $2 \leq p \leq n$; given a Borel measure with finite total mass on Ω there exists a SOLA $u \in W_0^{1,p-1}(\Omega, \mathbb{R}^N)$ to the p-Laplacian system with measure data*

$$-\text{div}\,(|Du|^{p-2}Du) = \mu \quad \text{in } \Omega$$

such that $|Du|^{p-1} \in \mathcal{M}^{\frac{n}{n-1}}(\Omega)$.

See also [45] for an earlier result in model case (2.11).

The results of the last two theorems are sharp—see for instance the discussion in Remark 2.6—and are the nonlinear analog of the linear estimates (1.8), (1.11) and (1.25). In particular, the sharpness of the result in (2.33) can be tested by looking at the nonlinear fundamental solution in (2.12). When looking at the case where μ is a measure, there is an interesting phenomenon here. The result in (2.33) is actually sharp only when the measure, in a certain sense, concentrates on points. The principle is that the more the measure concentrates, the less integrable is the solution found, while if the measure diffuses, the integrability improves. This is made rigorous in the following theorem.

Theorem 2.8 [74] *Under the assumptions (2.3) with $2 \leq p \leq n$, assume also that the measure μ satisfies the density condition*

$$|\mu|(B_R) \leq cR^{n-\theta}, \quad p \leq \theta \leq n \tag{2.35}$$

for every ball $B_R \subset \mathbb{R}^n$ with radius R. Then every SOLA $u \in W_0^{1,p-1}(\Omega)$ to (2.7) is such that

$$|Du|^{p-1} \in \mathcal{M}_{\text{loc}}^{\frac{\theta}{\theta-1}}(\Omega). \tag{2.36}$$

The previous result is also sharp, as shown by the examples in [82]. Note that (2.35) implies that the measure cannot concentrate on sets with Hausdorff dimension smaller than $n - \theta$; in particular, the Dirac measure case is excluded as soon as $\theta < n$. Moreover, when $\theta < n$ the gradient integrability in (2.36) is always better that the one in (2.33), while (2.36) reduces to (2.33)—up to localization—when $\theta = n$. The restriction $p \leq \theta$ is natural; indeed when $\theta < p$, a measure satisfying (2.35) belongs to the dual space $W^{-1,p'}$ and this case falls in the realm of standard monotonicity methods that in turn provide standard energy solutions $u \in W_0^{1,p}(\Omega)$. Of particular interest is the borderline case $\theta = p$, which gives $Du \in \mathcal{M}^p_{\mathrm{loc}}$, and connects to the aforementioned dual case $\theta < p$, when $Du \in L^p$. For more results involving density conditions, as well as for other borderline cases of Marcinkiewicz and Lorentz spaces, we refer to [76].

Theorems 2.6, 2.7 and 2.8 give optimal regularity results in terms of gradient integrability. On the other hand equations of the type $(2.7)_1$ involve a second order operator, and therefore it is natural to ask whether or not the gradient of solutions enjoy higher regularity properties as for instance higher differentiability. This indeed happens when the right hand side μ is for instance a smooth function. One obstruction is given by the fact already in the case $\triangle u = \mu \in L^1$ we have that in general $Du \notin W^{1,1}$. The approach presented in [74] bypasses this fact by means of the use of fractional Sobolev spaces, and shows that although $Du \notin W^{1,1}$, we still have that Du is differentiable with every degree less than one. Let us recall a few definitions.

Definition 2.1 For a bounded open set $A \subset \mathbb{R}^n$ and $k \in \mathbb{N}$, parameters $\sigma \in (0,1)$ and $q \in [1,\infty)$, the fractional Sobolev space $W^{\sigma,q}(A,\mathbb{R}^k)$ is defined requiring that $w \in W^{\sigma,q}(A,\mathbb{R}^k)$ iff the following Gagliardo-type norm is finite:

$$\|w\|_{W^{\sigma,q}(A)} := \left(\int_A |w(x)|^q\,dx\right)^{1/q} + \left(\int_A\int_A \frac{|w(x)-w(y)|^q}{|x-y|^{n+\sigma q}}\,dx\,dy\right)^{1/q}.$$

To view the previous definition in a more intuitive way, the reader may think of $W^{\sigma,q}$-functions as those function having "derivatives of order σ", in turn integrable with exponent q. Roughly writing, this means

$$[w]^q_{\sigma,q;A} := \int_A\int_A \frac{|w(x)-w(y)|^q}{|x-y|^{n+\sigma q}}\,dx\,dy \approx \int_A |D^\sigma w|^q\,dz \quad 0<\sigma<1. \tag{2.37}$$

