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PREFACE

Preface Issue 3-2012

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Although the problem of finding periodic solutions of Hamiltonian systems is easily explained, it is in many cases quite difficult to solve and in most cases still open. This question—among others—gave rise to the development of symplectic geometry. In this context, Alan Weinstein conjectured in 1978 that every hypersurface of the standard symplectic $\mathbb{R}^{2n}$ of so called *contact type* admits a closed orbit. Being of contact type generalises the notion of an object being star-shaped. The survey article of Frederica Pasquotto provides "A short history of the Weinstein conjecture". Some cases of this conjecture have been solved while in its full generality, it is still open.

Herbert von Kaven from Detmold in North Rhine-Westphalia established the eponymous foundation which awards usually every year the von Kaven prize to a young and promising mathematician. The decision is made by the German Research Foundation's (DFG) mathematics review board. During the 2011 annual meeting of "Deutsche Mathematiker-Vereinigung", that year's von Kaven prize was awarded to the DFG-Heisenberg-fellow Christian Sevenheck. His field of research is complex algebraic geometry and in particular mirror symmetry. Sometimes it is possible to find correspondences between two seemingly completely different situations, the so called A-model and the B-model, which may arise e.g. in algebraic geometry and string theory respectively. It may be possible to use results from the B-model to obtain results in the A-model, once these connections have been better understood. Details can be found in Christian Sevenheck's survey article "Mirror symmetry, singularity theory and non-commutative Hodge structures".

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One year ago, in Issue 3-2011, one could find the survey article by Simon Brendle on "Sphere Theorems in Riemannian Geometry", which was dedicated to Wilhelm Klingenberg who had died on the 14th October 2010. One of the main achievements of Wilhelm Klingenberg was his contribution to proving the topological sphere theorem: If the sectional curvatures of a compact simply connected n-dimensional Riemannian manifold are contained in the interval $(\frac{1}{4}, 1]$, then the manifold is homeomorphic to a sphere. In the current issue, Jost-Hinrich Eschenburg gives a personal view and a number of remarks on further fundamental scientific achievements of Wilhelm Klingenberg, his Ph.D. advisor.

The book reviews section is concerned with new publications on toric varieties, nonabelian algebraic topology and the nonlinear theory of water waves.

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SURVEY ARTICLE

A Short History of the Weinstein Conjecture

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Abstract A very natural and fundamental question (both from a historical and a mathematical point of view) is that of existence of periodic orbits of Hamiltonian flows on a fixed energy hypersurface. In 1978 Alan Weinstein conjectured that a geometric property of the hypersurface under consideration would provide a sufficient condition for the existence of such orbits. He called hypersurfaces with this property *hypersurfaces of contact type*. This article briefly describes the history of the Weinstein Conjecture, which has been one of the major driving forces behind the development of symplectic geometry at the end of the twentieth century, leading to some of the most fruitful interactions between analysis, geometry and topology. Weinstein's Conjecture has been proved in a number of significant cases but remains, in its most general form, an extremely interesting and challenging open problem.

Keywords Hamiltonian system · Periodic orbit · Symplectic form · Contact form · Reeb flow · Almost complex structure

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Many interesting physical phenomena are described by the solutions of a system of *Hamiltonian differential equations*. In their most familiar form, the Hamilton equa-

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tions look as follows:

$$\dot{p} = -\frac{\partial H}{\partial q}, \qquad \dot{q} = \frac{\partial H}{\partial p},$$

where H is the energy function and p and q are vectors: in the case of the motion of a planet or a particle, these vectors represent velocity (momentum) and position of the moving object, respectively. Due to an energy conservation principle, solutions lie on a given *energy surface*, that is, a level set of the energy function. Among the first conservative systems to receive a lot of attention and to be thoroughly investigated were the systems describing the motions of planets, stars and other astronomical objects. It is therefore not surprising that the question of existence of periodic solutions (or orbits) of such systems should arise so naturally and within a short time should come to play a prominent role in Hamiltonian mechanics. Solutions displaying a recurrent behavior are very important from a mathematical point of view, since it is precisely these solutions which correspond to the critical points of the *action functional*.

Symplectic geometry provides the right mathematical formalism to study these kind of systems and for this reason it plays an important role in the search for periodic orbits. Symplectic structures made their first appearance in connection with classical mechanical systems, but they make it possible to extend the study of Hamiltonian systems from the classical setting of Euclidean space to spaces with non-trivial geometry and topology (for example, with curvature and holes). The Weinstein Conjecture introduces a sufficient condition that leads to existence of periodic orbits on a fixed energy hypersurface: this condition is expressed in terms of the geometry of the energy hypersurface and makes use of the symplectic structure with which the ambient phase space is endowed. This conjecture has reinforced the relation between Hamiltonian dynamics and symplectic geometry and has contributed enormously to the development of symplectic geometry in the last thirty years.

The techniques and results which are currently available (and which I will discuss below) apply to bounded (*compact*) energy level sets. In practice, unbounded energy hypersurfaces arise very naturally. A good example of this is the following partial differential equation (Fisher-Kolmogorov equation):

$$\frac{\partial u}{\partial t} = -\frac{\partial^4 u}{\partial x^4} + \alpha \frac{\partial^2 u}{\partial x^2} - F(u).$$

For different choices of α and F, it describes physical phenomena as diverse as: water waves in shallow water, pulse propagation in optical fibers, geological folding of stone layers. Time-independent solutions of this equation satisfy

$$u'''' - \alpha u'' + F(u) = 0.$$

This equation also admits a Hamiltonian formulation, leading to a system with unbounded energy level sets. The existence of periodic orbits on unbounded energy level sets has scarcely been studied so far, while it does pose some very interesting geometric and topological problems.

Fig. 1 The 2-dimensional sphere with the Hamiltonian vector field X_H associated to the function 'height' by the standard area form. The level sets of H are meridians on the sphere

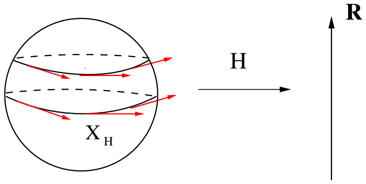
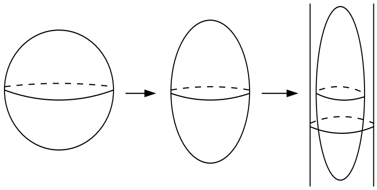


Fig. 2 A $2n$-dimensional ball can be squeezed inside a cylinder of smaller radius in a volume-preserving way, but this cannot be done symplectically



1 Symplectic Geometry

Suppose we consider a *manifold* M (think of spheres or hypersurfaces). At first we are interested in the *topology* of the manifold, that is, properties which are invariant under continuous deformations. At some point we may also want to introduce some additional structure: a *Riemannian structure* (or metric) g, for example, prescribes at every point an *inner product*, which gives us the notions of distance, length, and angles. If we are considering a system of Hamiltonian differential equations as above, with the energy function H we can associate a vector field X_H, called the *Hamiltonian vector field*, which prescribes direction and speed at each point for the solutions of the system of equations. In dimension 2 an area form suffices for this purpose, in higher dimension we need a two-form ω which is *closed* ($d\omega = 0$) and *nondegenerate*. The latter means that the n-fold exterior product $\omega^{\wedge n}$ is a volume form. We call ω a *symplectic form* and the integral of the corresponding volume form *symplectic volume*. A symplectic form on M sets up an isomorphism between vector fields and 1-forms: the vector field X_H is uniquely defined by the identity $\omega(X_H, -) = -dH(-)$. Finally, it is interesting to notice that a symplectic form ω defines an anti-symmetric product of vectors: this necessarily vanishes on all 1-dimensional subspaces, so instead of 1-dimensional measurements we have 2-dimensional measurements (Fig. 1).

While every compact orientable surface admits an area form, in higher dimension not every manifold is symplectic. The dimension of the manifold needs to be even, but this condition is far from being sufficient: spheres of dimension greater than two are not symplectic. Not every volume-preserving diffeomorphism or embedding also preserves the symplectic form: it was Gromov who clarified this point with his Non-Squeezing Theorem[1] [8]. This theorem states that a standard ball in $\mathbb{R}^{2n}$ cannot be symplectically embedded into a cylinder of the form $B^2(r) \times \mathbb{R}^{2n-2}$ (where $B^2(r)$ is the 2-disc with radius r) if r is smaller than the radius of the ball. This is of course in contrast with the situation we encounter if we consider volume-preserving embeddings, see also Fig. 2.

[1]This result is sometimes referred to as the *principle of the symplectic camel.*

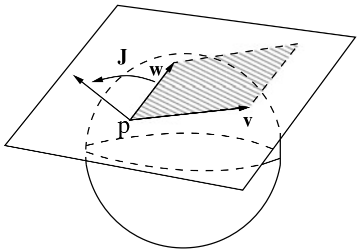


Fig. 3 The tangent space at the point p consists of vectors representing the velocity of some curve lying on the surface and going through p. Using g, ω and J we can speak of lengths, angles, areas, complex multiplication

Basic examples of symplectic manifolds are:

- Euclidean space $\mathbb{R}^{2n}$, with coordinates $(p_1, \ldots, p_n, q_1, \ldots, q_n)$ and the standard symplectic form

$$\omega_0 = \sum_{i=1}^{n} dp_i \wedge dq_i .$$

  This example is also *universal*, in the sense that up to a choice of suitable local coordinates, it provides a local model for any other $2n$-dimensional symplectic manifold;
- cotangent bundles over Riemannian manifolds are symplectic manifolds: we can think of them as phase spaces of Hamiltonian dynamical systems, with the base manifold playing the role of the configuration space. In this case the symplectic form is $\omega = -d\lambda$, where λ is the canonical 1-form defined, in local cotangent bundle coordinates (x, ξ), by $\Sigma_{i=1}^{n} \xi_i \, dx_i$.

Symplectic geometry can also be thought of as a more flexible version of complex geometry.[2] In order to further clarify some concepts, we can look at surfaces, which always admit both a metric and a symplectic structure: for a pair of tangent vectors v and w the metric g measures the length of the vectors and the angle between them. The symplectic form ω measures the area of the parallelogram spanned by v and w. We can introduce another operation, denoted by J, which rotates tangent vectors counterclockwise by $\pi/2$, giving us a notion of *complex multiplication* (see Fig. 3). Such a rotation leaves all angle, length and area measurements invariant, so we say that it is *compatible* with the metric and the symplectic structure. The relation expressing the compatibility of g, ω and J is

$$\omega(v, Jw) = g(v, w).$$

We call J a *tamed* almost complex structure if $\omega(v, Jv) > 0$ for all $v \neq 0$.

In higher dimension, every symplectic manifold still admits a complex structure on its tangent bundle (we call this an *almost complex structure* on the manifold),

[2] According to the Oxford English Dictionary, the word *symplectic* was introduced by Weyl, who proposed to substitute the name ‘complex group’ by the corresponding Greek adjective ‘symplectic’. On a trip to Asia, Klaus Niederkrüger learned that the Chinese character for symplectic is one whose standard meaning is ‘hot, spicy’, so that, at least in China, symplectic geometry is ‘hot geometry’!

which gives a notion of complex multiplication for tangent vectors. In particular, one can always find almost complex structures which are *compatible* with the symplectic structure, meaning that the two can be combined as above to produce a Riemannian metric. A complex structure on the manifold, that is, a system of complex local coordinates with holomorphic transition maps, induces an almost complex structure in a canonical way: symplectic manifolds which carry a compatible complex structure are called *Kähler manifolds*. Not every almost complex structure arises in this way: Kähler manifolds form a proper subclass of symplectic manifolds and are important objects in algebraic geometry (all smooth projective varieties, for instance, are Kähler manifolds).

A great revolution in symplectic geometry was brought about by the realization that curves whose differential is complex linear with respect to an almost complex structure compatible with a symplectic structure (*pseudo-holomorphic or J-holomorphic curves*) are almost as good as honest holomorphic curves in a complex manifold. In particular, J-holomorphic curves with respect to a tamed almost complex structure are minimal surfaces. The basis for this revolution were laid by Gromov in his 1985 paper [8], and Floer carried it further by using J-holomorphic curve techniques in his celebrated proof of the Arnold Conjecture [6]. In its original formulation, this conjecture reads: a *symplectomorphism* (i.e., a diffeomorphism which preserves the symplectic form) that is generated by a time-dependent Hamiltonian vector field should have as many fixed points as a function on the manifold must have critical points.

2 Periodic Orbits of Hamiltonian Systems

Given a symplectic manifold (M, ω) and a smooth Hamiltonian function $H : M \to \mathbb{R}$, one is interested in the existence of solutions of the associated Hamiltonian system of differential equations

$$\dot{x}(t) = X_H\big(x(t)\big),$$

where $x : I \to M$ is a path in M. In classical mechanics, the symplectic manifold under consideration is the standard symplectic Euclidean space, so we can write $x(t) = (p(t), q(t))$ and the Hamiltonian vector field has the well-known form $(-\frac{\partial H}{\partial q}, \frac{\partial H}{\partial p})$. The fundamental remark here is that existence of such solutions on a given *regular* energy surface $S = H^{-1}(c)$ is completely determined by the underlying hypersurface and the symplectic structure and does not depend on the function H. The reason is that, if H and G are two Hamiltonian functions having S as a (regular) level set, X_H and X_G coincide up to reparametrization. In other words, these vector fields are both sections of the so called *characteristic line bundle* $\mathcal{L}_S$, which is defined as the kernel of the restriction of the symplectic form to S,

$$\mathcal{L}_S = \big\{v \in TS : \omega(v, w) = 0 \text{ for all } w \in TS\big\} = \ker(\omega|_{TS}),$$

and hence have the same integral curves.

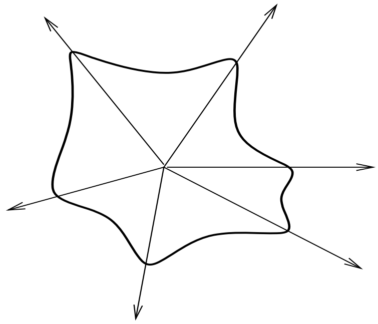


Fig. 4 A starshaped hypersurface is everywhere transverse to a radial vector field

The question that has generated some of the most interesting recent developments in Hamiltonian dynamics and symplectic topology has been that of existence of *closed characteristics* of $\dot{x}(t) = X_H(x(t))$ on a given energy level S. Independence of the particular choice of the Hamiltonian H implies that it is meaningful to pose the question as follows: given (M, ω) and S, does S admit closed characteristics?

The first important global existence results were proved by Rabinowitz and Weinstein for starlike, respectively convex, hypersurfaces in the standard symplectic Euclidean space $\mathbb{R}^{2n}$. Here is the statement of Rabinowitz's result ([12]):

Theorem 1 (Rabinowitz, 1978) *Every starshaped level set of a Hamiltonian function H in the standard symplectic $\mathbb{R}^{2n}$ admits a periodic orbit of X_H.*

Since starshaped is a very special condition, it is natural to try and isolate the essential geometric properties of this class of hypersurfaces. The following remark already points in the right direction: if $S \subset (\mathbb{R}^{2n}, \omega_0)$ is starshaped with respect to the origin, then it is everywhere transverse to the vector field

$$Y = \frac{1}{2}\sum_{i=1}^{n}\left(p_i\frac{\partial}{\partial p_i} + q_i\frac{\partial}{\partial q_i}\right).$$

Transversality means that at a given point x on the hypersurface, the vector $Y(x)$ does not lie in the tangent space to the hypersurface at that point. Notice that Y satisfies the condition $\mathcal{L}_Y\omega_0 = \omega_0$, where $\mathcal{L}_Y$ denotes the Lie derivative with respect to Y. A vector field with this property is called a *Liouville field*. The flow of a Liouville vector field preserves the symplectic form up to an exponential term or, in other words, it expands the symplectic volume (Fig. 4).

3 The Weinstein Conjecture

A similar observation probably inspired Weinstein when he introduced the notion of a hypersurface of *contact type*, a generalization of starlike, with the property of being invariant under symplectic diffeomorphisms. A compact hypersurface S in a

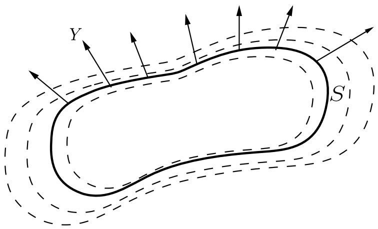


Fig. 5 The flow of the Liouville vector field gives a *foliation* of a neighborhood of S by smooth hypersurfaces diffeomorphic to S: the dynamics is the same on each hypersurface (almost existence $\Rightarrow$ existence)

symplectic manifold (M, ω) is called of *contact type* if there exists a Liouville vector field Y, defined in a neighborhood of S, which is everywhere transverse to the hypersurface. With this notion, in [16] he could formulate his famous conjecture.