In order to get higher regularity for Du it is therefore natural to require more regularity of the vector field $a(\cdot)$, and we shall therefore consider assumption (2.4) rather than (2.3). Moreover, we shall consider an unavoidable—for the type of results eventually derived—Lipschitz regularity assumption on $x \mapsto a(x,z)$:

$$\omega(R) \le R. \tag{2.38}$$

Theorem 2.9 [74] *Under the assumptions* (2.4) *with* $2 \le p \le n$ *and* (2.38) *every SOLA* $u \in W_0^{1,p-1}(\Omega)$ *to the problem* (2.7) *is such that*

$$Du \in W^{\frac{1-\varepsilon}{p-1},p-1}_{\mathrm{loc}}(\Omega,\mathbb{R}^n) \quad \textit{for every } \varepsilon > 0. \tag{2.39}$$

In particular, when $p=2$ *it holds that*

$$Du \in W^{1-\varepsilon,1}_{\mathrm{loc}}(\Omega,\mathbb{R}^n) \quad \textit{for every } \varepsilon > 0. \tag{2.40}$$

This result essentially provides what should be considered as *the maximal regularity for measure data problems*. In the realm of Sobolev spaces the sharpness of (2.40) has already been discussed, but in general (2.39) is always sharp too. Indeed, let us recall that the Sobolev embedding in the fractional spaces gives

$$W^{\sigma,q}_{\mathrm{loc}} \hookrightarrow L^{nq/(n-\sigma q)}_{\mathrm{loc}} \quad \text{provided } \sigma q < n$$

(see for instance [6]). Therefore assuming that $Du \in W^{1/(p-1),p-1}_{\mathrm{loc}}$ would imply $|Du|^{p-1} \in L^{n/(n-1)}_{\mathrm{loc}}$, which is on the other hand impossible as the nonlinear Green's function G_p in (2.12)—i.e. the unique SOLA to (2.11)—does not enjoy such an integrability property. We only remark that non-integer differentiability exponents as in (2.39) are typical of problems with p-growth, especially in the degenerate case; see for instance [71]. A parabolic version of Theorem 2.9 has been recently given in [11], and as a corollary implies some of the results of [16].

Remark 2.10 (Fully nonlinear equations) The results presented in this section concern the divergence form case (3.5). A very deep, nonlinear Calderón-Zygmund theory for fully non linear problems of the type

$$F(x, D^2u) = \mu \tag{2.41}$$

is available, being a fundamental contribution of Caffarelli—see [26, 27]. For obvious reasons the phenomena and the techniques involved for the case (2.41), where solutions are to be understood in the viscosity sense, are quite different from the divergence form/variational case (1.1), where distributional solutions can be used. For this reason we do not touch the theory available for operators of the type (2.41). An advantage of divergence form problems as (1.1) is given by the fact that results do exist also for systems, while in the case (2.41) the theory is strictly scalar, since there has not been found any suitable vectorial analog of the concept of viscosity solution up to now. A bridge between the viscosity methods and quasilinear structures has been anyway built in [28], a paper that eventually inspired the proof of many results for divergence form operators, see for instance [3, 25].

3 Pointwise Estimates and Consequences

In the previous section we have seen how Theorem 2.6 extends the classical linear regularity results (1.8), (1.11) and (1.25) to the nonlinear setting. While the latter actually follow from the pointwise representation formulas (1.6), the original proofs for Theorem 2.6 make only use of integral estimates. At a first sight, it would appear impossible to get an analogue of the more stringent estimates (1.6) in the nonlinear case, since representation formulas are very much linked to the specific structure of the equation. The aim of this section is to report on a series of results that instead show that *pointwise estimates in terms of potentials hold for nonlinear problems too*. We remark that for the ease of exposition in this section we again restrict to the superquadratic case $p \geq 2$.

Since we are going to deal with local results, we need a suitable, truncated version of the classical Riesz potentials defined in (1.5), that is

$$\mathbf{I}^{\mu}_{\beta}(x,R) := \int_0^R \frac{\mu(B(x,\varrho))}{\varrho^{n-\beta}}\,\frac{d\varrho}{\varrho}, \tag{3.1}$$

and we note that the inequality $\mathbf{I}^{\mu}_{\beta}(x,R) \lesssim I_\beta(\mu)(x)$ holds whenever μ is a non-negative measure. Now, although estimates (1.6) could be still possible for nonlinear equations of the type $(2.7)_1$ when $p=2$, they certainly do not hold when $p \geq 2$ since they clearly do not respect the homogeneity properties of the equation. Indeed, if we consider a solution to $\operatorname{div}(|Du|^{p-2}Du) = \mu$ with $p \neq 2$, we see that $\tilde{u} = c^{1/(p-1)}u$—and *not* cu—solves $\operatorname{div}(|D\tilde{u}|^{p-2}D\tilde{u}) = c\mu$ for $c \neq 0$. Therefore, in order to hope for a nonlinear analog of relations (1.6) we need to consider a suitable family of nonlinear potentials, which encode in their structure the peculiar scaling of equations of p-Laplacian type.

Definition 3.1 Let μ be Borel measure with finite total mass on $\mathbb{R}^n$; the nonlinear Wolff potential is defined by

$$\mathbf{W}^{\mu}_{\beta,p}(x,R) := \int_0^R \left(\frac{|\mu|(B(x,\varrho))}{\varrho^{n-\beta p}}\right)^{1/(p-1)} \frac{d\varrho}{\varrho}, \quad \beta \in (0, n/p] \tag{3.2}$$

whenever $x \in \mathbb{R}^n$ and $0 < R \leq \infty$.

We immediately notice that for a suitable choice of the parameter β, p Wolff potentials reduce to Riesz potentials, i.e. $\mathbf{I}^{\mu}_{\beta} \equiv \mathbf{W}^{\mu}_{\beta/2,2}$. Wolff potentials play a crucial role in nonlinear potential theory and in the description of the fine properties of solutions to nonlinear equations in divergence form; for this we refer to [7, 8] and in particular to the famous paper by Hedberg & Wolff [50].

An important fact about Wolff potentials is that their behavior can be in several aspects recovered from that of Riesz potentials. Indeed, the following pointwise inequality holds:

$$\mathbf{W}^{\mu}_{\beta,p}(x,\infty) \leq c I_\beta \left\{ \left[I_\beta(|\mu|) \right]^{1/(p-1)} \right\}(x) =: c\mathbf{V}_{\beta,p}(|\mu|)(x). \tag{3.3}$$

The nonlinear potential $\mathbf{V}_{\beta,p}(\mu)(x_0)$ appearing in the right hand side of the previous inequality—often called the Havin-Maz'ya potential of μ—is a classical object in nonlinear potential theory, and together with the bound (3.3) comes from the pioneering work of Adams & Meyers and Havin & Maz'ya; see also [7, 8, 49]. Estimate (3.3) allows to derive all types of local estimates starting by the properties of the Riesz potential.

The first fundamental result connecting Wolff potentials to solutions of nonlinear equations in divergence form is due Kilpeläinen & Malý [57]; another approach has been later offered by Trudinger & Wang [86].

Theorem 3.1 [63, 86] *Let $u \in C^0(\Omega) \cap W^{1,p}(\Omega)$ be a weak solution to* (1.1), *under the assumptions* (2.3) *with $2 \leq p \leq n$, where μ is a Borel measure with finite* *total mass. Then there exists a constant* $c \equiv c(n, p, \nu, L) > 0$ *such that the pointwise estimate*

$$|u(x)| \leq c \fint_{B(x,R)} (|u| + Rs)\, dy + c\mathbf{W}^{\mu}_{1,p}(x, R) \tag{3.4}$$

holds whenever $B(x, R) \subseteq \Omega$.

See also [42] for yet another proof; note that for $p = 2$ we have that $\mathbf{W}^{\mu}_{1,p} \equiv \mathbf{I}^{|\mu|}_2$ and we recover a local analog of the first estimate in (1.6). Theorem 3.1 allows, amongst other things, to recover in a local way all the integrability results known for u via the properties of the Wolff potentials—see also (3.3). Moreover, several applications have been given: the proof of a boundary Wiener criterion for nonlinear equations [63] is a major instance. Applications to the solvability equations with right hand side with critical growth [78] are another important one.

Remark 3.1 For the sake of clarity, since it is our aim to emphasize on the regularity aspect, in this section we are presenting all the results in the form of a priori estimates for more regular solutions, $u \in C^0, C^1$ or the like. In turn, such estimates allow, via the usual approximation arguments for instance described in Sect. 2.1, to recover estimates for SOLA, or for all the other kinds of solutions considered in the literature when the right hand side is not the dual of $W^{1,p}_0$; see for instance the discussion in [42, 51, 63].

The possibility of extending pointwise potential estimates to the gradient has remained an open issue discussed rather intensively for a long while, and an answer came only recently. The first result in this direction is contained in [77] and is due to the author of this paper. Indeed, in [77] a precise local analog of the second estimate in (1.6) has been given; an announcement of the result already appears in [41]. We start by the simplest case, namely we consider the equation

$$-\mathrm{div}\, a(Du) = \mu \tag{3.5}$$

under the assumptions

$$\nu|\lambda|^2 \leq \langle \partial a(z)\lambda, \lambda\rangle, \qquad |a(0)| + |\partial a(z)| \leq L, \tag{3.6}$$

with the same notation of (2.4). The result is

Theorem 3.2 [77] *Let* $u \in C^1(\Omega)$ *be a solution to* (3.5), *under the assumptions* (3.6), *with* μ *being a Borel measure with finite total mass. Then there exists a constant* $c \equiv c(n, \nu, L)$ *such that whenever* $\xi \in \{1, \dots, n\}$ *the pointwise estimate*

$$|D_\xi u(x)| \leq c \fint_{B(x,R)} |D_\xi u|\, dy + c\mathbf{I}^{|\mu|}_1(x, R) \tag{3.7}$$

holds whenever $B(x, R) \subseteq \Omega$.