Conjecture 1 (Weinstein, 1978) *Every compact hypersurface of contact type in a symplectic manifold admits at least one closed characteristic.*

A remarkable feature of hypersurfaces of contact type—which may serve as an indication that this condition is indeed a plausible one to guarantee existence of periodic orbits—is that we can always find a one-parameter family of such hypersurfaces which share the same dynamics. Recall that, thanks to independence of this problem of the choice of Hamiltonian function, what we mean here by 'dynamics' on a hypersurface S is the time evolution of the system associated to any function H having S as a regular level set. More precisely, if S is of contact type in (M, ω) and Y is a transverse Liouville field for S, we can use the flow of Y to define an embedding

$$\Psi : S \times (-\epsilon, \epsilon) \to M$$

such that the hypersurfaces $S_t := \Psi(S \times \{t\})$ are not only diffeomorphic, but their characteristic line bundles are isomorphic. It follows that if S is of contact type and one can prove existence of a periodic orbit sufficiently close to S, a periodic orbit must also exist on S. In other words, *almost existence* results automatically imply existence results for periodic orbits on S (Fig. 5).

Viterbo proved the Weinstein Conjecture for compact hypersurfaces in $\mathbb{R}^{2n}$ [15].

Theorem 2 (Viterbo, 1986) *The Weinstein Conjecture holds for compact hypersurfaces of contact type in the standard symplectic space* $\mathbb{R}^{2n}$.

In his proof, he makes use of the freedom to choose a suitable Hamiltonian function and identifies periodic orbits with critical points of the corresponding *Hamiltonian action functional*, which takes the form

$$\mathcal{A}(x) = -\int_0^1 \big(p(t)\dot{q}(t)\,dt - H\big(p(t), q(t)\big)\big)\,dt$$

with respect to the splitting of a loop $x(t) = (p(t), q(t)) \in \mathbb{R}^n \times \mathbb{R}^n$. While non-compact hypersurfaces occur very naturally as energy levels (higher order Lagrangian problems, singular potentials, Lorentzian geodesic problem...), Viterbo's result does

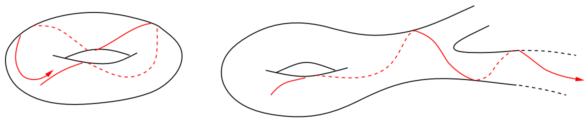


Fig. 6 On bounded energy level sets, every orbit is almost periodic, while on unbounded levels Poincarè recurrence fails and orbits can 'run off to infinity'

not hold anymore if we remove the compactness assumption. An easy example of this failure is the infinite cylinder in $\mathbb{R}^{2n}$ defined as the 1-level set of the function $\Sigma_{i=1}^n p_i^2$. It is probably intuitively clear that existence of periodic orbits on non-compact hypersurfaces is a harder question than existence on compact hypersurfaces. *Poincaré's recurrence theorem* gives this intuition a precise mathematical formulation, by stating that on a compact energy surface, every orbit eventually returns very close to its initial point (i.e., is *almost periodic*) (Fig. 6).

In [3] we were able to formulate a set of geometric and topological conditions (implying, in particular, the contact type condition) that led to a proof of the existence of periodic orbits for the case of non-compact *mechanical hypersurfaces* in $\mathbb{R}^{2n}$, that is, hypersurfaces arising as level sets of Hamiltonian functions of the form $H(p,q) = \frac{1}{2}|p|^2 + V(q)$, consisting of kinetic and potential energy. For these mechanical systems, periodic orbits can be detected as critical points of the *Lagrangian action functional*

$$\mathcal{A}(q) = -\int_0^1 \left(\frac{1}{2}\left|\dot{q}(t)\right|^2 - V\big(q(t)\big)\right) dt,$$

which is defined on the space of loops $q(t)$ in $\mathbb{R}^n$ (the configuration space). One necessary condition for proving existence of a critical point of such a functional is the so called *Palais-Smale condition*, which ensures that sequences along which the functional is bounded and its derivative tends to zero have convergent subsequences. In the case of compact mechanical hypersurfaces, compactness takes care of this condition, in the non-compact case we need to require the *potential energy* V to satisfy additional asymptotic growth conditions (the result in [3] holds, for instance, in the case of asymptotically quadratic potential energy). It is interesting to remark that these growth conditions imply, in particular, that the hypersurfaces must be of contact type. The other ingredient which goes into the proof of the existence of critical points of the action functional is a *linking argument*. In this setup, existence of a critical (saddle) point of the action functional is proved by producing two subsets of the function space which link and along which the functional satisfies appropriate estimates.

4 Contact Manifolds and Reeb Dynamics

A different step in the direction of generalizing the study of existence of periodic orbits was that of focusing on the properties of a hypersurface of contact type that could be considered independently of its embedding in a symplectic manifold. Recall that a *contact structure* on a compact, orientable manifold of dimension $2n-1$ is a maximally non-integrable hyperplane distribution ξ. In other words, if $\xi = \ker\alpha$ locally, then $\alpha \wedge d\alpha^{n-1}$ is a non-vanishing top form. Let S be a hypersurface of contact type in the symplectic manifold (M,ω) and Y a Liouville vector field for S: then the kernel of the 1-form λ defined by $\lambda(-) = \omega(Y,-)$ is a contact structure on S. The kernel of $d\lambda$ is one-dimensional, so we can uniquely define a vector field X_λ (called the *Reeb vector field*) as a generator of this kernel, normalized so that $\lambda(X_\lambda) = 1$. If H is a Hamiltonian function having S as a level set, the periodic orbits of the Hamiltonian vector field X_H on S coincide with the closed trajectories of the Reeb vector field, so the Weinstein Conjecture for hypersurfaces of contact type can be restated as a conjecture on the existence of periodic orbits of the Reeb flow on contact manifolds.

Conjecture 2 (Intrinsic version of the Weinstein conjecture) *For every closed odd-dimensional manifold N with contact form λ, the Reeb vector field X_λ admits a closed orbit.*

In this more general form, the conjecture was first proved by Hofer in [9] for the three-sphere.

Theorem 3 (Hofer, 1993) *The Weinstein Conjecture holds for the* 3*-sphere* $\mathbb{S}^3$.

More recently, the conjecture was proved for closed 3-dimensional manifolds, a result due to Taubes ([14]).

Theorem 4 (Taubes, 2007) *The Weinstein Conjecture holds for any closed* 3*-dimensional manifold.*

In fact, in [9] Hofer also proved that the conjecture holds for any *overtwisted* contact 3-manifold and for contact 3-manifolds with non-vanishing second fundamental group. Different generalizations of Hofer's result to $(2n+1)$-dimensional contact manifolds were achieved by Albers and Hofer in [2] and by Niederkrüger and Rechtman in [10]. In particular, the authors of [10] generalize the statement about contact 3-manifolds with non-vanishing second fundamental group by proving that one finds at least one periodic contractible Reeb orbit if there exists an embedded $(n+1)$-dimensional submanifold which represents a non-trivial homology class and such that the contact structure induces an *open book decomposition*. A beautiful introduction to open books and their various applications can be found in [13], in the Appendix by E. Winkelnkemper.

In recent literature one finds yet another version of the Weinstein Conjecture. A contact manifold is said to satisfy the *strong Weinstein Conjecture* if for any form defining the contact structure there exists a finite collection of closed Reeb orbits x_i, $i = 1,\dots,k$, representing 1-dimensional homology classes x_i such that $\Sigma_{i=1}^k [x_i] = 0$ (such a collection of Reeb orbits is also called a *null-homologous Reeb link*). The notion of strong Weinstein Conjecture was introduced in [1], where it is proved for *planar* contact structures. A planar contact structure ξ is a contact structure supported by an open book decomposition whose pages have genus zero. Geiges and Zehmisch in [7] prove the strong Weinstein Conjecture for various higher-dimensional contact manifolds, in particular for contact type hypersurfaces in cotangent bundles of the form $T^*(Q \times \mathbb{S}^1)$, where Q is any closed manifold.

The proofs of Hofer's result and corresponding generalizations and the proofs of the results on the strong Weinstein Conjecture are based on *J-holomorphic curves*.

5 *J*-Holomorphic Curves

Let J be an almost complex structure on a manifold M: this gives us a notion of complex multiplication for tangent vectors, that is, $J^2 v = -v$ for any tangent vector v on M. The local model for such a structure is the standard almost complex structure $J_0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ on $\mathbb{R}^{2n}$ (under the identification with $\mathbb{C}^n$, this is just multiplication by i.)

Let Σ be a Riemann surface with holomorphic structure j. A map $u : \Sigma \to M$ is called a *J-holomorphic curve* if its differential is complex linear: $du \circ j = J \circ du$. On the level of tangent spaces we have, for each $x \in \Sigma$, a commutative diagram

$$\begin{array}{ccc} T_x\Sigma & \xrightarrow{\ j_x\ } & T_x\Sigma \\ {\scriptstyle du(x)}\big\downarrow & & \big\downarrow{\scriptstyle du(x)} \\ T_{u(x)}M & \xrightarrow{\ J_{u(x)}\ } & T_{u(x)}M \end{array}$$

If we decompose du into its complex linear and anti-linear part, we see that u is J-holomorphic if and only if the anti-linear part

$$\bar{\partial}_J u = \frac{1}{2}(du + J \cdot du \cdot j)$$

vanishes. After choosing holomorphic coordinates $z = s + it$ on Σ, this condition takes the form

$$\partial_s u + J(u)\partial_t u = 0.$$

Hence, for $\mathbb{C}^n$ with almost complex structure given by multiplication by i, we recover the standard Cauchy-Riemann equations for maps $u : \mathbb{C} \to \mathbb{C}^n$. This explains why the equations defining J-holomorphic curves are often called *perturbed Cauchy-Riemann equations*.

Given a contact manifold (N, λ), the *symplectization* of N is the manifold $M = \mathbb{R} \times N$ with symplectic form $\omega = d(e^t\lambda)$, where t is the coordinate on $\mathbb{R}$. The above-mentioned result by Hofer, proving the Weinstein Conjecture for the 3-sphere, is based on the following idea: we fix a suitable compatible almost complex structure J on M and look at J-holomorphic curves in $\mathbb{R} \times N$, that is, we consider proper J-holomorphic maps

$$F : (\Sigma, j) \longrightarrow (\mathbb{R} \times N, J),$$

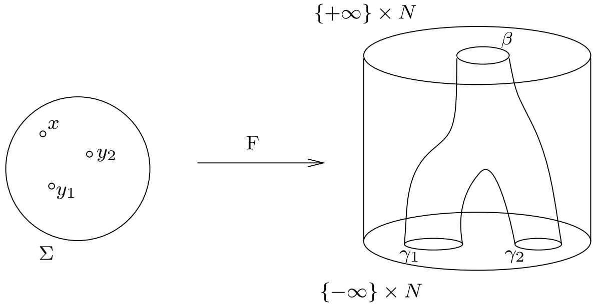


Fig. 7 A holomorphic sphere with 3 punctures in the symplectization of the contact manifold N

where Σ now denotes a closed Riemann surface with finitely many points removed ('punctures'). The energy of such a curve is defined as the integral of $d\lambda$ over its image. Figure 7 gives an impression of how this image might look like.

Under suitable assumptions, for instance *finite energy*, the image of F near each puncture converges asymptotically (at $+\infty$ and $-\infty$) to a cylinder over a closed Reeb orbit of N. Therefore, from an existence result for solutions of the perturbed Cauchy-Riemann equations (which is a set of partial differential equations) one can deduce existence of periodic solutions of an ordinary differential equation. At first this might not seem the most natural way to approach the problem (studying a PDE rather than an ODE usually does not make life easier!), but it has proved to be very efficient in overcoming certain difficulties which make other methods not applicable.

While in Hofer's work the construction of suitable solutions of the perturbed Cauchy-Riemann equations was achieved with constructive methods, in more recent years a homology theory (*contact homology*) has appeared in the background of this construction. Taking infinite-dimensional Morse theory as a model, one can associate with a contact manifold (N, λ) an algebraic object (a ring or an algebra) which is generated by the periodic Reeb orbits, modulo some relations which are obtained by looking at J-holomorphic curves connecting these orbits in the symplectization of N. It follows from [5] that these curves form compact, 0-dimensional moduli spaces and therefore it is possible to "count" them. If the algebraic object defined in this way is not isomorphic to the coefficient ring, then the Reeb field X_λ has to have a periodic orbit (a precise statement is provided by [4, Proposition 3.6]). Very schematically, thus, we can view this homology theory as combining the geometric and topological information about some contact manifold (N, λ) and its contact structure to obtain an algebraic object, which in turn can tell us something about the Reeb dynamics on N.

This scheme has been greatly developed in recent years (by Bourgeois, Eliashberg, and Hofer to name but a few) and it has started to work pretty well in the case of compact contact manifolds, but it will be a real challenge to make it work in the case of non-compact energy surfaces, a case which is extremely interesting from a mathematical point of view, as well as from the point of view of the applications in dynamics.

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SURVEY ARTICLE

Mirror Symmetry, Singularity Theory and Non-commutative Hodge Structures

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Abstract We review a version of the mirror correspondence for smooth toric varieties with a numerically effective anticanonical bundle. We give a precise description of the so-called B-model, which involves the Gauß-Manin system of a family of Laurent polynomials. We show how to derive from these data a variation of non-commutative Hodge structures and describe general results on period maps and classifying spaces for these generalized Hodge structures. Finally, we explain a version of mirror symmetry as an isomorphism of Frobenius manifolds.

Keywords Quantum cohomology · Frobenius manifold · Gauß-Manin system · Hypergeometric $\mathcal{D}$-module · Toric variety · Landau-Ginzburg model · Mirror symmetry · Non-commutative Hodge structure

Mathematics Subject Classification (2010) 14J33 · 14M25 · 14D07 · 32S40 · 53D45

1 Introduction

The aim of this survey is to describe how classical constructions from singularity theory enter into mirror symmetry. The latter subject evolves from string theory, but has become over the last 20 years one of the main branches of research in pure mathematics, connecting various areas like algebraic, symplectic and differential geometry, integrable systems, linear differential equations, homological algebra and so on. Due to the complexity of the subject, we will limit this survey to a particular aspect of the

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mirror symmetry picture, in which linear differential equations with complex coefficients (also called $\mathcal{D}$-modules) play a central role.

Let us start this introduction with a well-known motivating example which does not come from physics, but which is a very classical problem in enumerative algebraic geometry. Enumerative geometry is concerned with the question of counting geometric objects of a certain type that satisfy some extra conditions. Here we will be interested in the number of curves in the plane passing through some prescribed set of points. To be more precise, we will look at algebraic curves, that is, vanishing loci of a single polynomial in two variables, second, we will work over the complex numbers, that is, we take polynomials with complex coefficients to be sure that our vanishing loci are really 1-dimensional as topological spaces (we do not want things like $\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 + 1 = 0\} = \emptyset$), and last we will actually look at projective curves, that is, we consider the zero locus in the projective plane $\mathbb{P}^2$ of a *homogeneous* polynomial in three variables. This is the usual approach in algebraic geometry that excludes pathological facts like parallel lines with no intersections. The degree of such a polynomial, henceforth called the degree of the curve, will be a fixed positive integer denoted by d. The problem will consists in determine the number of such curves passing through some fixed points. It is easy to see that we need $3d - 1$ points in general position to have a chance that the number of curves through these points is finite. For small values of d these numbers are known via classical methods of algebraic geometry:

- $d = 1$ and $d = 2$: The number of lines through two points as well as the number of quadrics through 5 general points is known to be one since antiquity.
- The number of cubics (curves of degree three) through 8 general points is 12 (Steiner, 1848).
- For $d = 4$, Zeuthen (1873) computed the number of quartics through 11 points in general position to be 620.

However, for higher d there is no general method to calculate theses number. It came as a surprise when Kontsevich produced a formula that allows one to do this calculation for all d. He used in an essential way ideas from string theory. The precise result is as follows.

Theorem (Kontsevich, 1994) *Denote by N_d the number of rational curves of degree d passing through $3d - 1$ points in $\mathbb{P}^2$ which lie in general position. Then the following recursive formula holds.*

$$
\begin{aligned}
N_d = & \sum_{d_1+d_2=d;d_1,d_2\geq 1} d_1^2 d_2^2 \binom{3d-4}{3d_1-2} N_{d_1} N_{d_2} \\
& - \sum_{d_1+d_2=d;d_1,d_2\geq 1} d_1^3 d_2 \binom{3d-4}{3d_1-1} N_{d_1} N_{d_2}
\end{aligned}
$$

The meaning of this theorem is that we only need to know the first two numbers N_d and then we can calculate all others recursively by the computer. Without giving the details of the proof, let us just outline the strategy: First one re-interprets the numbers

N_d as a so-called Gromov-Witten invariant (precise definitions and properties are below in Sect. 2). These give rise to the quantum product on the cohomology space of $\mathbb{P}^2$. One of the main properties of the latter is its associativity, and Kontsevich's proof consists in deriving the above recursive formula from that property.