A proof of it will be outlined in the next section. The extension to the case $p \geq 2$ involves nonlinear Wolff potentials, and works under the natural assumptions in (2.4), together with a Dini regularity condition on the coefficients $x \mapsto a(x, z)$. Indeed, when considering (2.4) in this case we shall assume that

$$\int_0 [\omega(\varrho)]^{2/p} \frac{d\varrho}{\varrho} < \infty. \tag{3.8}$$

Theorem 3.3 [42] *Let $u \in C^1(\Omega)$ be a weak solution to* (1.1) *under the assumptions* (2.4) *with $p \geq 2$ and* (3.8) *being enforced, where μ is a Borel measure with finite total mass. Then there exists a constant $c \equiv c(n, p, \nu, L, L_1, \omega(\cdot)) > 0$ such that the pointwise estimate*

$$|Du(x)| \leq c \fint_{B(x,R)} (|Du| + s)\, dy + c\mathbf{W}^{\mu}_{1/p,p}(x, R) \tag{3.9}$$

holds whenever $B(x, R) \subseteq \Omega$.

An interesting fact is that, when applied to the *the model case equation*

$$-\text{div}(|Du|^{p-2}Du) = \mu, \tag{3.10}$$

Theorem 3.3 allows to give a unified approach to all the gradient integrability estimates of the papers [10, 17, 18, 36, 39, 52, 53, 85]. Indeed, although in such papers more general assumptions are considered—for instance vectorial cases and measurable coefficients are also allowed in some of them—an *ad hoc* technique had to be developed in every case according to the type of regularity in question, while here the single estimate (3.9) suffices to catch all types of regularity results, both for energy solutions (higher regularity) and very weak ones (low regularity). In particular, Theorem 2.6 follows as a corollary for the model case (3.10), *together with a series of refined borderline cases which had remained as an open issue*. For instance, borderline cases of estimates in Marcinkiewicz and Lorentz spaces follow as corollary:

$$\mu \in \mathcal{M}^{(p^*)'}_{\text{loc}} \quad \Longrightarrow \quad Du \in \mathcal{M}^{p}_{\text{loc}} \quad p \leq n \tag{3.11}$$

and this—as well as [76, Theorem 2]—settles a delicate open problem raised several times in the literature [10, 15, 56] (the only result available was for the case $p = n$, were $Du \in \mathcal{M}^n$ and it has been settled in [39]). We note that the result in (3.11) is delicate since it is exactly the borderline case between the dual and the subdual range; compare with the discussion in Sect. 2.3 and recall (2.14). Finally, since estimate (3.9) is pointwise, sharp local integral estimates follow, whereas they did not seem to be easily achieved with the global methods developed earlier.

We note that a Dini type continuity requirement of the type in (3.8) is actually unavoidable in that even in the case of a plain linear equation as

$$\text{div}(A(x)Du) = 0$$

solutions are not Lipschitz if the elliptic matrix $A(x)$ has just continuous-but-not-Dini-continuous entries [55]. Moreover, the degree of preciseness of estimate (3.9) can be measured observing that in the case of the problem (2.11) we have that

$$\fint_{B(x,R)} |Du|\, dx + \mathbf{W}^{\delta}_{1/p,p}(x, R) \leq c|Du(x)|$$

holds for a suitable constant c, whenever $|x| = R > 0$; see [42, Remark 6.2].

Remark 3.2 We anyway stress that the primary significance of Theorems 3.2 and 3.3 relies in showing that *pointwise gradient potential estimates are in fact possible*, something which *was even believed to be false* by some experts in nonlinear potential theory.

The techniques for proving Theorem 3.3 actually open the way to finer results for establishing gradient continuity properties. Indeed, assuming for instance that $\mathbf{W}^{\mu}_{1/p,p} \in L^{\infty}$ then (3.9) implies that Du is locally bounded (here we are not talking about a priori regular solutions of course). In this situation we note that, by the very definition of Wolff potential and the absolute continuity of the integral the following converge:

$$\lim_{R\to 0} \mathbf{W}^{\mu}_{1/p,p}(x, R) = \lim_{R\to 0} \int_0^R \left[\frac{|\mu|(B(x,\varrho))}{\varrho^{n-1}}\right]^{1/(p-1)} \frac{d\varrho}{\varrho} = 0 \tag{3.12}$$

holds almost everywhere and equiboundedly. Something more can be said if (3.12) holds in a stronger sense:

Theorem 3.4 [43] *Let $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ be a weak solution to* (1.1) *under the assumptions* (2.4) *with $p \geq 2$ and* (3.8) *being enforced, where μ is a Borel measure with finite total mass. Assume that the convergence in* (3.12) *holds locally uniformly in Ω; then Du is continuous.*

Remark 3.3 By a well-known theorem of Hedberg & Wolff [50] the validity of the inequality

$$\int_{\Omega} \mathbf{W}^{\mu}_{1,p}(x, 1)\, d|\mu|(x) < \infty \tag{3.13}$$

is sufficient to deduce that μ belongs to the dual of $W^{1,p}_0(\Omega)$ and therefore it is not restrictive to consider usual energy solutions $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ in Theorem 3.4, without involving SOLA. Indeed, note that (3.13) is obviously implied by $\mathbf{W}^{\mu}_{1/p,p} \in L^{\infty}$.

Theorem 3.4 tells that a threshold between gradient boundedness and gradient continuity can be established by the rate of convergence to zero of Wolff potentials, and this in turn allows us to derive a series of borderline cases for gradient continuity. We hereby mention a couple of them. A well-known result of Lieberman [67] states that if the right hand side measure satisfies a density condition of the type

$|\mu|(B_R) \le cR^{n-1+\varepsilon}$ for some $\varepsilon > 0$ then the gradient is continuous. A borderline case of this result now follows as a consequence of Theorem 3.4: If the measure satisfies $|\mu|(B_R) \le cR^{n-1}h(R)$ and

$$\int_0 [h(\varrho)]^{1/(p-1)} \frac{d\varrho}{\varrho} < \infty$$

then Du is continuous. From Lieberman's result it easily follows that

$$\mu \in L^{n+\varepsilon} \quad \text{for some } \varepsilon > 0 \quad \Longrightarrow \quad Du \text{ is continuous.} \tag{3.14}$$

At this point Theorem 3.4 also provides borderline continuity results in the framework of rearrangement invariant function spaces. It is the right moment to recall the definition of the so called Lorentz spaces $L(t,q)(\Omega)$, with $1 \le t < \infty$ and $0 < q \le \infty$. When $q < \infty$ a measurable map g belongs to $L(t,q)(\Omega)$ iff

$$\|g\|^q_{L(t,q)(\Omega)} := q \int_0^\infty \left(\lambda^t |\{x \in \Omega \,:\, |g(x)| > \lambda\}|\right)^{q/t} \frac{d\lambda}{\lambda} < \infty. \tag{3.15}$$

For $q = \infty$ Lorentz spaces are defined as Marcinkiewicz spaces $L(t,\infty)(\Omega) \equiv \mathcal{M}^t(\Omega)$ which have been already introduced in Definition 1.2 (let formally $q \to \infty$ in (3.15)). The local variant of such spaces is then obtained by saying $g \in L(t,q)(\Omega)$ locally iff $g \in L(t,q)(\Omega')$ whenever $\Omega' \Subset \Omega$ is a subset. Lorentz spaces are in most of the cases Banach spaces—for instance when $t > 1$ or when $t = q = 1$—when equipped with a norm essentially equivalent to the quantity in (3.15) [83, Theorems 3.21–3.22]. The spaces $L(t,q)(\Omega)$ "decrease" in the first parameter t, while they increase in q. Moreover, they "interpolate" Lebesgue spaces as the second parameter q "tunes" t in the following sense: whenever $0 < q < t < r \le \infty$ we have, with continuous embeddings, that the following strict inclusion hold:

$$L^r \equiv L(r,r) \subset L(t,q) \subset L(t,t) \subset L(t,r) \subset L(q,q) \equiv L^q. \tag{3.16}$$

A particular case of the last relations was already given in (1.13). Useful references for Lorentz spaces are for instance [47, 83]. To have a closer feeling on what kind of growth conditions Lorentz spaces are bound to describe, it could be useful to observe that a function as

$$\frac{1}{|x|^{n/t} \log^\beta |x|}$$

with $1 \le t < n$ and $\beta > 0$, belongs to $L(t,q)(B(0,1))$ provided $q > 1/\beta$. Compare this with (1.12).

A borderline case of (3.14) can be now stated via Lorentz spaces, and it is a less immediate consequence of Theorem 3.4:

Theorem 3.5 [43] *Let $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ be a weak solution to (1.1) under the assumptions (2.4) with $p \ge 2$ and (3.8) being enforced. Assume that $\mu \in L(n, 1/(p-1))(\Omega)$ locally; then Du is continuous.*

In fact the proof of the previous theorem relies on the fact that when $\mu \in L(n, 1/(p-1))(\Omega)$ then the convergence in (3.12) holds locally uniformly in Ω. The previous result is a borderline condition of (3.14) in that by (3.16) we have $L^{n+\varepsilon} \subset L(n, 1/(p-1))$ whenever $\varepsilon > 0$.