There is a similar problem in enumerative geometry where classical methods fail to produces results beyond the easiest cases. Namely, it concerns the number of curves of fixed degree on special three-dimensional complex manifold, called Calabi-Yau. Let us give here for future use the precise definition of these and some related varieties.

Definition 1.1 Let X be a smooth and projective algebraic variety over the complex numbers. Let n be the dimension of X. Denote by K_X its canonical bundle, by definition, $K_X := \bigwedge^n \Omega^1_X$ is the top exterior power of the cotangent bundle, hence, as the latter is of rank n, a line bundle. Then we call X

1. Calabi-Yau iff $K_X \cong \mathcal{O}_X$, that is, if it is the trivial line bundle.
2. Fano iff $-K_X$ is ample, that is, if there in an embedding $i : X \hookrightarrow \mathbb{P}^N$ such that for some $n \in \mathbb{N}$ we have $K_X^{\otimes n} = i^*\mathcal{O}_{\mathbb{P}^N}(-1)$.
3. numerically effective (nef) or sometimes also weak Fano if the intersection of $-K_X$ with any curve is non-negative (recall that to the line bundle $-K_X$ we can associate a divisor which have a well defined intersection product with curves).

Notice that both Calabi-Yau varieties and Fano varieties are nef.

The most prominent example of a Calabi-Yau threefold is a hypersurface of degree five in $\mathbb{P}^4$ (that this satisfies the Calabi-Yau condition is an immediate consequence of the adjunction formula). For those, [3] gave a formula predicting of the number of curves of fixed degree. This formula is not achieved by direct computation of Gromov-Witten invariants or by using formal properties of the quantum product. Rather, a basic principles of mirror symmetry comes into play: one is given a pair of Calabi-Yau manifolds (or a family thereof), and the quantum cohomology of one of these varieties (called A-model) can be obtained from much more classical invariants (like period integrals) of the other (family of) Calabi-Yau manifold(s), called B-model. To be more precise, the so-called quantum $\mathcal{D}$-module (see Sect. 3 below) of the given Calabi-Yau manifold can be reconstructed as a classical variation of Hodge structures of the mirror family. Let us notice however that in contrast to the case of curves in $\mathbb{P}^2$ discussed above, the enumerative meaning of the Gromov-Witten invariants of the quintic is less clear: The corresponding numbers should be seen as the "virtual number" of curves of fixed degree on X. However, for some degrees they actually coincide with the true numbers, and even this limited result gives interesting enumerative information that was not available by classical algebraic geometry.

It soon turned out that this picture can be extended to Fano and nef varieties. One striking difference to the Calabi-Yau case is that the mirror is no longer a family of compact varieties, but rather an affine morphism, e.g. a family of Laurent polynomials. These are called Landau-Ginzburg models. One aspect that is central to this survey is to explain which kind of Hodge-like structures survive in this more general correspondence. The objects which generalize usual Hodge structures in the correct way that is needed for these more general mirror correspondences are nowadays called non-commutative Hodge structures. This name stems from a (conjectural) relation to the so-called homological mirror symmetry, which very roughly speaking expresses the above mentioned correspondence between A- and B-model through an equivalence of certain categories. However, we are not going to touch upon any aspects of homological mirror symmetry in this survey. Rather, we emphasize the fact that non-commutative Hodge structures, in contrast to classical ones, are certain systems of linear differential equations. For that reason, one may try to express the mirror correspondence by identifying such differential systems on both sides. This is possible at least in the case where the variety on the A-side is toric, then the rather well understood theory of hypergeometric differential equations comes into play.

The structure of this article is as follows: We recall the very basics of quantum cohomology in Sect. 2, where we restrict to the easiest case of genus zero invariants for convex manifolds. This is not quite sufficient for all examples that we are interested in but avoids technical difficulties. Next (Sect. 3) we describe the so-called quantum $\mathcal{D}$-module which defines the differential system on the A-side. We proceed in Sect. 4 by a detailed description of the mirror of a toric Fano resp. numerically effective variety, and show how to define and calculate its associated *Gauß-Manin-system*. The latter will ultimately be the mirror partner for the quantum $\mathcal{D}$-module, this correspondence is worked out in Sect. 7. The Sect. 6 gives definition and some important results on non-commutative Hodge structures and describe how they appear in mirror symmetry.

2 Quantum Cohomology of Smooth Projective Varieties

We recall in this section the definition of the quantum cohomology ring of a smooth projective variety. As there are many excellent sources available (e.g. [5, 9, 23]), we mainly fix the notation for the later sections. Throughout the first two sections, we will denote by X a smooth projective variety having only even-dimensional cohomology classes.

Before giving precise definitions, let us point out the main idea of the construction of Gromov-Witten invariants and of the quantum product. Suppose that we are interested in an enumerative problem associated to X like those mentioned in the introduction, that is, counting the number of curves on X of a certain degree satisfying some incidence conditions, e.g., passing through some subspaces of X. These subspaces define homology classes on X, and as X is compact, they correspond via Poincaré duality to some cohomology classes. Now the idea is to construct a moduli space of all maps from $\mathbb{P}^1$ to X of a certain degree (this is the degree of the curves to be considered). Our $\mathbb{P}^1$ will also be equipped with some points, called markings, which are allowed to vary in the moduli space. The markings define maps from the moduli space to X (one for each marking) and our enumerative invariants will be obtained by pulling back the aforementioned cohomology classes to the moduli space and then integrating them against the fundamental class of the moduli space. It can be shown that in favorite situations, this construction (with some technical modifications, e.g., in order to obtain compact moduli spaces we need to consider also maps from certain singular curves) can be carried out and the invariants thus defined indeed have the enumerative meaning that we are looking for. Let us notice however that the general construction is very involved, in particular, the ordinary fundamental class of the moduli space may not be the right one (because the moduli space may have the "wrong" dimension, i.e., its dimension is higher than the so-called "expected one"). In order to circumvent this problem, one needs to construct a *virtual fundamental class*, and this uses rather advanced techniques like stacks, obstruction theories etc. However, as already mentioned above, we will concentrate on the simple case of genus zero Gromov-Witten invariants of *convex* varieties (see below), where these sophisticated techniques are unnecessary.

We now start with a precise description of the moduli spaces involved. First we need to describe what kind of curves on X we are looking at.

Definition 2.1 (Stable map) Let C be a reduced projective curve of genus zero with at most nodal singularities, i.e., singular points that are locally given by an equation $x \cdot y = 0$. Suppose that $x_1, \ldots, x_n \in C$ are distinct smooth points. Let $\beta \in H^2(X, \mathbb{Z})/Tors$. A stable map $f : C \to X$ is a projective morphism such that $f_*([C]) = \beta$ and such that each smooth component of C that is contracted by f to a point in X has at least three marked points.

Theorem 2.2 *Let X be convex, i.e., for all maps $f : \mathbb{P}^1 \to X$, we have that $H^1(\mathbb{P}^1, f^*(TX)) = 0$, and $\beta \in H^2(X, \mathbb{Z})/Tors$. Then there exists a coarse moduli space $\overline{\mathcal{M}}_{(0,n)}(X, \beta)$ of stable maps which is a projective variety of dimension $n + \dim(X) + \int_\beta c_1(X) - 3$ with at most orbifold singularities. In particular, it carries a well-defined fundamental cycle $[\overline{\mathcal{M}}_{(0,n)}(X, \beta)]$ of degree equal to the dimension of $\overline{\mathcal{M}}_{(0,n)}(X, \beta)$.*

For the definitions and basic results below, we will for simplicity of the exposition always suppose that X is convex. Notice however that not all examples that are going to occur later satisfy this assumption, e.g., the Hirzebruch surfaces $\mathbb{F}_1$ and $\mathbb{F}_2$ (see end of Sect. 4) are not convex.

We chose once and for all a basis of the cohomology space $H^*(X, \mathbb{C})$ consisting of homogeneous classes $T_0, T_1, \ldots, T_r, T_{r+1}, \ldots, T_s$ such that $T_0 = 1 \in H^0(X, \mathbb{C})$ is the Poicaré dual of the fundamental class and such that $T_1, \ldots, T_r$ form a basis of $H^2(X, \mathbb{C})$.

Definition 2.3 (GW-invariant) Let $\alpha_1, \ldots, \alpha_n$ be cohomology classes in $H^*(X, \mathbb{C})$, then we define the *correlator* or *Gromov-Witten invariant* to be

$$\langle \alpha_1, \ldots, \alpha_n \rangle_{0,n,\beta} := \int_{[\overline{\mathcal{M}}_{(0,n)}(X,\beta)]} ev_1^*(\alpha_1) \cup \cdots \cup ev_n^*(\alpha_n).$$

Here for $i \in \{1, \ldots, n\}$ the map $ev_i : \overline{\mathcal{M}}_{(0,n)}(X, \beta) \to X$ sends a class of a stable map $(C, [x_1, \ldots, x_n], f : C \to X) \in \overline{\mathcal{M}}_{(0,n)}(X, \beta)$ to $f(x_i) \in X$.

Definition 2.4 (Quantum cohomology) Denote by $\mathbb{L}_{\text{eff}}$ the set of effective homology classes in $H_2(X, \mathbb{Z})$/Tors, i.e., classes represented by curves. Denote by

$g : H^*(X,\mathbb{C}) \times H^*(X,\mathbb{C}) \to \mathbb{C}$ the Poincaré pairing, which is symmetric (recall that we suppose $H^*(X,\mathbb{C}) = H^{2*}(X,\mathbb{C})$) and non-degenerate. For any triple of cohomology classes $\alpha, \gamma, \eta \in H^*(X,\mathbb{C})$, we define the big quantum product, denoted by $\alpha \circ_\eta \gamma$, by its values under g on any class $\delta \in H^*(X,\mathbb{C})$ using the formula

$$g(\alpha \circ_\eta \gamma, \delta) := \sum_{n\geq 0;\, \beta\in\mathbb{L}_{\mathrm{eff}}} \frac{1}{n!}\langle \alpha, \gamma, \underbrace{\eta, \ldots, \eta}_{n\text{-times}}, \delta\rangle_{0,n+3,\beta} \in H^*(X,\mathbb{C}) \tag{1}$$

Remark There is a technical obstacle in the definition of the quantum product: Formula (1) does not make sense as such, as we are considering an infinite sum over *both* n and β. Hence it does not even define a formal sum. This problem is usually solved by splitting the contributions of the different homology classes β in $\mathbb{L}_{\mathrm{eff}}$ using the so-called Novikov ring. However, the above definition makes sense once we know that there is a domain in the parameter space on which the quantum product is convergent. Throughout this survey, we will use this assumption without further mentioning. More precise results on the convergence of the quantum product can be found, e.g., in [18].

Let us summarize very briefly some of the most important properties of the quantum product.

Proposition 2.5 *Consider the big quantum product* $\circ$ *as above.*

1. $\circ$ *is symmetric, associative with unit* $1 \in H^0(X,\mathbb{C})$.
2. *Gromov-Witten invariants have a special behavior with respect to degree* 2 *classes. More precisely, suppose that* $\alpha_1 \in H^2(X,\mathbb{C})$, *then we have that*

$$\langle \alpha_1, \ldots, \alpha_k\rangle_{0,n,\beta} = \left(\int_\beta \alpha_1\right) \cdot \langle \alpha_2, \ldots, \alpha_k\rangle_{0,n-1,\beta}$$

   This implies that we can rewrite the definition of the quantum product, that is, formula (1) *by decomposing a class* $\eta \in H^*(X,\mathbb{C})$ *into a sum* $\eta = \eta' + \eta''$ *with* $\eta' \in H^2(X,\mathbb{C})$ *and* $\eta'' \in H^{\neq 2}(X,\mathbb{C})$. *Then we have*

$$g(\alpha \circ_\eta \gamma, \delta) := \sum_{n\geq 0;\, \beta\in\mathbb{L}_{\mathrm{eff}}} \frac{e^{\eta'(\beta)}}{n!}\big\langle \alpha, \gamma, \underbrace{\eta'', \ldots, \eta''}_{n\text{-times}}, \delta\big\rangle_{0,n+3,\beta} \in H^*(X,\mathbb{C}) \tag{2}$$

3. *A convenient way to collect all Gromov-Witten invariants is the so-called* (*genus zero*) *potential, this is the formal function on* $H^*(X,\mathbb{C})$ *defined by*

$$\mathcal{F}(\underline{t}) := \sum_{n\geq 3,\, \beta\in H_2(X,\mathbb{Z})} \frac{1}{n!}\langle \underbrace{\underline{t}, \ldots, \underline{t}}_{n\text{-times}}\rangle_{0,n,\beta}$$

   Here $\underline{t} = (t_0, t_1, \ldots, t_s)$ *are the coordinates on* $H^*(X,\mathbb{C})$ *corresponding to the choice of a homogeneous basis* $T_0, T_1, \ldots, T_s$ *from above.*

4. *The associativity of the quantum product can be very nicely expressed using the Gromov-Witten potential. It is equivalent to the following system of partial differential equations satisfied by $\mathcal{F}$, which are called WDVV-equations (after Witten, Dijkgraaf, E. Verlinde and H. Verlinde):*

$$\sum_{e,f=0}^{s} (\partial_i \partial_j \partial_e \mathcal{F}) \cdot g^{ef} \cdot (\partial_f \partial_k \partial_l \mathcal{F}) = \sum_{e,f=0}^{s} (\partial_k \partial_j \partial_e \mathcal{F}) \cdot g^{ef} \cdot (\partial_f \partial_i \partial_l \mathcal{F})$$

for any $i, j, k, l \in \{0, \ldots, s\}$, where $(g^{ef})_{e,f \in \{0,\ldots,s\}} := (g(T_e, T_f))^{-1}$.

For many computations, it is sufficient to calculate only a restricted set of Gromov-Witten invariants, which involve the moduli space of stable maps from curves with only three marked points (the so-called three point invariants or correlators). A basic result due to Kontsevich and Manin (see [21]) says that often the (big) quantum product can be reconstructed from the small one. The precise definition of the small quantum product is as follows.

Definition 2.6 (The small quantum product) Let, as before, α, γ and δ be arbitrary classes in $H^*(X, \mathbb{C})$ and take η to be in $H^2(X, \mathbb{C})$. Then we define

$$g(\alpha \star_\eta \gamma, \delta) := \sum_{\beta \in \mathbb{L}_{\text{eff}}} e^{\eta(\beta)} \langle \alpha, \gamma, \delta \rangle_{0,3,\beta} \tag{3}$$

Remarks The divisor axiom and the formula (2) that it implies show that we can naturally define the quantum product and the potential on the space $H^0(X, \mathbb{C}) \oplus (H^2(X, \mathbb{C})/2\pi i H^2(X, \mathbb{Z})) \oplus H^{>2}(X, \mathbb{C})$ (where $2\pi i H^2(X, \mathbb{Z})$ acts on $H^2(X, \mathbb{C})$ by translation), which is a product of an affine space with a torus. Namely, the Gromov-Witten invariants resp. the potential depend on the coordinates $t_1, \ldots, t_r$ on $H^2(X, \mathbb{C})$ only via the exponentials $e^{\eta'(\beta)}$, so that by putting $q_a := e^{t_a}$ $(a = 1, \ldots, r)$, we obtain a function (resp. a tensor) in the variables $t_0, q_1, \ldots, q_r, t_{r+1}, \ldots, t_s$.

At several places below, we will have to use the fact that the quantum product (written in the above q-coordinates) carry an inherent grading. More precisely, consider the first Chern class of X, i.e., the first Chern class of the tangent bundle of X, this is a cohomology class of degree two and hence can be written as $c_1(X) = \sum_{a=1}^{r} d_a T_a$. Write $\deg(T_i) = k$ iff $T_i \in H^{2k}(X, \mathbb{C})$ and put

$$\begin{aligned} \deg(q_a) &= 2 \cdot d_a \\ \deg(t_i) &= 2 - 2 \cdot \deg(T_i) \end{aligned} \tag{4}$$

Then the potential $\mathcal{F}$ has a certain homogeneity property with respect to this grading (to be more precise, the quantum part

$$\mathcal{F}^{\text{quant}}(\underline{t}) := \sum_{n \geq 3, \beta \in H_2(X,\mathbb{Z}) \setminus \{0\}} \frac{1}{n!} \langle \underbrace{\underline{t}, \ldots, \underline{t}}_{n\text{-times}} \rangle_{0,n,\beta}$$

is homogeneous of degree $2(3 - \dim(X))$).