We close this section with a non-local version of the gradient estimate of Theorem 3.3. The point is now the following: when switching from Theorem 3.1 to Theorem 3.3 we pass from assumptions (2.3) to (2.4). The main difference here is not only in the differentiability of the vector field $a(\cdot)$ with respect to the gradient variable, but mainly in the fact that in (2.4) we assume a continuous rather than just a measurable dependence on the coefficients x. The main model we have in mind here is the p-Laplacian equation with coefficients given by $-\mathrm{div}(\gamma(x)|Du|^{p-2}Du) = \mu$. It is traditionally an important point in regularity—since the pioneering paper of De Giorgi [33]—proving results for equations with merely measurable coefficients. It is clear that an estimate as (3.9) cannot hold under assumptions (2.3), as in this case the maximal gradient regularity of solutions to equations as $\mathrm{div}\, a(x, Du) = 0$ is in general only given by

$$Du \in L^{p+\delta} \tag{3.17}$$

for some $\delta > 0$. This is essentially a consequence of Gehring's lemma and $\delta \equiv \delta(n, p, \nu, L)$ is a universal exponent depending only on the ellipticity properties of the operator; see [23]. On the other hand something remains; more precisely a non-local version of estimate (3.9) still holds yielding level sets information rather than a pointwise one. Moreover, such an estimate is bound to provide regularity results in accordance to the maximal gradient regularity in (3.17), in that it will provide in the best possible case gradient estimates in L^q with $q < p + \delta$, where δ is exactly the exponent in (3.17) given by Gehring's lemma. Before stating the result we need to recall a few facts. We recall the definition of the (restricted) fractional maximal function operator relative to a cube $Q_0 \subseteq \mathbb{R}^n$; this is defined as

$$M^*_{\beta, Q_0}(g)(x) := \sup_{Q \subseteq Q_0,\ x \in Q} |Q|^{\beta/n} \fint_Q |g(y)|\, dy, \quad \beta \in [0, n), \tag{3.18}$$

where the sup is taken with respect all the cubes Q contained in Q_0; all the cubes here have sides parallel to the coordinate axes. A similar definition can be given when g is replaced by a measure in an obvious way. When $\beta = 0$ this is essentially a local variant of the classical Hardy-Littlewood operator; in this case we shall abbreviate $M^*_{\beta, Q_0} \equiv M^*_{Q_0}$.

Theorem 3.6 [76] *Let $u \in W^{1,p}(\Omega)$ be a weak solution to (1.1) under the assumptions (2.3) with $p \geq 2$, where μ with a Borel measure with finite total mass. Let $Q_{2R} \Subset \Omega$ be a cube and let $M^* \equiv M^*_{Q_{2R}}$ denote the restricted maximal operator with respect to Q_{2R}. There exist constants $\delta \equiv \delta(n, p, L/\nu) > 0$ and $A \equiv A(n, p, L/\nu) > 1$ such that: For every $T > 1$ there exists $\varepsilon \equiv \varepsilon(n, p, \nu, L, T) \in (0, 1)$ such that*

$$\begin{aligned} &\left|\left\{x \in Q_R : M^*((|Du| + s))(x) > AT\lambda\right\}\right| \\ &\quad \leq T^{-(p+\delta)}\left|\left\{x \in Q_R : M^*((|Du| + s))(x) > \lambda\right\}\right| \end{aligned}$$

$$+\left|\left\{x \in Q_R : \mathbf{W}^{\mu}_{1/p,p}(x, 2\sqrt{n}R) > \varepsilon\lambda\right\}\right| \qquad (3.19)$$

holds whenever

$$\lambda \geq c(n)T^{p+\delta} \fint_{Q_{2R}} (|Du| + s)\, dx.$$

The previous result, although not explicitly stated in [76], is anyway implicit in the proof of [76, Theorem 11], and can be obtained by a different choice of the parameters and of the dyadic cubes used there. Inequality (3.19) roughly tells that "up to a $L^{p+\delta}$-correction" given by the level set appearing in the intermediate line of (3.19), the level sets of $M^*((|Du|+s))$, and therefore those of $|Du|$ in Q_R, are controlled by the level sets of $\mathbf{W}^{\mu}_{1/p,p}$. In turn this implies a local control of the norm $\|Du\|_X$ in terms of $\|\mathbf{W}^{\mu}_{1/p,p}\|_X$ in virtually every rearrangement invariant function space X strictly larger than $L^{p+\delta}$ (roughly: rearrangement invariant function space are all those function spaces the membership to is determined by measuring the decay of the measure of the level sets of the functions; Lebesgue, Lorentz, and Orlicz spaces belong to this class). The use of the restricted maximal operator allows to obtain a suitable localization of the estimates.

Remark 3.4 An interesting fact of Theorem 3.6 is that it refers to a structural/universal regularization property of the class of operators in question

$$u \mapsto -\operatorname{div} a(x, Du). \qquad (3.20)$$

Indeed, the exponent $\delta > 0$ in Theorem 3.6 is exactly the one given by Gehring's lemma and describing the maximal gradient regularity of solutions to homogeneous equations asserted in (3.17). As already mentioned this exponent only depends on the structural monotonicity properties of the vector field $a(\cdot)$ described in (2.3), that is on the parameters n, p, ν, L. In other words, Theorem 3.6 also refers to a certain rigidity properties of the regularity theory involving the operator in (3.20).