Example As a well known and instructive example, we are going to compute here the small quantum product of the projective spaces. The advantage of this case is that the mirror correspondence with the Landau-Ginzburg model can be very explicitly written down (see Sect. 7 below) and this motivates also the general mirror constructions for toric varieties, as explained below.

Let $\mathbb{P}^n$ be the n-dimensional projective space. It is well-known that its ordinary cohomology ring $H^*(\mathbb{P}^n, \mathbb{Z})$ is isomorphic to $\mathbb{Z}[p]/(p^{n+1})$. Here p denotes the degree 2 cohomology class which is Poincaré dual to the class of a hyperplane $H \subset \mathbb{P}^n$. In particular (this is true for any smooth toric variety), the cohomology is generated as a ring by its degree two classes and hence we have $H^*(\mathbb{P}^n, \mathbb{Z}) = H^{2*}(\mathbb{P}^n, \mathbb{Z})$, that is, only even dimensional cohomology classes do appear.

The small quantum cohomology ring is by definition a finitely generated algebra over $\mathbb{C}[q^{\pm}] := \mathbb{C}[q, q^{-1}]$, where q is the coordinate on $H^2(\mathbb{P}^n, \mathbb{C})/2\pi i H^2(\mathbb{P}^n, \mathbb{Z})$ corresponding to the choice of p as generator of $H^2(\mathbb{P}^n, \mathbb{Z})$. It is graded by $\deg(p) = 2$ and $\deg(q) = 2c_1(\mathbb{P}^n) = 2(n+1)$. As a $\mathbb{C}[q^{\pm}]$-module, it is isomorphic to $\mathbb{C}[p]/(p^{n+1}) \otimes \mathbb{C}[q^{\pm}]$. The degree preserving property of the quantum product tells us that for any $k \in \{1, \ldots, n\}$, we have

$$\underbrace{p \star \cdots \star p}_{k\text{-times}} \stackrel{!}{=} \underbrace{p \cup \cdots \cup p}_{k\text{-times}} =: p^k$$

so that it suffices to compute $p^{\star(n+1)}$ which equals $p^{\star n} \star p = p^n \star p$. Notice that here we do not have to put a parameter as index to the small quantum product as we consider it as a family of algebras (see the remark after Definition 3.1 below for a more conceptual explanation). We use the definition of the quantum product, i.e., formula (3), saying that we have to compute for any class $\gamma \in H^*(\mathbb{P}^n, \mathbb{C})$ the expression

$$\begin{aligned} g\big(p^n \star p, \gamma\big) &= \sum_\beta q^{p(\beta)} \langle p^n, p, \gamma\rangle_{0,3,\beta} \\ &= \sum_\beta q^{p(\beta)} \left(\int_{[\mathcal{M}_{0,3}(\mathbb{P}^n, \beta)]} ev_1^*\big(p^n\big) \otimes ev_2^*(p) \otimes ev_3^*(\gamma) \right) \end{aligned}$$

Notice that β is always a integer multiple of the class of a line $[L] \in H_2(\mathbb{P}^n, \mathbb{Z})$ dual to p, so that we can rewrite the sum as $\sum_{d \geq 0} \langle p^n, p, \gamma\rangle_{0,3,d[L]} \cdot q^d$. The correlators in this sum are nonzero only if the degrees of the classes p^n, p and γ add up to the dimension of $\mathcal{M}_{0,3}(\mathbb{P}^n, d[L])$, that is, to $3 + \dim(\mathbb{P}^n) + \int_{d \cdot [L]} (n+1) \cdot \mathrm{PD}([H]) - 3 = n + d \cdot (n+1)$. Hence only classes γ of degree $\deg(\gamma) = (n+1)d - 1$ can give a nonzero correlator. Since $\deg(\gamma) \leq n$, we arrive at the conclusion that d can only take the value 1 and then $\deg(\gamma) = n$. Hence we are left with $\langle p^n, p, p^n\rangle_{0,3,[L]}$, and this is the number of lines in $\mathbb{P}^2$ through two generic points meeting a generic hypersurface. Obviously, there is only one such line, so that we finally arrive at the conclusion that $g(p^{\star(n+1)}, \gamma) = q$ if $\gamma = p^n$ and 0 else, meaning that the relation $p^{\star n+1} = q$ holds in the small quantum cohomology ring, i.e., we have the isomorphism

$$SQH\big(\mathbb{P}^n\big) = \big(H^*(X, \mathbb{C}), \star\big) \cong \mathbb{C}\big[p, q^{\pm}\big] / \big(p^{n+1} - q\big) \tag{5}$$

of $\mathbb{C}[q^{\pm}]$-algebras.

The following rather obvious remark will be of fundamental importance in the sequel. The above description of the small quantum cohomology of $\mathbb{P}^n$ shows that it corresponds to a vector bundle on $\mathbb{C}^*$ (the coordinate on $\mathbb{C}^*$ being q) of rank $n+1$, equipped with a commutative and associative multiplication. There is a canonical extension of that bundle to a bundle over $\mathbb{C}_q$, i.e., over the limit point $q=0$, which is simply given as the $\mathbb{C}[q]$-algebra $\mathbb{C}[p,q]/(p^{n+1}-q)$. Even more, the fibre of this extended bundle on $q=0$ is nothing but the restriction of this algebra to $q=0$, i.e., the zero-dimensional Gorenstein ring $\mathbb{C}[p]/p^{n+1} \cong (H^*(\mathbb{P}^n, \mathbb{C}), \cup)$. This limit behavior is not accidental, the point $q=0$ is called the large radius limit, and it is one of the most prominent features of the quantum product that it degenerates to the ordinary cup product at this limit point. A basic philosophy in the later sections of this paper is to express the mirror correspondence by objects defined on partial compactifications of the parameter spaces which include the large radius limit.

3 Givental's Approach: Quantum $\mathcal{D}$-Module and J-Function

One of the central ideas in quantum cohomology that we are going to exploit is that the relations encoded in the WDVV-equation can be rewritten as a system of linear differential equations. This is usually called the quantum $\mathcal{D}$-module. We will encounter some general $\mathcal{D}$-modules below as the so-called Gauß-Manin systems, however, the quantum $\mathcal{D}$-module is merely a vector bundle with a connection. We recall the basic definitions.

Definition 3.1 Let M be a complex manifold M and $E \to M$ be a holomorphic vector bundle.

1. A connection on E is a $\mathbb{C}$-linear map
$$\nabla : E \longrightarrow E \otimes \Omega^1_M$$
satisfying the Leibniz rule $\nabla(f \cdot s) = f \cdot \nabla(s) + s \otimes df$ for any local sections $f \in \mathcal{O}_M$ and $s \in E$. It is called flat if moreover the $\mathcal{O}_M$-linear map (called curvature) $\nabla^{(2)} \circ \nabla$ vanishes, where $\nabla^{(2)} : E \otimes \Omega^1_M \to E \otimes \Omega^2_M$ is defined by $\nabla^{(2)}(s \otimes \omega) := \nabla(s) \wedge \omega - s \otimes d\omega$.
2. Let $D \subset M$ be a simple normal crossing divisor (in most cases that we will meet below, D is simply smooth). Then a meromorphic bundle F is by definition a locally free sheaf of $\mathcal{O}_M(*D)$-modules, and a meromorphic connection on E resp. F is a $\mathbb{C}$-linear operator $\nabla : E \to E \otimes \Omega^1_M(*D)$ resp. $\nabla : F \to F \otimes \Omega^1_M(*D)$ satisfying the Leibniz rule as above. Any meromorphic connection on E defines a meromorphic connection on $E(*D) = E \otimes \mathcal{O}_M(*D)$.
3. A meromorphic connection on a holomorpic bundle $E \to M$ is said to have a logarithmic pole along D if it takes values in $E \otimes \Omega^1(\log D)$. Here $\Omega^1(\log D)$ is the sheaf of differential forms with logarithmic poles along D. Locally, by choosing coordinates $z_1, \ldots, z_k, t_{k+1}, \ldots, z_m$ on M such that $D = \{z_1 \cdot \ldots \cdot z_k = 0\}$, the sheaf $\Omega^1_M(\log D)$ is freely generated over $\mathcal{O}_M$ by the forms $dz_1/z_1, \ldots, dz_k/z_k$ and $dz_{k+1}, \ldots, dz_m$.

4. A connection (E, ∇) which is logarithmic with respect to a smooth divisor $D \subset M$ defines a residue endomorphism Φ^{res} on the restriction $E_{|D}$, locally, if $z_1 = 0$ is the equation of the divisor, it is given by the class of $z_1 \nabla_{\partial_{z_1}} \in \mathcal{E}nd_{\mathcal{O}_D}(E_{|D})$. However, Φ^{res} does not depend on the choice of coordinates. Similarly, but only if we fix a coordinate system $(z_1, \dots, z_m)$ on M such that $D = \{z_1 = 0\}$, the residue connection ∇^{res} can be defined on $E_{|D}$ in the following way: If $\underline{e} = (e_1, \dots, e_n)$ is a local basis of E, then ∇ is written as

$$\nabla(\underline{e}) = \underline{e} \cdot \left(A_1(z_1, \dots, z_m) \frac{dz_1}{z_1} + \sum_{i \geq 2}^{m} A_i(z_1, \dots, z_m) \right)$$

then ∇^{res} is defined with respect to $\underline{e}$ by the matrix $A_1(0, z_2, \dots, z_m)$.

Notice that there is a natural generalization of the notion of the residue endomorphism which applies to meromorphic connections with pole order two (more precisely, with Poincaré rank one), these are the so-called Higgs fields. We refer to [29, Sect. 0.14c] for details.

We can now define the small quantum $\mathcal{D}$-module. It corresponds to the small quantum cohomology ring, and will be sufficient for our purpose. Before giving the formal definition, let us make a remark on the definition of the quantum product (both the big and the small one). A priori, it is defined as a product of two cohomology classes, denoted by α and γ above, depending on a third class, denoted by η (this class has to be of degree two for the small quantum product). An elegant way to eliminate this dependence in the notation is to consider a trivial vector bundle with fibre $H^*(X, \mathbb{C})$ on either $H^*(X, \mathbb{C})$ (big quantum product) or $H^2(X, \mathbb{C})$ (small quantum product). Then any cohomology class gives a (constant) section of this bundle, and we can consider the (small/big) quantum product as a commutative and associative multiplication with unit on this vector bundle. On each fibre of this bundle at a base point η, this multiplication gives back the product $\alpha \circ_\eta \gamma$ resp. $\alpha \star_\eta \gamma$ from above. Below, we will always adopt this convention and write the quantum product as a product of two classes $\alpha \circ \gamma$ resp. $\alpha \star \gamma$.

Definition 3.2 Let X be smooth projective satisfying $H^*(X, \mathbb{C}) = H^{2*}(X, \mathbb{C})$. Choose a homogeneous basis $T_0, T_1, \dots, T_r, T_{r+1}, \dots, T_s$ of the cohomology as above. Let $M \subset H^2(X, \mathbb{C})$ be an open subset (with coordinates $t_1, \dots, t_r$ corresponding to the basis vectors $T_1, \dots, T_r$) on which the small quantum product is convergent. Consider the projection $\pi : \mathbb{P}^1_z \times M \twoheadrightarrow M$, where we chose z to be a fixed coordinate on the chart of $\mathbb{P}^1$ centered at 0. Let $G := H^*(X, \mathbb{C}) \times M \to M$ be the trivial vector bundle on M with fibre $H^*(X, \mathbb{C})$. It comes equipped with a flat connection ∇^{fl} corresponding to the given trivialization, i.e., such that $\nabla^{fl}(s) = 0$ for any section $s : M \to H^*(X, \mathbb{C}) \times M$ sending any point in M to a constant value $\gamma \in H^*(X, \mathbb{C})$. Put $F := \pi^* G \to \mathbb{P}^1_z \times M$, and define the *Givental connection* ∇^{Giv} on F as follows:

$$\nabla^{\mathrm{Giv}}_{\partial_{t_k}}(s) := \nabla^{fl}_{\partial_{t_k}}(s) - \frac{1}{z} T_k \star s$$
$$\nabla^{\mathrm{Giv}}_{\partial_z}(s) := \frac{1}{z}\left(\frac{E \star s}{z} + \mu(s)\right) \tag{6}$$

where $s \in F$, $\mu \in Aut_{\mathbb{C}}(H^*(X, \mathbb{C}))$ is the grading operator that takes the value $k \cdot \gamma$ on any class $\gamma \in H^{2k}(X, \mathbb{C})$ and the vector field E is defined as $E := \sum_{i=0}^{s}(1 - \frac{\deg(T_i)}{2}) t_i \partial_{t_i} + \sum_{a=1}^{r} k_a \partial_{t_a}$, where $\sum_{a=1}^{r} k_a T_a = c_1(X)$ (see also (4))

It follows from formula (6) that ∇^{Giv} has poles along $z = \{0, \infty\}$. Its restriction to $\mathbb{C}^*_z \times M$ thus defines a holomorphic connection operator.

By a slight abuse of notation, the object $(F, \nabla^{\mathrm{Giv}})$ is called the quantum $\mathcal{D}$-module.

The following is one of the main properties of the Givental connection. It is follows easily from the basic properties of the quantum product. In particular, it implies the WDVV differential equations for the Gromov-Witten potential and hence the associativity of the quantum product.

Proposition 3.3 *The Givental connection is flat, that is, the linear operator* $\nabla^2 : F \to \Omega^2_{\mathbb{P}^1_z \times M} \otimes F$ *vanishes.*

Remark By its very definition, the quantum $\mathcal{D}$-module is a vector bundle on $\mathbb{P}^1 \times M$ and the connection has a logarithmic pole along $\{\infty\} \times M$ and a pole of Poincaré rank one along $\{0\} \times M$. Hence we can consider the residue connection ∇^{res} and the residue endomorphism of ∇^{Giv} along $\{\infty\} \times M$ as well as the Higgs field of ∇^{Giv} along $\{0\} \times M$. It also follows from the definition, i.e., from formula (6) that the residue connection is nothing but ∇^{fl}, the residue endomorphism is the grading operator μ and the Higgs field is given by the quantum multiplication. This observation is very useful when studying the object corresponding to the Givental connection via mirror symmetry.

A (rather simple) variant of the Riemann-Hilbert correspondence tells us that the restriction $(F, \nabla^{\mathrm{Giv}})_{|\mathbb{C}^*_z \times M}$ defines a local system of flat sections. There is a canonical way to construct such flat sections, starting from the so-called J-function. This is a cohomology valued function, defined in terms of the so-called gravitational descendents.

Definition 3.4 (J-function and fundamental solutions)

1. Let $\alpha_1, \ldots, \alpha_n \in H^*(X, \mathbb{C})$. Define the genus zero gravitational descendant invariants by

$$\langle \alpha_1 \psi_1^{k_1}, \ldots, \alpha_n \psi_n^{k_n} \rangle_{0,n,\beta} := \int_{[\overline{\mathcal{M}}_{(0,n)}(X,\beta)]} \psi_1^{k_1} \cup ev_1^*(\alpha_1) \cup \cdots \cup \psi_n^{k_n} \cup ev_n^*(\alpha_n).$$

where ψ_i are line bundles on $\overline{\mathcal{M}}_{(0,n)}(X,\beta)$ defined in such a way that their fibre at the point $[C,(x_1,\dots,x_n),f]\in\overline{\mathcal{M}}_{(0,n)}(X,\beta)$ (here C is the projective curve, $x_1,\dots,x_n$ its marked points and $f:C\to X$ the stable map from C to X) *is* the cotangent line $T^*_{x_i}C$. A more precise definition can be found, e.g., in [27, Definition 4.1].

2. Write $\eta'=\sum_{a=1}^r t_aT_a$ and define the $H^*(X_\Sigma,\mathbb{C})$-valued power series J by

$$J\left(\eta',z^{-1}\right):=e^{\frac{\eta'}{z}}\cdot\left[1+\sum_{\substack{\beta\in\mathrm{Eff}_{X_\Sigma}\setminus\{0\}\\ j=0,\dots,s}}e^{\eta'(\beta)}\left\langle\frac{T_j}{z-\psi_1},1\right\rangle_{0,2,\beta}T^j\right].$$

here the gravitational descendent GW-invariant $\langle\frac{T_j}{z-\psi_1},1\rangle_{0,2,\beta}$ has to be understood as the formal sum $\sum_{k\geq 0}z^{-k-1}\langle T_j\psi_1^k,1\rangle_{0,2,\beta}$ and $T^0,T^1,\dots,T^s$ is the basis of $H^*(X_\Sigma,\mathbb{C})$ which is g-dual to $T_0,T_1,\dots,T_s$.

Theorem 3.5 *We have*

$$z\nabla^{\mathrm{Giv}}(\partial_{t_k}J)=0,$$

that is, the partial derivatives of J form a fundamental system of solutions of the (relative) Givental connection.

Remark The derivatives of the J-function are not flat with respect to the "vertical connection" $\nabla^{\mathrm{Giv}}_{\partial_z}$. However, one can obtain truly flat sections from the J-function by an easy twist, taking into account the logarithmic pole of ∇^{Giv} on F along $z=\infty$.

Givental has shown how to obtain all interesting Gromov-Witten invariants for nef *toric* varieties. In fact, Giventals result is broader, as it concerns the case of complete intersections in toric varieties. Such a subvariety is not necessarily toric itself. In particular, it includes the case of the quintic Calabi-Yau hypersurface in $\mathbb{P}^4$ mentioned in the introduction. We will restrict to toric varieties in the sequel. A thorough discussion of the B-model of a complete intersection would require the introduction of more notions, from which we refrain here.

The next section contains a detailed reminder on toric geometry, however, let us state Givental's main result here in order to finish the discussion of the quantum $\mathcal{D}$-module. We need one object from toric geometry, which is the so-called I-function.

Definition 3.6 Let X_Σ be a smooth projective toric variety and write $D_1,\dots,D_m$ for its torus invariant divisors. Define I to be the $H^*(X_\Sigma,\mathbb{C})$-valued formal power series

$$I=e^{\eta'/z}\cdot\sum_{\underline{l}\in\mathbb{L}_{\mathrm{eff}}}q^{\underline{l}}\cdot\prod_{i=1}^m\frac{\prod_{\nu=-\infty}^0([D_i]+\nu z)}{\prod_{\nu=-\infty}^{l_i}([D_i]+\nu z)}\in H^*(X_\Sigma,\mathbb{C})[z][\![q_1,\dots,q_r]\!][\![z^{-1}]\!].$$

Now Givental's theorem takes the following form.

Theorem 3.7 ([8, Theorem 0.1]) *Let X_Σ be a smooth projective toric variety with a numerical effective anticanonical bundle $-K_{X_\Sigma}$. There is a formal coordinate change $\kappa \in (\mathbb{C}[[q_1, \ldots, q_r]])^r)$ called the mirror map, such that*

$$I = (\mathrm{id}_{\mathbb{C}_z} \times \kappa)^* J$$

If X_Σ is Fano, that is, if $-K_{X_\Sigma}$ is ample, then $\kappa = \mathrm{id}$.

4 Landau-Ginzburg Models of Toric Fano Varieties

In the sequel of this survey, we will concentrate on the case of a toric variety with a nef anticanonical divisor. We will describe how to associate a certain family of Laurent polynomials to such a variety. These are called Landau-Ginzburg models. Our presentation below follows mainly [19].

Whereas these Landau-Ginzburg models can be obtained very explicitly in a rather elementary way in the toric case, it is a largely unsolved problem how to construct them for more general varieties. Most of the known construction somehow come back to the toric case, like the technique of toric degenerations [2]. In order to orient the reader, let us give some reminders on basics of toric geometry that are relevant for the present paper. As a basic reference for the facts discussed below, the reader may consult [6] or the more recent [4].

Let $N = \bigoplus_{k=1}^n \mathbb{Z}n_k$ be a free abelian group of rank n and $\Sigma \subset N \otimes \mathbb{R}$ be a fan. This means that Σ is a collection of cones $\{\sigma \in \Sigma\}$, where any σ is a strongly convex (i.e., $\sigma \cap (-\sigma) = \{0\}$) polyhedral cone (i.e., $\sigma = \sum \mathbb{R}_{\geq 0} b_i$ for some b_i's in N). Being a fan means that for any $\sigma \in \Sigma$, any face of σ is again a cone in Σ, and that for any two cones $\sigma, \tau \in \Sigma$, the intersection $\sigma \cap \tau$ is a face of both τ and σ. The fan Σ defines a toric variety X_Σ. Recall that X_Σ is covered by affine charts $X_\sigma := \mathrm{Spec}\, \mathbb{C}[M \cap \sigma^\vee]$, here $M := \mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$ and $\sigma^\vee := \{m \in M \otimes \mathbb{R} \mid m(n) \geq 0\ \forall n \in N \otimes \mathbb{R}\}$, and that X_Σ is obtained from these affine pieces by gluing X_σ and X_τ along $X_{\sigma \cap \tau}$.

We will suppose for simplicity that the fan Σ is *smooth* and *complete*, which means by definition that any cone $\sigma \in \Sigma$ can be generated by elements b_i which can be completed to a $\mathbb{Z}$-basis of N and that the *support* $\mathrm{Supp}(\Sigma) = \bigcup_{\sigma \in \Sigma} \sigma$ is all of $N \otimes \mathbb{R}$. It is well-known that this translates into X_Σ being smooth and complete. The smoothness condition can by weakened by requiring Σ to be only *simplicial*, which means that the generators of each cone are linearly independent over $\mathbb{Q}$. In this case X_Σ can have quotient singularities, i.e., it is the underlying topological space of a an orbifold. The question how to extend the results presented here to the orbifold case is in the focus of current reasearch on quantum cohomology and mirror symmetry, however, we will restrict to the smooth case in this survey for simplicity.

We have an exact sequence

$$0 \longrightarrow \mathbb{L} \longrightarrow \mathbb{Z}^{\Sigma(1)} \longrightarrow N \longrightarrow 0 \tag{7}$$

where $\Sigma(1)$ are the one-dimensional cones of Σ, called rays, the last map sends a generator e_i of $\mathbb{Z}^{\Sigma(1)}$ to a primitive integral generator $b_i \in N$ of a ray, and where the lattice $\mathbb{L}$ is the free submodule of $\mathbb{Z}^{\Sigma(1)}$ of relations between the elements $b_i \in N$. Dualizing yields the sequence

$$0 \longrightarrow M \longrightarrow \mathbb{Z}^{\Sigma(1)} \longrightarrow \mathbb{L}^\vee \longrightarrow 0.$$

It is well known (see, e.g., [6, p. 106]) that for a smooth toric manifold X_Σ, we have $H^2(X_\Sigma, \mathbb{Z}) \cong \mathbb{L}^\vee$. Inside $\mathbb{L}^\vee \otimes \mathbb{R}$ we have the cone $K(X_\Sigma)$ of *Kähler classes*, which can be defined by saying that $a \in K(X_\Sigma)$ iff $a(\beta) \geq 0$ for all effective 1-cycles in $H_2(X_\Sigma, \mathbb{R})$ (The latter set of cycles also forms a cone, called the Mori cone). We write $K^0(X_\Sigma)$ for the interior of $K(X)$, i.e., for all elements $a \in \mathbb{L}^\vee$ with $a(\beta) > 0$. Write $D_i \in \mathbb{L}^\vee$ for the components of the map $\mathbb{L} \hookrightarrow \mathbb{Z}^{\Sigma(1)}$, then the anticanonical divisor $-K_{X_\Sigma}$ is $\sum_{i=1}^m D_i \in \mathbb{L}^\vee$. As we already mentioned in Definition 1.1, X_Σ is called a Fano variety iff $-K_{X_\Sigma}$ is ample, i.e., if it lies in $K^0(X_\Sigma)$. If $-K_{X_\Sigma} \in K(X_\Sigma)$, then X_Σ is nef . Notice that a Calabi-Yau manifold (i.e., $K_{X_\Sigma} = 0$) is obviously nef, however, it is easy to see that in this case the defining fan can never be complete.

The projection $\mathbb{Z}^{\Sigma(1)} \twoheadrightarrow N$ is given by a matrix $(a_{ki})_{k=1,\ldots,n; i=1,\ldots,m}$ with respect to the basis (n_k) of N. Moreover, we will chose once and for all a basis $(p_a)_{a=1,\ldots,r}$ of $\mathbb{L}^\vee$ (with $r = m - n$ and $m = |\Sigma(1)|$) which consists of nef classes (i.e., classes lying inside of $K(X)$) and such that the anti-canonical class $-K_{X_\Sigma}$ lies in the cone $\sum_{a=1}^r \mathbb{R}_{\geq 0} p_a$. Then the map $\mathbb{L} \hookrightarrow \mathbb{Z}^{\Sigma(1)}$ is given by a matrix $(m_{ia})_{i=1,\ldots,m; a=1,\ldots,r}$ with respect to the dual basis $(p_a^\vee)$.

Applying the functor $\mathrm{Hom}_\mathbb{Z}(-, \mathbb{C}^*)$ (where $\mathbb{Z}$ acts on $\mathbb{C}^*$ by exponentiating) to the exact sequence 7 yields

$$1 \longrightarrow \mathrm{Hom}_\mathbb{Z}\big(N, \mathbb{C}^*\big) \cong \big(\mathbb{C}^*\big)^n \xrightarrow{\ \alpha\ } \big(\mathbb{C}^*\big)^{\Sigma(1)} \xrightarrow{\ \beta\ } \mathrm{Hom}_\mathbb{Z}\big(\mathbb{L}, \mathbb{C}^*\big) \cong \big(\mathbb{C}^*\big)^r \longrightarrow 1 \tag{8}$$

where $\alpha(y_1, \ldots, y_k) = (w_i := \prod_{k=1}^n y_k^{a_{ki}})_{i=1,\ldots,m}$ and $\beta(w_1, \ldots, w_m) = (q_a := \prod_{i=1}^m w_i^{m_{ia}})_{a=1,\ldots,r}$, here $(q_a)_{a=1,\ldots,r}$ are the coordinates on $\mathrm{Hom}_\mathbb{Z}(\mathbb{L}, \mathbb{C}^*)$ corresponding to the basis (p_a) of $\mathbb{L}^\vee$, $(w_i)_{i=1,\ldots,m}$ are the standard coordinates on $(\mathbb{C}^*)^{\Sigma(1)}$ and $(y_k)_{k=1,\ldots,m}$ are the coordinates on $\mathrm{Hom}_\mathbb{Z}(N, \mathbb{C}^*)$ corresponding to the basis $(n_k^\vee)$ of M.

Definition 4.1 Let $W = \sum_{i=1}^m w_i$. The *Landau-Ginzburg* model of X_Σ is defined to be the restriction of W to the fibres of the map $\beta : (\mathbb{C}^*)^{\Sigma(1)} \to (\mathbb{C}^*)^r$.

Notice that the choice of the basis (p_a) of $\mathbb{L}^\vee$ and hence the isomorphism $\mathrm{Hom}_\mathbb{Z}(\mathbb{L}, \mathbb{C}^*) \cong (\mathbb{C}^*)^r$ is part of the data of the Landau-Ginzburg model, as the map $(\mathbb{C}^*)^{\Sigma(1)} \to \mathrm{Hom}_\mathbb{Z}(\mathbb{L}, \mathbb{C}^*)$ depend only on $\Sigma(1)$, not on Σ itself.

The following construction allows us to rewrite the restriction of W to the fibres of β as a family of Laurent polynomials. Chose a section $l : \mathbb{L}^\vee \to \mathbb{Z}^{\Sigma(1)}$ of the projection $l : \mathbb{Z}^{\Sigma(1)} \twoheadrightarrow \mathbb{L}^\vee$, given, with respect to the above bases, by a matrix (l_{ia}). This yields a section (denoted abusively by the same letter)

$$l : \big(\mathbb{C}^*\big)^r \longrightarrow \big(\mathbb{C}^*\big)^{\Sigma(1)} \tag{9}$$

which sends $(q_1, \dots, q_r)$ to $(w_i := \prod_{a=1}^r q_a^{l_{ia}})$. Then putting $\Psi : \mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{C}^*) \times (\mathbb{C}^*)^r \to (\mathbb{C}^*)^{\Sigma(1)}$ where $\Psi(\underline{y}, \underline{q}) := (w_i := \prod_{a=1}^r q_a^{l_{ia}} \cdot \prod_{k=1}^n y_k^{b_{ki}})_{i=1,\dots,r}$ yields a coordinate change on $(\mathbb{C}^*)^m$ such that β becomes the projection $p_2 : \mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{C}^*) \times (\mathbb{C}^*)^r \twoheadrightarrow (\mathbb{C}^*)^r$. Then we put

$$\begin{aligned} \widetilde{W} := W \circ \Psi : \mathrm{Hom}_{\mathbb{Z}}\big(N, \mathbb{C}^*\big) \times \big(\mathbb{C}^*\big)^r \quad &\longrightarrow \quad \mathbb{C} \\ (y_1, \dots, y_k, q_1, \dots, q_a) \quad &\longmapsto \quad \sum_{i=1}^m \prod_{a=1}^r q_a^{l_{ia}} \cdot \prod_{k=1}^n y_k^{b_{ki}} \end{aligned}$$

which is a family of Laurent polynomials on $\mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{C}^*)$ parameterized by $(\mathbb{C}^*)^r$.

Recall [22] that a single Laurent polynomial $\widetilde{W}_{\underline{q}} := \widetilde{W}(-, \underline{q}) \in \mathcal{O}_{\mathrm{Hom}(N,\mathbb{C}^*)}$ is called convenient iff 0 lies in the interior of its Newton polyhedron, and non-degenerate iff for any proper face τ of its Newton polyhedron, the Laurent polynomial $(\widetilde{W}_{\underline{q}})_\tau = \sum_{b_i \in \tau} \prod_{a=1}^r q_a^{l_{ia}} \cdot \prod_{k=1}^n y_k^{b_{ki}}$ does not have any critical point on $\mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{C}^*)$. If we consider the whole family $\widetilde{W}$, the following holds

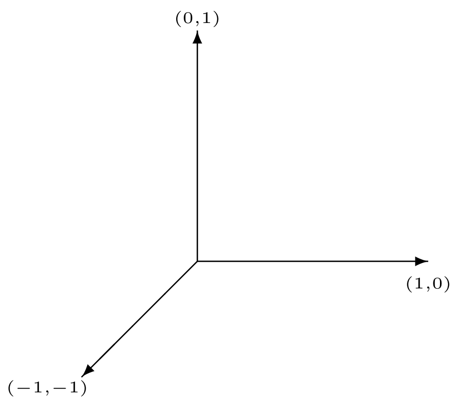
Proposition 4.2

1. *$\widetilde{W}_{\underline{q}}$ is convenient for any $\underline{q} \in (\mathbb{C}^*)^r$.*
2. *There is an algebraic subvariety $Z \subset (\mathbb{C}^*)^r$ such that $\widetilde{W}_{\underline{q}}$ is non-degenerate for all $\underline{q} \notin Z$. Write $\mathcal{M}^0 := (\mathbb{C}^*)^r \backslash Z$.*
3. *If X_Σ is Fano, then $Z = \emptyset$.*
4. *If X_Σ is weak Fano, then there exists an $\epsilon > 0$, such that for all $\underline{q} \in \mathrm{Hom}_{\mathbb{Z}}(\mathbb{L}, \mathbb{C}^*) \subset \mathbb{C}^r$ with $|q| < \epsilon$, we have $\underline{q} \notin Z$. Here the inclusion $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{L}, \mathbb{C}^*) \subset \mathbb{C}^r$ and the metric $|\cdot|$ refer to the chosen coordinates (q_a) on $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{L}, \mathbb{C}^*)$.*

Examples In order to make the above construction more transparent, let us consider some simple but important examples.