The following sharp regularity result is now one of the possible consequences of Theorem 3.6 in that $L^{p+\delta} \subset L(p, q(p-1))$ for every $q > 0$.

Theorem 3.7 [76] *Assume that* (2.3) *holds with* $2 \leq p < n$, *and* $0 < q \leq \infty$, $\mu \in L^1(\Omega)$ *and let* $u \in W^{1,p-1}_0(\Omega)$ *be the unique SOLA to the problem* (2.7). *It holds that*

$$\mu \in L\left((p^*)', q\right) \quad \Longrightarrow \quad Du \in L(p, q(p-1)) \quad \textit{locally in } \Omega. \qquad (3.21)$$

The result in (3.11) is actually a particular case of (3.21)—take $q = \infty$; as already indicated after (3.11), the implication in (3.21) is very delicate since the spaces involved let the problem "oscillate" around the dual/subdual case (the previous result implies that solutions are energy one when $q \leq p'$, and very weak otherwise). Again, Theorem 3.7 solves a problem raised several times before in the literature [10, 15, 56], which could not be solved for instance via rearrangement techniques that cover the case $\mu \in L(t, q)$ only for $t < (p^*)'$—immediately covered also by Theorem 3.6—and cannot moreover achieve local regularity (see for instance [9, 10]).

Remark 3.5 A stronger version of (3.19) holds in that we may replace $\mathbf{W}^{\mu}_{1/p,p}$ by the 1-fractional maximal operator $[M^{*}_{1,Q_{2R}}(\mu)]^{1/(p-1)}$.

Remark 3.6 (Parabolic pointwise estimates) Some of the results presented in this section have already seen a few extensions to the parabolic case as long as the case $p = 2$. For a parabolic version of Theorems 3.1 and 3.3 we again refer to [42] while for a version of Theorem 3.6 we refer to [12].

Remark 3.7 Further extensions of the potential estimates of Theorem 3.1 and 3.3 are possible and are concerned with anisotropic operators as

$$-\mathrm{div}\,(|Du|^{p(x)-2}Du) = \mu.$$

In this case the pointwise estimates found make use of adapted nonlinear Wolff potentials as

$$\mathbf{W}^{\mu}_{\beta,p(\cdot)}(x,R) := \int_0^R \left(\frac{|\mu|(B(x,\varrho))}{\varrho^{n-\beta p(x)}}\right)^{1/(p(x)-1)} \frac{d\varrho}{\varrho}.$$

We refer to [69] and [22] and also recall Remark 2.9. In this context we also mention the recent work [34].

4 A Fractional Approach to Regularity

This final section aims at outlining an approach to Theorem 3.2 which we think has its own interest in that it displays a method which connects in a natural way the maximal regularity of Theorem 2.6 and the classical pointwise regularization techniques known for homogeneous equations. Introducing this approach was indeed one of the objectives of [77].

Aiming at the explanation of a general point of view, let us recall how the local L^{∞}-character of the gradient of solutions $u \in W^{1,2}(\Omega)$ to homogeneous equations of the type

$$\mathrm{div}\, a(Du) = 0 \tag{4.1}$$

can be obtained. Looking at this is of course relevant since estimate (3.7) obviously provides a gradient-L^{∞}-bound when $\mu = 0$. The local boundedness of Du now follows in two steps

1) One first differentiates (4.1), proving that $Du \in W^{1,2}_{\mathrm{loc}}(\Omega)$.

2) Thanks to 1) one observes that, whenever $\xi \in \{1,\ldots,n\}$, the gradient component $v := D_{\xi}u$ is an energy solution to the linear elliptic equation with measurable coefficients given by

$$\mathrm{div}(A(x)Dv) = 0, \qquad A(x) := a_z(Du(x)).$$

At this stage the boundedness of $v \equiv D_{\xi}u$ follows applying an iteration method, as for instance the one devised in the pioneering work of De Giorgi [33]. This is in turn based on the use of *Caccioppoli's inequalities on level sets*. Defining

$$(w-k)_+ := \max\{w-k, 0\}, \qquad (w-k)_- := \max\{k-w, 0\}$$

we have that inequalities of the type

$$\int_{B_{R/2}} |D(D_\xi u - k)_+|^2\, dx \leq \frac{c}{R^2} \int_{B_R} |(D_\xi u - k)_+|^2\, dx \tag{4.2}$$

and similar variants, for instance involving $(D_\xi u - k)_-$, hold whenever $k \in \mathbb{R}$. In turn, the iteration of such inequalities yields the boundedness of $D_\xi u$. In such an iteration, *one controls the level sets of* $D_\xi u$ via the higher order derivatives $D(D_\xi u - k)_+$ and Sobolev embedding theorem, building a geometric iteration in which, at every step, the rate of convergence is dictated by the Sobolev embedding exponent.