1. The mirror of projective spaces, see Fig. 1.

Fig. 1 The fan of $\mathbb{P}^2$



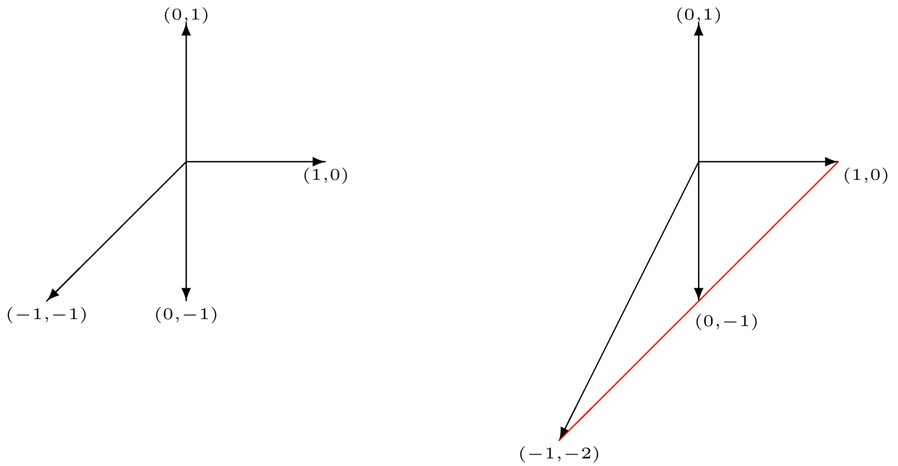


Fig. 2 The fans of $\mathbb{F}_1$ and $\mathbb{F}_2$

The fan of $\mathbb{P}^n$ consists of $n+1$ rays (see Fig. 1 for the case $n=2$), namely, the standard vectors e_i for $i=1,\dots,n$ in $\mathbb{Z}^n$ and the additional vector $\sum_{i=1}^n -e_i$. Hence the exact sequence (7) reads

$$0 \longrightarrow \mathbb{L} \cong \mathbb{Z} \xrightarrow{\begin{pmatrix}1\\ \vdots\\ \vdots\\ 1\end{pmatrix}} \mathbb{Z}^{n+1} \xrightarrow{\begin{pmatrix}1 & 0 & \dots & \dots & -1\\ 0 & 1 & \dots & \dots & -1\\ \vdots & \vdots & \ddots & \dots & -1\\ 0 & 0\dots & 0 & 1 & -1\end{pmatrix}} \mathbb{Z}^n \longrightarrow 0$$

where we have chosen a basis of $\mathbb{L}^\vee$ corresponding to the Poincaré dual of a hyperplane. Hence by dualizing and tensoring with $\mathbb{C}^*$ we obtain the Landau-Ginzburg model given by the restriction of the linear function $W = w_0 + \dots + w_n$ to the fibres of the fibration $\beta : (\mathbb{C}^*)^{n+1} \to \mathbb{C}^*$, which sends $(w_0,\dots,w_n)$ to $w_0 \cdot \dots \cdot w_n$. Choosing the section $l : \mathbb{Z} \cong \mathbb{L}^\vee \to \mathbb{Z}^{n+1}$, $l(m) = (m,0,\dots) \in \mathbb{Z}^n$ we obtain that $\widetilde{W}_{\mathbb{P}^n}(y_1,\dots,y_n,q) = y_1 + \dots + y_n + \frac{q}{y_1 \cdot \dots \cdot y_n}$.

2. The mirror of the Hirzebruch surfaces $\mathbb{F}_1$ and $\mathbb{F}_2$:

   Recall that for any $k \in \mathbb{N}$, the projective bundle $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(k) \oplus \mathcal{O}_{\mathbb{P}^1})$ is called the k-th Hirzebruch surface and denoted by $\mathbb{F}_k$. However, these surfaces are Fano only for $k=0$ (this is the rather trivial case $\mathbb{P}^1 \times \mathbb{P}^1$) and $k=1$. For $k=2$, $\mathbb{F}_2$ has a nef anticanonical divisor, but this is no longer true for the higher $\mathbb{F}_k$'s. Hence we can construct Landau-Ginzburg models for $\mathbb{F}_0$, $\mathbb{F}_1$ and $\mathbb{F}_2$. Let us concentrate on the last two cases. These are toric varieties defined by the fans shown in Fig. 2.

   The exact sequence (7) takes the following form for $\mathbb{F}_1$

$$0 \longrightarrow \mathbb{L} \cong \mathbb{Z}^2 \xrightarrow{\begin{pmatrix}1 & 0\\ 0 & 1\\ 1 & 0\\ -1 & 1\end{pmatrix}} \mathbb{Z}^4 \xrightarrow{\begin{pmatrix}1 & 0 & -1 & 0\\ 0 & 1 & -1 & -1\end{pmatrix}} \mathbb{Z}^2 \longrightarrow 0$$

so that the Landau-Ginzburg model is the following two parameter family of Laurent polynomials:

$$\widetilde{W}_{\mathbb{F}_1} = x + y + \frac{q_1 \cdot q_2}{xy} + \frac{q_2}{y}$$

where we have chosen the section of the map $\mathbb{Z}^4 \to \mathbb{L}^\vee$ given by the matrix

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

For $\mathbb{F}_2$, we have the following exact sequence

$$0 \longrightarrow \mathbb{L} \cong \mathbb{Z}^2 \xrightarrow{\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ -2 & 1 \end{pmatrix}} \mathbb{Z}^4 \xrightarrow{\begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -2 & -1 \end{pmatrix}} \mathbb{Z}^2 \longrightarrow 0$$

and we obtain:

$$\widetilde{W}_{\mathbb{F}_2} = x + y + \frac{q_1 \cdot q_2^2}{xy^2} + \frac{q_2}{y}$$

where the section is given by the matrix

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 2 \\ 0 & 1 \end{pmatrix}.$$

Notice that for $q_1 = \frac{1}{4}$, $\widetilde{W}_{\mathbb{F}_2}(x, y, \frac{1}{4}, q_2)$ is degenerate: the Laurent polynomial $x + \frac{q_2^2}{4xy^2} + \frac{q_2}{y}$ has critical points on the torus $(\mathbb{C}^*)^2$. This reflects the fact that $\mathbb{F}_2$ is nef but not Fano, and can be seen on its fan from the fact that there is a lattice point on the boundary of the convex hull defined by the rays of the fan of $\mathbb{F}_2$ (the point $(1, -1)$ on the red line).

5 Gauß-Manin Systems and Hypergeometric Differential Equations

In this section we describe how to associate a system of differential equations to the Landau-Ginzburg models defined above. These systems will ultimately be equal to the quantum $\mathcal{D}$-module, and this is precisely the kind of mirror correspondences we are interested in. However, the differential equations we are going to consider are interesting in their own right, and have been studied since a long time. They are related to the classical *Gauß-Manin connection* but they are more general in two respects: First, one has to take into account singularities which occur at the critical points of the Laurent polynomials. The corresponding object is called Gauß-Manin system, and is constructed in a functorial way using the general notion of direct image in the category of $\mathcal{D}$-modules. Equivalently, and this is the point of view that we are going to adapt below, it is obtain as a twisted de Rham cohomology group. The second difference to the classical setup is that in order to match with the quantum $\mathcal{D}$-module, we have to consider a variant of the Gauß-Manin system, which is obtained by a partial Fourier transformation. The solutions of the transformed systems can be obtained as oscillating integrals, whereas the original Gauß-Manin system consists of differential equations satisfied by period integrals over vanishing cycles. We will not explain in detail this more analytic point of view, one can find in [12, Chap. 8] some explanations for the related case of germs of functions with isolated singularities. Notice also that the construction described below is carried out in the analytic category in [19].

In order to establish the mirror correspondence via differential equations satisfied by oscillating integrals, one needs to have a concrete description of these $\mathcal{D}$-modules. Luckily, such a description is available in the toric case, and the systems obtained are said to have a hypergeometric structure. Hypergeometric functions and hypergeometric differential equations have a long history, starting at least with Gauß. We will not review here these developments (one may consult, e.g., [33] for some classical aspects of the theory). Instead, we start with the following definition of the so-called GKZ-systems (after Gelfand, Kapranov and Zelevinski) taken from [7] and [10] (see also the more recent reference [1]). Any system of hypergeometric equation can be rewritten as a GKZ-system (or a reduction of it). We also discuss the main properties of these $\mathcal{D}$-modules, and for that purpose we recall some of the most important notions related to algebraic $\mathcal{D}$-modules.

Definition 5.1 (GKZ- or A-hypergeometric system) Consider a lattice $\mathbb{Z}^n$ and vectors $\underline{a}_1, \ldots, \underline{a}_m \in \mathbb{Z}^n$ which we also write as a matrix $A = (\underline{a}_1, \ldots, \underline{a}_n)$. Moreover, let $\beta = (\beta_1, \ldots, \beta_n) \in \mathbb{C}^n$. Write $\mathbb{L}$ for the module of relations of A and $\mathcal{D}_{\mathbb{C}^m}$ for the sheaf of rings of algebraic differential operators on $\mathbb{C}^m$ (where we choose $x_1, \ldots, x_m$ as coordinates). Define

$$\mathcal{M}_A^\beta := \mathcal{D}_{\mathbb{C}^s} / \big((\Box_{\underline{l}})_{\underline{l} \in \mathbb{L}} + (Z_k)_{k=1,\ldots t} \big), \tag{10}$$

where

$$\Box_{\underline{l}} := \prod_{i: l_i < 0} \partial_{x_i}^{-l_i} - \prod_{i: l_i > 0} \partial_{x_i}^{l_i}$$

$$Z_k := \sum_{i=1}^{s} b_{ki} x_i \partial_{x_i} + \beta_k$$

$\mathcal{M}_A^\beta$ is called hypergeometric system.

Notice that although the definition of $\mathcal{M}_A^\beta$ involves infinitely many operators (one for each $\underline{l} \in \mathbb{L}$ plus the finite number of operators Z_k), the denominator of in formula (10) is of course generated by a finite number of elements of $\mathcal{D}_{\mathbb{C}^s}$. However, and this is one important feature of the theory of GKZ-systems, in order to generate the ideal $(\Box_{\underline{l}})_{\underline{l}\in\mathbb{L}}$, it is in general not sufficient to take operators $\Box_{\underline{l}}$ where $\underline{l}$ runs through a basis of $\mathbb{L}$.

Next we are going to describe some general properties of GKZ-systems. In order to do this, we first recall some basic notions from the general theory of algebraic $\mathcal{D}$-modules. As an example for a reference, the interested reader may consult [17] for the proofs and many more results.

Definition-Lemma 5.2 *Let X be a smooth algebraic variety over $\mathbb{C}$, $\mathcal{D}_X$ the sheaf of algebraic differential operators on X and $\mathcal{M}$ be a coherent $\mathcal{D}_X$-module. Consider the usual filtration $F_\bullet$ on $\mathcal{D}$ by orders of operators. The associated graded (sheaf of) rings $gr_\bullet(\mathcal{D})$ equals the structure sheaf on the cotangent bundle T^*X.*

1. *A filtration $F_\bullet\mathcal{M}$ is good if it is compatible with $F_\bullet\mathcal{D}$, i.e., if $F_k\mathcal{D}\cdot F_l\mathcal{M}\subset F_{k+l}\mathcal{M}$ holds for all k,l and if this is an equality for l sufficiently large and if moreover we have that $gr_\bullet(\mathcal{M})$ is $gr_\bullet(\mathcal{D})=\mathcal{O}_{T^*X}$-coherent.*
2. *The characteristic variety* $\mathrm{char}(\mathcal{M})$ *of a $\mathcal{M}$ is the reduced support of $gr_\bullet(\mathcal{M})$ in T^*X. This subvariety does not depend on the choice of the good filtration $F_\bullet\mathcal{M}$.*
3. *$\mathcal{M}$ is holonomic iff* $\mathrm{char}(M)$ *is a Lagrangian subvariety of $T^*\mathbb{C}^m$ for its natural symplectic structure, that is, iff the restriction of the symplectic form to all tangent spaces of smooth points of* $\mathrm{char}(M)$ *vanishes. Equivalently, $\mathcal{M}$ is holonomic iff $\mathcal{E}xt^p_{\mathcal{D}_X}(\mathcal{M},\mathcal{D}_X)=0$ for all $p\neq n$.*
4. *Let $\pi: T^*X\to X$ be the canonical projection. Let* $\mathrm{char}(\mathcal{M})=\bigcup C_i$ *be the decomposition of* $\mathrm{char}(\mathcal{M})$ *into irreducible components. Suppose that the zero section $T^*_X X$ of T^*X is a component of* $\mathrm{char}(\mathcal{M})$ *and that it is equal to C_1. Define the singular support* $\mathrm{Sing}(\mathcal{M})$ *to be* $\pi(\mathrm{char}(M)\backslash C_1)$, *if* $T^*_X X\not\subset \mathrm{char}(\mathcal{M})$, *then* $\mathrm{Sing}(\mathcal{M})=\mathrm{Supp}(\mathcal{M})$. *The restriction $\mathcal{M}_{|X\backslash C_1}$ is $\mathcal{O}_X$-locally free of rank k ($k=0$ if* $\mathrm{Supp}(\mathcal{M})\subsetneq X$), *and k is called the holonomic rank of $\mathcal{M}$. It is equal to the dimension of the space of (say, holomorphic) local solutions of $\mathcal{M}$ near a point in* $X\backslash\mathrm{Sing}(\mathcal{M})$.
5. *$\mathcal{M}$ is regular if its restriction to any curve $C\subset X$ is so, and this last condition can be reduced to the usual condition of regularity for linear systems of differential equations in one variable. A precise definition of regularity can be found, e.g., in* [17, *Chap.* 6].

With all these notions in mind, we can describe the main properties of the GKZ-systems.

Proposition 5.3 *Let A, β and $\mathcal{M}^\beta_A$ be as above.*

1. *$\mathcal{M}^\beta_A$ is holonomic for any A and any β. For generic β, the holonomic rank of $\mathcal{M}^\beta_A$ is $n!\cdot\mathrm{vol}(\Delta(\underline{a}_1,\ldots,\underline{a}_n))$, where $\Delta(\underline{a}_1,\ldots,\underline{a}_n)$ denotes the convex hull of $\underline{a}_1,\ldots,\underline{a}_n$ in $\mathbb{R}^n$ and $\mathrm{vol}(-)$ is the normalized volume, which takes the value 1 on the hypercube $[0,1]^n\subset\mathbb{R}^n$. In particular, if β is generic, then $n!\cdot\mathrm{vol}(\Delta(\underline{a}_1,\ldots,\underline{a}_n))$ is the dimension of the solution space of the differential system defined by $\mathcal{M}^\beta_A$ at a generic point of $\mathbb{C}^m$.*

2. $\mathcal{M}_A^\beta$ *is regular if and only if* $\underline{a}_1, \ldots, \underline{a}_n$ *is contained in an affine hyperplane of* $\mathbb{Z}^n$.
3. *The singular locus equals the degeneracy locus of the Laurent polynomial* $\sum_{i=1}^m x_i \cdot \underline{y}^{\underline{a}_i}$, *where* $\underline{y}^{\underline{a}_i} := \prod_{k=1}^n y_k^{a_{ki}}$, *i.e., is equal to*

$$\left\{(\lambda_1, \ldots, \lambda_m) \in \mathbb{C}^m \middle| \forall \tau \in \partial\Delta(\underline{a}_1, \ldots, \underline{a}_n), \right.$$

$$\left. \sum_{j:\underline{a}_j \in \tau} \lambda_j \underline{y}^{\underline{a}_j} \text{ has a critical point in } \left(\mathbb{C}^*\right)^n \right\}$$

The differential systems that will appear in the mirror correspondence that we are going to explain are variants of special GKZ-systems. First, one starts with a regular GKZ-system, this is achieved by forcing the columns of the defining matrix to be contained in an affine hyperplane (see point (2) in the above proposition). Next there are two modifications to be carried out: A restriction of the parameter space, this corresponds to the chosen embedding l from (9), and finally a Fourier-Laplace transformation which introduces irregular singularities. Let us explain these steps in some more detail. For the restriction just mentioned, we have to use the inverse image functor of $\mathcal{D}$-modules, which we do not explain here (see again [17] for details).

Definition-Lemma 5.4

1. *For a given matrix* $A \in M(n \times m, \mathbb{Z})$ *with columns* $\underline{a}_1, \ldots, \underline{a}_m$, *let* $\widetilde{\underline{a}}_i := (1, \underline{a}_i) \in \mathbb{Z}^{n+1}$ *for* $i = 1, \ldots, m$ *and* $\widetilde{\underline{a}}_0 = (1, \underline{0})$. *Write* $\widetilde{A}$ *for the matrix with columns* $\widetilde{\underline{a}}_0, \widetilde{\underline{a}}_1, \ldots, \widetilde{\underline{a}}_m$ *and consider the hypergeometric systems* $\mathcal{M}_{\widetilde{A}}^\beta$ *for* $\beta \in \mathbb{C}^{n+1}$.
2. *For any* $\mathbb{C}[\lambda_0, \ldots, \lambda_m]\langle\partial_0, \ldots, \partial_m\rangle$*-module* $\mathcal{M}$, *define* $\mathrm{FL}_{\lambda_0}^{z^{-1}}(\mathcal{M})[z]$ *to be the operation of replacing* ∂_0 *by* z^{-1}, λ_0 *by* $z^2\partial_z$ *and by inverting* z^{-1}, *i.e., by tensoring with* $\mathbb{C}[z^\pm, \lambda_1, \ldots, \lambda_m]$ *over* $\mathbb{C}[z^{-1}, \lambda_1, \ldots, \lambda_m]$. *Then* $\mathrm{FL}_{\lambda_0}^{z^{-1}}(\mathcal{M})[z]$ *is a* $\mathbb{C}[z^\pm, \lambda_1, \ldots, \lambda_m]\langle\partial_z, \partial_1, \ldots, \partial_m\rangle$*-module.*
3. *Consider the chosen section* $l : \mathrm{Hom}_{\mathbb{Z}}(\mathbb{L}, \mathbb{C}^*) \cong (\mathbb{C}^*)^r \to (\mathbb{C}^*)^{\Sigma(1)} \cong (\mathbb{C}^*)^m$ *from* (9). *Define*

$$\mathcal{QM}_{\widetilde{A}} := \left((\mathrm{id}_z, l)^+ \mathrm{FL}_{\lambda_0}^{z^{-1}}\left(\mathcal{M}_{\widetilde{A}}^{(1,0)}\right)[z]\right)$$

then $\mathcal{QM}_{\widetilde{A}}$ *is given as the quotient of* $\mathbb{C}[z^\pm, q_1^\pm, \ldots, q_r^\pm]\langle\partial_z, \partial_{q_1}, \ldots \partial_{q_r}\rangle$ *by the left ideal generated by*

$$\widetilde{\Box}_{\underline{l}} := \prod_{a:p_a(\underline{l})>0} q_a^{p_a(\underline{l})} \prod_{i:l_i<0} \prod_{\nu=0}^{-l_i-1} \left(\sum_{a=1}^r m_{ia} z q_a \partial_{q_a} - \nu z\right)$$

$$- \prod_{a:p_a(\underline{l})<0} q_a^{-p_a(\underline{l})} \prod_{i:l_i>0} \prod_{\nu=0}^{l_i-1} \left(\sum_{a=1}^r m_{ia} z q_a \partial_{q_a} - \nu z\right)$$

for any $\underline{l} \in \mathbb{L}$ *and by the single operator*

$$z^2\partial_z - \sum_{a=1}^{r} K_{X_\Sigma}(p_a^\vee) q_a z \partial_{q_a}$$

4. *Denote by* ${}_0\mathcal{QM}_{\widetilde{A}} \subset \mathcal{QM}_{\widetilde{A}}$ *the* $\mathbb{C}[z, q_1^\pm, \ldots, q_r^\pm]$*-subalgebra generated by* $z^2\partial_z$ *and* $zq_a\partial_{q_a}$ *where* $a = 1, \ldots, r$. *Then* ${}_0\mathcal{QM}_{\widetilde{A}}$ *is* $\mathbb{C}[z, q_1^\pm, \ldots, q_r^\pm]$*-free of rank* $n! \cdot \mathrm{vol}(\Delta(\underline{a}_1, \ldots, \underline{a}_m))$ *and it comes equipped with a connection operator with a pole of Poincaré rank one along* $z = 0$.

There are many results in the literature concerning solutions of hypergeometric differential equations. In our setup, the I-function introduced above (Definition 3.6) will yield (cohomology valued) solutions of the GKZ-systems defined by a toric variety.

Proposition 5.5 *Put* $\widetilde{I} := z^{K_{X_\Sigma}} \cdot z^\mu \cdot I$ (μ *is the grading operator on cohomology classes*) *and write* $\widetilde{I} = \sum_{t=0}^{s} \widetilde{I}_t \cdot T_t$, *where* $T_0, T_1, \ldots, T_s$ *of* $H^*(X_\Sigma, \mathbb{C})$ *is a homogeneous basis of* $H^*(X_\Sigma, \mathbb{C})$ *as above. Then the components* $\widetilde{I}_t$ *yield solutions of the differential system* $\mathcal{QM}_{\widetilde{A}}$ *over a subset of* $\mathbb{C}_z^* \times (\mathbb{C}^*)^r$ *on which the* I*-function is convergent. For a precise statement, see* [27, *Proposition* 3.12 *and Corollary* 3.13].

The next step is to explain how we can associate a (version of a) GKZ-system to the Landau-Ginzburg models defined in Sect. 4. As mentioned in the beginning of this section, this is done using the so-called twisted de Rham cohomology, which is a version of the more general Gauß-Manin system. Here is the corresponding definition, which is simplified to fit to our purpose.

Definition 5.6 Let U, K be a smooth affine algebraic varieties with $\dim_\mathbb{C}(U) = n$ and $\varphi = (F, pr) : U \times K \to \mathbb{C} \times K$ be an affine morphism, where $F \in \mathcal{O}_{U\times K}$ and $pr : U \times K \to K$ is the projection. Let z be a new variable and consider the following complex of $\mathcal{O}_{U\times K}$-modules with a $\mathcal{O}_{\mathbb{C}\times K}$-linear differential

$$\left(\Omega^\bullet_{pr}[z], zd - d\varphi\wedge\right)$$

where $\Omega^\bullet_{pr} := \Omega^\bullet_U \otimes_{\mathcal{O}_U} \mathcal{O}_{K\times U}$ are the differential forms relative to the projection map pr. Call $H^n(\varphi) := H^n(\Omega^\bullet_{pr}[z], zd - d\varphi\wedge)$ the twisted de Rham cohomology of φ (more precisely, it is the de Rham complex of U with the differential twisted by φ). Notice that in the examples we are interested in, all other cohomology groups $H^i(\varphi)$ for $i \neq 0$ will vanish. Moreover, define a connection operator $\nabla : H^n(\varphi) \to H^n(\varphi) \otimes \Omega^1_{\mathbb{C}\times U}(*\{0\} \times U)$ by

$$\nabla_{\partial_z}(\omega) := -z^{-2} \cdot F \cdot \omega$$
$$\nabla_X(\omega) := \mathrm{Lie}_X(\omega) + z^{-1} \cdot X(F)\omega$$

where $\omega \in \Omega^k_{pr}$ and $X \in \mathcal{T}_K$ and where the above formulas are extended to the whole module $H^n(\varphi)$ by the Leibniz rule.

An appropriate version of the GKZ-system can be used to compute the twisted de Rham complex of the Landau-Ginzburg models we are interested in. Hence, let $(\widetilde{W}, pr) : (\mathbb{C}^*)^n \times (\mathbb{C}^*)^r \to \mathbb{C} \times (\mathbb{C}^*)^r$ be a morphism as in Sect. 4. Then we have the following.

Theorem 5.7 ([27, Corollary 3.3], [19]) *There is an isomorphism of $\mathcal{O}_{\mathbb{C}_z \times (\mathbb{C}^*)^r}$-modules with connections on $\mathbb{C}_z \times (\mathbb{C}^*)^r$*

$$H^n(\widetilde{W}, pr) \cong {}_0\mathcal{QM}_{\widetilde{A}}. \tag{11}$$

${}_0\mathcal{QM}_{\widetilde{A}}$ *(and hence also $H^n(\widetilde{W}, pr)$) is locally free of rank $n! \cdot \mathrm{vol}(\Delta(\underline{a}_1, \ldots, \underline{a}_m))$ when restricted to the complement of the degeneracy locus Z defined in Proposition 4.2. In particular, both objects are locally free over the whole of $\mathbb{C}_z \times (\mathbb{C}^*)^r$ if X_Σ is Fano.*

Examples Using the last theorem, we can give an explicit expression for the twisted de Rham cohomology for the examples considered in Sect. 4.

1. $X_\Sigma = \mathbb{P}^n$: Recall that $h^2(\mathbb{P}^n) = 1$ and hence $(\widetilde{W}_{\mathbb{P}^n}, pr) : (\mathbb{C}^*)^n \times \mathbb{C}^* \to \mathbb{C} \times \mathbb{C}^*$; $(x_1, \ldots, x_n, q) \mapsto x_1 + \cdots + x_n + q/(x_1 \cdot \ldots \cdot x_n)$. Then we have

$$H^n(\widetilde{W}, pr) \cong \frac{\mathbb{C}[z, q^{\pm}]\langle z^2\partial_z, qz\partial_q\rangle}{((zq\partial_q)^{n+1} - q, z^2\partial_z + (n+1)q\partial_q)} \tag{12}$$

2. $X_\Sigma = \mathbb{F}_1$: We have $h^2(\mathbb{F}_1) = 2$, $(\widetilde{W}_{\mathbb{F}_1}, pr) : (\mathbb{C}^*)^2 \times (\mathbb{C}^*)^2 \to \mathbb{C} \times (\mathbb{C}^*)^2$; $(x, y, q_1, q_2) \mapsto x + y + \frac{q_1 \cdot q_2}{xy} + \frac{q_2}{y}$ and

$$H^n(\widetilde{W}, pr)$$
$$\cong \frac{\mathbb{C}[z, , q_1^{\pm}, q_2^{\pm}]\langle z^2\partial_z, z\partial_{q_1}, q_2 z\partial_{q_2}\rangle}{((q_1 z\partial_{q_1})^2 - q_1(q_2 z\partial_{q_2} - q_1 z\partial_{q_1}), q_2 z\partial_{q_2} \cdot q_1 z\partial_{q_1} - q_2, z^2\partial_z + q_1 z\partial_{q_1} + 2q_2 z\partial_{q_2}))}$$

3. $X_\Sigma = \mathbb{F}_2$: We have $h^2(\mathbb{F}_1) = 2$, $(\widetilde{W}_{\mathbb{F}_1}, pr) : (\mathbb{C}^*)^2 \times (\mathbb{C}^*)^2 \to \mathbb{C} \times (\mathbb{C}^*)^2$; $(x, y, q_1, q_2) \mapsto x + y + \frac{q_1 \cdot q_2}{xy^2} + \frac{q_2}{y}$ and

$$H^n(\widetilde{W}, pr) \cong \mathbb{C}\big[z, q_1^{\pm}, q_1^{\pm}\big]\big\langle z^2\partial_z, z\partial_{q_1}, q_2 z\partial_{q_2}\big\rangle / I$$

   where I is the left ideal generated by

$$(q_1 z\partial_{q_1})^2 - q_1(q_2 z\partial_{q_2} - 2q_1 z\partial_{q_1})(q_2 z\partial_{q_2} - 2q_1 z\partial_{q_1} - 1)$$
$$(zq_2\partial_{q_2})(zq_2\partial_{q_2} - 2zq_1\partial_{q_1}) - q_2$$
$$z^2\partial_z + 2q_2 z\partial_{q_2}$$

6 Non-commutative Hodge Structures

In this section, which can be read almost independently of the other parts of the text, we will discuss some results on abstract non-commutative Hodge structures (called ncHodge structures for short in the sequel). The ultimate aim is to use ncHodge structures very much like ordinary ones, in particular, one would like to study period maps, Torelli problems etc. However, for the moment these kind of techniques are available only for a restricted class of ncHodge-structures (namely, the so-called *regular* ones). However, these are in a certain sense the building blocks for more general (irregular) ncHodge structures, which, as we will see, occur in mirror symmetry. In that sense any result for the regular case will certainly also be of importance for ncHodge structures defined by Landau-Ginzburg models.

We start with the very definition of a non-commutative Hodge structure. For simplicity, we suppress any notion of weights, that is, we consider only ncHodge structures of weight 0. Such structures almost never exists in (commutative or non-commutative) geometry, however, they are technically slightly simpler to treat and the adaption to the general case is not very difficult. We also omit the grading present in the definition in [20], as we are not going to discuss the (conjectural) construction of an ncHodge structures from a category, as described in loc.cit.

Definition 6.1 (*ncHodge structure*, [15, 20, 31]) A real resp. rational non-commutative Hodge structure (of weight 0) consists of the following data:

1. An algebraic vector bundle $\mathcal{H}$ on $\mathbb{C}_z$ (z being a fixed coordinate on $\mathbb{C}$) of rank μ.
2. A $\mathbb{K}$-local system $\mathcal{L}$ on $\mathbb{C}^*$ (with $\mathbb{K}$ being either $\mathbb{R}$ or $\mathbb{Q}$), together with an isomorphism
$$\mathrm{iso} : \mathcal{L} \otimes_{\mathbb{K}} \mathcal{O}_{\mathbb{C}^*} \to \mathcal{H}_{|\mathbb{C}^*}$$
such that the connection ∇ induced by iso has a pole of order at most 2 at $z = 0$ and a regular singularity at $z = \infty$.
3. A polarizing symmetric form $P : \mathcal{L} \otimes j^*\mathcal{L} \to \underline{\mathbb{K}}_{\mathbb{C}^*}$ (where $j(z) = -z$), which induces a non-degenerate pairing
$$P : \mathcal{H} \otimes_{\mathcal{O}_{\mathbb{C}}} j^*\mathcal{H} \to \mathcal{O}_{\mathbb{C}}$$
In particular, we have an induced non-degenerate pairing
$$[P] : \mathcal{H}/z\mathcal{H} \times \mathcal{H}/z\mathcal{H} \to \mathbb{C}.$$
4. There is an isomorphism
$$\mathcal{H} \otimes_{\mathcal{O}_C} \widehat{\mathcal{O}}_{\mathbb{C}}[*\{0\}] \cong \bigoplus_{i=1}^{k} (\mathcal{R}_i, \nabla_i) \otimes e^{u_i/z}$$
where $u_1, \ldots, u_k \in \mathbb{C}$, where $\widehat{\mathcal{O}}_{\mathbb{C}}$ denotes the completion of $\mathcal{O}_{\mathbb{C}}$ at $z = 0$ and where $(\mathcal{R}_i, \nabla_i)$ are formal meromorphic bundles (i.e., locally free $\widehat{\mathcal{O}}_{\mathbb{C}}(*\{0\})$-modules) equipped with a connection with *regular* singularity at $z = 0$.

   If all u_i in this decomposition are equal to zero, then $\mathcal{H} \otimes \mathcal{O}_{\mathbb{C}}(*0)$ is a regular $\mathcal{D}_{\mathbb{C}_z}$-module, and we say that the $(\mathcal{H}, \mathcal{L}, \mathrm{iso}, P)$ is regular in this case.
5. Consider the morphism $\gamma : \mathbb{P}^1 \to \mathbb{P}^1$ given by $\gamma(z) = 1/\overline{z}$, and glue the bundle $\mathcal{H}$ on $\mathbb{C}$ and the bundle $\overline{\gamma^*\mathcal{H}}$ on $\mathbb{P}^1 \backslash \{0\}$ via an identification of the local systems

$\mathcal{L} \otimes_{\mathbb{K}} \underline{\mathbb{C}}$ and $\overline{\gamma^*\mathcal{L} \otimes_{\mathbb{K}} \underline{\mathbb{C}}}$ on $\mathbb{C}^*$. Using the flat structure, it suffices to define this identification on S^1 only and here it is given by complex conjugation called τ, that is, by conjugation with respect to $\mathbb{K}$-structure $\mathcal{L}$ in $\mathcal{L}_{\mathbb{C}}$. Call the resulting holomorphic bundle $\widehat{\mathcal{H}} \to \mathbb{P}^1$. Then we call $\mathcal{H}$ pure iff $\widehat{\mathcal{H}} \cong \mathcal{O}_{\mathbb{P}^1}^{\mu}$ and pure polarized if it is pure, and if the hermitian form

$$h := P(-, \tau-) : H^0\big(\mathbb{P}^1, \widehat{\mathcal{H}}\big) \times H^0\big(\mathbb{P}^1, \widehat{\mathcal{H}}\big) \longrightarrow \mathbb{C}$$

is positive definite. Notice that τ induces an anti-linear involution on the space of global sections of the trivial bundle $\widehat{\mathcal{H}}$.

The following result gives a partial explanation of the term "ncHodge": namely, it shows how ordinary Hodge structures can be seen as ncHodge-structures.

Proposition 6.2 ([20, Lemma 2.9], [15, Sect. 5]) *The functor sending a real resp. rational Hodge structure* $(V_{\mathbb{K}}, F^\bullet V_{\mathbb{C}}, w)$ *to the ncHodge structure given by* $\mathcal{H} := \bigoplus_{k\in\mathbb{Z}} z^{-k} F^k V_{\mathbb{C}} \subset V_{\mathbb{C}} \otimes_{\mathbb{C}} \mathbb{C}[z, z^{-1}]$ (*where* $\mathcal{L} \cong V_{\mathbb{K}} \subset V_{\mathbb{C}} \otimes_{\mathbb{C}} \mathbb{C}[z, z^{-1}]$) *is fully faithful. Its image are the ncHodge structures where* ∇ *has a logarithmic pole* (*i.e., a pole of order* 1) *on* $\mathcal{H}$ *at zero and such that the monodromy of* ∇ *is trivial.*

Remark We will not give any more details on the origins of the name non-commutative Hodge structures. The basic idea (which is described in [20] but which is for the moment merely a collection of conjectures) is that one can find such structures starting from a certain triangulated category. The object which is supposed to underly a non-commutative Hodge structure is the so-called negative cyclic homology of this category. In some special cases, one expects to get back ncHodge structures constructed directly from geometric input data via this general categorical setup. See, e.g. [32] for the case of isolated hypersurface singularities (the example from Theorem 6.9 below).