Applying such an argument to (3.5) seems to be difficult, as 1) immediately fails due to the lack of differentiability of Du even in the simplest case given by $-\triangle u = \mu$. On the other hand Theorem 2.9 tells us that although full differentiability fails in general, fractional derivatives survive. More precisely, with the notation in (2.37), (2.40) gives that

$$[Du]_{1-\varepsilon,1;\Omega'} = \int_{\Omega'}\int_{\Omega'} \frac{|Du(x) - Du(y)|}{|x-y|^{n+1-\varepsilon}}\, dx\, dy < \infty \tag{4.3}$$

holds for every $\varepsilon \in (0,1)$, and every subdomain $\Omega' \Subset \Omega$ relatively compact with respect to Ω. The previous quantity is intuitively the L^1-norm of the "$(1-\varepsilon)$-order derivative" of Du, roughly denotable by $D^{1-\varepsilon}Du$. The inequality in (4.3) lets us think that *Caccioppoli type inequality* (4.2) *should be replaced by a fractional order version*, and using the L^1-norm, rather than the L^2-one. Indeed we have the following theorem, that we again state for simplicity in the form of an a priori estimate (this can be again removed via an approximation scheme, and by considering suitable definitions of solutions). Needless to say, what matters here is the precise form of the a priori estimate.

Theorem 4.1 (Non-local Caccioppoli inequality, [77]) *Under the assumptions of Theorem* 3.2, *whenever* $\xi \in \{1,\dots,n\}$, $k \geq 0$, *and whenever* $B_R \subseteq \Omega$ *is a ball with radius* R, *the inequality*

$$[(|D_\xi u| - k)_+]_{\sigma,1;B_{R/2}} \leq \frac{c}{R^\sigma} \int_{B_R} (|D_\xi u| - k)_+\, dx + cR^{1-\sigma}|\mu|(B_R), \tag{4.4}$$

holds for every $\sigma < 1/2$, *where the constant* c *depends only on* $n, L/\nu, \sigma$.

See (2.37) for more notation. Comparing (4.4) and (4.2), *Theorem* 4.1 *tells us that for quasilinear equations Caccioppoli's inequalities are a robust tool that keeps holding at intermediate derivatives/integrability levels.* From second derivatives in L^2 we switch to $1+\sigma$ derivatives in L^1 according to the following scheme:

	classical (4.2)	fractional (4.4)
integrability	$L^2 - L^2$	$L^1 - L^1$
differentiability	$1 \longrightarrow 2$	$1 \longrightarrow 1+\sigma$

We do think that the idea of using non-local Caccioppoli inequalities instead of the usual ones is interesting in itself as it leads to certain types of iterations which work without fully differentiating the equation. In turn, this could apply to all those problems with a lack of full differentiability. We explicitly note that *a fractional Caccioppoli inequality has been indeed derived for a problem which has formally integer order.*

The idea is now rather natural: inequality (4.4) serves to start an iteration in which at each stage we control the level set of $D_\xi u$ via the fractional derivative $D^\sigma(D_\xi u)$ and the fractional version of Sobolev embedding theorem. We come up again with a geometric iteration where each step is governed by the fractional Sobolev embedding exponent. A point we want to emphasize is that inequality (4.4) contains all the information about the pointwise gradient estimate, no matter how small σ is taken. As a matter of fact in this last step we are not explicitly using the fact that u is a solution, but rather the fact that $D_\xi u$ satisfies (4.4). This is indeed the point of view that became classical in regularity theory after [33] and that here reappears on a fractional level. For this reason, we shall report the next result in an abstract way, i.e. solutions are not necessarily involved in the next statement.

Theorem 4.2 (De Giorgi's fractional iteration, [77]) *Let $v \in L^1(\Omega)$ be a function with the property that there exist $\sigma \in (0,1)$ and $c_1 \geq 1$, and a Borel measure μ with finite total mass, such that whenever $B_R \subseteq \Omega$ is a ball with radius R and $k \geq 0$, the inequality*

$$[(|v|-k)_+]_{\sigma,1;B_{R/2}} \leq \frac{c_1}{R^\sigma}\int_{B_R}(|v|-k)_+\,dx + c_1 R^{1-\sigma}|\mu|(B_R) \tag{4.5}$$

holds. Then the estimate

$$|v(x)| \leq c \fint_{B(x,R)} |v|\,dy + c\mathbf{I}_1^{|\mu|}(x,R) \tag{4.6}$$

holds whenever $B(x,R) \subseteq \Omega$, and x is a Lebesgue point of v; the constant c depends on c_1, n, σ.

Now Theorem 3.2 follows by Theorems 4.1 and 4.2: by Theorem 4.1 we have assumption (4.5) from Theorem 4.2 satisfied by $v \equiv D_\xi u$. In turn, applying Theorem 4.2 with such a choice of v we conclude with the desired pointwise gradient bound, that is (3.7).

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