A very important feature of the theory is the study of families of ncHodge structures, very much similar to the case of ordinary Hodge structures. The classical notion of *Griffiths transversality* is expressed as a certain pole order property of a family of ncHodge structures.

Definition 6.3 A variation of (pure resp. pure polarized) ncHodge structures on a complex manifold M is a vector $\mathcal{H}$ bundle on $\mathbb{C}_z \times M$, equipped with a connection operator with poles along $\{0\} \times M$ such that

1. For any vector field X on M, $\mathcal{H}$ is invariant under the operator $z\nabla_X$.
2. For any point $c \in M$, the restriction of $\mathcal{H}$ to $\mathbb{C}_z \times \{c\}$ is a (pure resp. pure polarized) ncHodge structure in the sense of Definition 6.1

One checks that for a variation of ncHodge structures coming from an ordinary one, that is, such that the restriction to each point in the parameter space lies in the essential image of the functor considered in Proposition 6.2, the pole order property (1) in the above definition is equivalent to the Griffiths transversality property of the variation of Hodge structures we started with.

A regular ncHodge structure has a set of discrete invariants, the so called spectral numbers. They will be used below for the construction of classifying spaces. We give the definition together with some of the main properties.

Definition-Lemma 6.4 *Let* $(\mathcal{H}, \mathcal{L}, \text{iso}, P)$ *be a regular ncHodge structure and suppose that the monodromy of the local system* $\mathcal{L}$ *is quasi-unipotent, i.e, that the monodromy operator* T *on the space of (flat multivalued) sections of* $\mathcal{L}$ *satisfies* $(T^n - Id)^m = 0$ *for some non-negative integers* m, n*. This condition is equivalent to the fact that the eigenvalues of* T *are roots of unity, and it is satisfied in virtually all examples coming from geometry.*

1. *Define the spectrum (which is actually an invariant of* $\mathcal{H}$ *and* ∇ *only) by*

$$\mathrm{Sp}(\mathcal{H}, \nabla) = \sum_{\alpha \in \mathbb{Q}} \dim_{\mathbb{C}} \left(\frac{\mathrm{Gr}_V^{\alpha} \mathcal{H}}{\mathrm{Gr}_V^{\alpha} z\mathcal{H}} \right) \cdot \alpha \in \mathbb{Z}[\mathbb{Q}],$$

   where $V^{\bullet}\mathcal{H}(*D)$ *is the canonical* V*-filtration, also called Kashiwara-Malgrange filtration, on the* $\mathcal{D}_{\mathbb{C}_z}$*-module* $\mathcal{H}(*0)$ *(see, e.g.* [11, *Sect.* 7.2]). *It induces a filtration* $V^{\bullet}\mathcal{H}$ *on the lattice* $\mathcal{H} \subset \mathcal{H}(*0)$*, which is used in the above definition through its graded parts* $\mathrm{Gr}_V^{\alpha}\mathcal{H}$*. Notice that the* V*-filtration is indexed by* $\mathbb{Q}$*, which corresponds to the quasi-unipotency of* $\mathcal{L}$*.*

   We also write $\mathrm{Sp}(H, \nabla)$ *as a tuple* $\alpha_1, \ldots, \alpha_\mu$ *of* μ *numbers (with* $\mu = \mathrm{rank}(H)$*), ordered by* $\alpha_1 \leq \cdots \leq \alpha_\mu$*.*
2. α *is a spectral number, that is, we have* $\dim_{\mathbb{C}}(\mathrm{Gr}_V^{\alpha}(\mathcal{H}/z\mathcal{H})) > 0$*, only if* $e^{-2\pi i \alpha}$ *is an eigenvalue of the monodromy operator* T*.*
3. *The spectrum satisfies* $\alpha_i = -\alpha_{\mu+1-i}$ *(more generally, if we allow weights, then we have* $\alpha_i + \alpha_{\mu+1-i} = w$ *for an ncHodge structure of weight* w*).*

A fundamental tool in the study of Hodge structures is the theory of classifying spaces and period maps associated to a variation of Hodge structures. A similar result exists in the non-commutative case, and can be expressed as follows.

Theorem 6.5

1. [16, *Theorem* 7.3] *Fix a quasi-unipotent* $\mathbb{K}$*-local system* $\mathcal{L}$ *on* $\mathbb{C}^*$ *and the polarizing form* $P : \mathcal{L} \otimes j^*\mathcal{L} \to \underline{\mathbb{K}}_{\mathbb{C}^*}$*. Fix also a rational number* α_1 *such that* $e^{-2\pi i \alpha_1}$ *is an eigenvalue of the monodromy of* $\mathcal{L}$*. Moreover, suppose that* $\alpha_1 \leq 0$ *(or, more generally, that* $\alpha_1 \leq \frac{w}{2}$*, where* w *is a fixed integer that will be the weight of the ncHodge structures to consider). Put*

$$\begin{aligned} \mathcal{M} := \Big\{ & (\mathcal{H}, \nabla_z) \,|\, \mathcal{H} \to \mathbb{C}_z \textit{ vector bundle}, \nabla_z \in \mathrm{Aut}_{\mathbb{C}}(\mathcal{H}_{|\mathbb{C}_z^*}) \text{ connection}, \\ & \big(z^2\nabla_z\big)(\mathcal{H}) \subset \mathcal{H}, \\ & \mathrm{Sp}(\mathcal{H}, \nabla) \subset [\alpha_1, -\alpha_1] \cap \mathbb{Q}, \exists \text{ iso} : \mathcal{L} \otimes_{\underline{\mathbb{K}}_{\mathbb{C}^*}} \mathcal{O}_{\mathbb{C}^*} \xrightarrow{\cong} \mathcal{H}_{|\mathbb{C}^*}, \\ & P(\mathcal{H}, \mathcal{H}) \subset z^w \mathcal{O}_{\mathbb{C}_z} \textit{ non-degenerate} \Big\} \end{aligned}$$

Then $\mathcal{M}$ is a projective variety which is stratified by locally closed smooth subvarieties parameterizing bundles with connection with fixed spectral numbers.

$\mathcal{M}$ comes equipped with a universal bundle $\mathcal{H}^{\mathcal{M}} \to \mathbb{C}_z \times \mathcal{M}$ with a relative connection ∇_z, and a polarizing form defined by P.

2. [16, *Sect.* 8] *Define*

$$\mathcal{M}^{pp} := \left\{x \in \mathcal{M} \mid \left(\mathcal{H}^{\mathcal{M}}, \mathcal{L}, \text{iso}, P\right)_{|\mathbb{C}\times\{x\}} \text{ is a pure polarized ncHodge structure}\right\}$$

(*this is an open subvariety of* $\mathcal{M}$), *then the tangent sheaf* $\Theta_{\mathcal{M}^{pp}}$ *of* $\mathcal{M}^{pp}$ *can be endowed with a positive definite hermitian metric* h, *which defines a distance function* d_h *on* $\mathcal{M}^{pp}$.

3. [16, *Theorem* 8.6] *The metric space* $(\mathcal{M}^{pp}, d_h)$ *is complete.*

The following result describes the period maps which are analogues of the classical period maps for variations of ordinary Hodge structures.

Proposition 6.6 *Let* $(\mathcal{H}, \mathcal{L}, \text{iso}, P)$ *be a variation of regular pure polarized ncHodge structures on a simply connected manifold* M, *and let* α_1 *be the smallest spectral number of the restriction of* $(\mathcal{H}, \nabla)$ *to a generic point of* M. *Then there is a period map* $\phi_{\text{ncHodge}} : M \to \mathcal{M}^{pp}$ *satisfying* $\phi^*_{\text{ncHodge}}(\mathcal{H}^{\mathcal{M}}) \cong \mathcal{H}$.

If the spectrum of $(\mathcal{H}, \nabla)$ *is constant on* M, *then the holomorphic sectional curvature* κ *of the metric* h *on* $\Theta_{\mathcal{M}^{pp}}$ *will be negative and bounded from above by a negative number on the image* $\text{Im}(d\phi_{\text{ncHodge}})$ *of the derivative* $d\phi_{\text{ncHodge}} : \mathcal{T}_M \to \phi^*_{\text{ncHodge}}\Theta_{\mathcal{M}^{pp}}$ *of the period map.*

Using standard tools from complex hyperbolic analysis, we obtain the following two consequences.

Corollary 6.7

1. [14, *Corollary* 4.5] *Let* $(\mathcal{H}, \mathcal{L}, \text{iso}, P)$ *be a variation of pure polarized regular ncHodge-structures on* $\mathbb{C}^n$ *with constant spectrum. Then the associated period map* ϕ_{ncHodge} *is constant, in other words* $\mathcal{H}$ *is stable under* ∇. *One says that* $(\mathcal{H}, \mathcal{L}, \text{iso}, P)$ *is a trivial variation of ncHodge structures in this case.*
2. [16, *Theorem* 9.5] *Let* X *be a complex manifold,* $Z \subset X$ *a complex space of codimension at least two. Suppose that the complement* $Y := X \backslash Z$ *is simply connected. Let* $(\mathcal{H}, \mathcal{L}, \text{iso}, P)$ *be a variation of pure polarized regular ncHodge-structures on the complement* Y *which has constant spectral numbers. Then this variation extends to the whole of* X, *with possibly jumping spectral numbers over* Z.

Remarks

1. The first statement from the above corollary even extends to the irregular case. However, if we do not suppose that the connection ∇ is regular along $z = 0$, then we need some kind of regularity along the boundary of the parameter space, e.g. along $\mathbb{P}^n \backslash \mathbb{C}^n$. Such a property exists, and is called *tameness* of the associated *harmonic bundle*. See [16, Corollary 6.3] for a precise statement.

2. The second statement treats extensions of regular ncHodge structures over codimension two subvarieties. The question how to extend a variation over a *divisor* is perhaps even more important. In that case, we need the full power of the limit statements for harmonic bundle, due to Mochizuki (see [25]). We also need to take care of the possible monodromy along the boundary divisor, this can be done by adding the structure of a lattice (i.e., a $\mathbb{Z}$-local subsystem of $\mathcal{L}$). A precise formulation of the result for extensions over divisors can be found in [16, Theorem 9.7].

The following fundamental theorem shows how ncHodge structures occur in geometry. It concerns a certain type of regular functions on smooth affine varieties, called cohomologically tame. Without giving the precise definition of this notion (see [28]) let us just mention that such functions have isolated critical points and satisfy moreover an assumption concerning their behavior at infinity (in the fibres of an appropriate compactification). In particular, convenient and non-degenerate Laurent polynomials, like the Landau-Ginzburg model of a toric Fano manifold are cohomologically tame.

Theorem 6.8 ([30]) *Let U be a smooth affine manifold and $f : U \to \mathbb{C}$ a cohomologically tame function. Then the twisted de Rham cohomology $H^n(f)$ underlies a pure polarized non-commutative Hodge structure.*

There is another important class of examples where the twisted de Rham cohomology can be equipped with an ncHodge structure. These are germs of holomorphic functions with isolated critical points also called isolated hypersurface singularities. However, and this is one of the main issues of study in this so-called local case, the corresponding structure does not necessarily satisfy the condition (5) in Definition 6.1. Nevertheless, we have the following result.

Theorem 6.9 ([15, Corollary 11.4]) *Let $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ be an isolated hypersurface singularity. Then for a sufficiently large real number r, the twisted de Rham cohomology of an appropriate representative of the germ $r \cdot f$ underlies a pure polarized ncHodge structure.*

7 Mirror Symmetry Statements

Using all the objects introduced above we can now express mirror correspondences as isomorphisms of systems of differential equations. This identification relies on Givental's theorem (Theorem 3.7 above), and yields an isomorphism of vector bundle with connections on $\mathbb{C}_z \times (\mathbb{C}^*)^r$. This is the first result of this section. However, from a physical point of view, we would like to express the mirror correspondence as an isomorphism of so-called *Frobenius manifolds*, which appear as moduli spaces of two dimensional topological field theory in physics. The cohomology of any, say smooth projective, variety carries a Frobenius structure which is defined precisely using the quantum multiplication. We introduce this notion here and show briefly (referring to [27] for more details) how to construct such a manifold from the Landau-Ginzburg model. The final result then says that there is an isomorphism of Frobenius manifolds between the A-model and the B-model, and this can be considered to be the culminating point of this version of mirror symmetry for smooth toric varieties with ample or nef anticanonical bundle.

Let us start with the very definition of a Frobenius manifold. For our purpose, we also need an extended version called logarithmic Frobenius manifold, which takes into account the degeneration behavior of the quantum multiplication at the large radius volume limit.

Definition 7.1 Let M be a complex manifold.

1. A Frobenius structure on M is given by two tensors $\circ \in (\Omega_M)^{\otimes 2} \otimes_{\mathcal{O}_M} \mathcal{T}_M$, $g \in (\Omega_M)^{\otimes 2}$ and two vector fields $E, e \in \mathcal{T}_M$ subject to the following relations.
   (a) $\circ$ defines a commutative and associative multiplication on $\mathcal{T}_M$ with unit e.
   (b) g is bilinear, symmetric and non-degenerate.
   (c) For any $X, Y, Z \in \mathcal{T}_M$, $g(X \circ Y, Z) = g(X, Y \circ Z)$.
   (d) g is flat, i.e., locally there are coordinates $t_1, \ldots, t_\mu$ on M such that the matrix of g in the basis $(\partial_{t_1}, \ldots, \partial_{t_\mu})$ is constant.
   (e) Write ∇ for the Levi-Civita connection of g, then the tensor $\nabla\circ$ is totally symmetric.
   (f) $\nabla(e) = 0$.
   (g) $\mathrm{Lie}_E(\circ) = \circ$, $\mathrm{Lie}_E(g) = D \cdot g$ for some $D \in \mathbb{C}$
2. Now suppose that $\dim_{\mathbb{C}}(M) > 0$ and let $D \subset M$ be a simple normal crossing divisor. Suppose that $(M \backslash D, \circ, g, e, E)$ is a Frobenius manifold. Then we say that it has a logarithmic pole along D (or that $(M, D, \circ, g, e, E)$ is a logarithmic Frobenius manifold for short) if $\circ \in \Omega^1_M(\log D)^{\otimes 2} \otimes \mathcal{T}_M(\log D)$, $g \in \Omega^1_M(\log D)^{\otimes 2}$, $E, e \in \mathcal{T}(\log D)$ and if g is non-degenerate on $\mathcal{T}_M(\log D)$. Here $\Omega^1(\log D)$ resp. $\mathcal{T}(\log D)$ are the sheaves of logarithmic differential forms resp. logarithmic vector fields along D.

The following is the basic result which explains why Frobenius structures enter into the mirror symmetry picture.

Theorem 7.2 (See, e.g., [24]) *Let X be smooth projective and convex (this last assumption is not essential). Define a multiplication $\circ$ on the tangent bundle of the cohomology space $H^*(X, \mathbb{C})$ as was done above before Definition 3.2. Define a constant (hence flat) pairing $g(-,-)$ on $TH^*(X, \mathbb{C})$ by the non-degenerate Poincaré metric on $H^*(X, \mathbb{C})$. Put $e = 1 \in H^0(X, \mathbb{C})$ and recall from the definition of the quantum $\mathcal{D}$-module (6) that $E = \sum_{i=0}^{s}(1 - \frac{\deg(T_i)}{2})t_i \partial_{t_i} + \sum_{a=1}^{r} k_a \partial_{T_a}$, where $\sum_{a=1}^{r} k_a T_a = c_1(X)$. Then the tuple $(H^*(X, \mathbb{C}), \circ, g, e, E)$ defines a formal germ of a Frobenius manifold at $t_0 = t_1 = \cdots = t_s = 0$.*

The fact that we do not know in general whether the quantum product is convergent forces us to restrict to formal germs in the last theorem. However, as explained above, we do not worry about convergence questions in this paper, so that we simply assume that there is a certain subspace of $H^*(X, \mathbb{C})$ on which we have a holomorphic Frobenius structure, and we also want this subspace to contain a neighborhood of the large radius limit. Then the divisor axiom for Gromov-Witten invariants yields

Lemma 7.3 ([26, Sect. 2.1.2]) *Let $U \subset H^0(X,\mathbb{C}) \oplus (H^2(X,\mathbb{C})/2\pi i H^2(X,\mathbb{Z})) \oplus H^{>2}(X,\mathbb{C})$ be a domain of convergence of the quantum product, and assume that a point $(t_0, \underline{q}, \underline{t})$ is contained in U if it is small enough in the standard hermitian metric of $\mathbb{C} \times (\mathbb{C}^*)^r \times \mathbb{C}^{s-r-1}$. Let $\overline{U} \subset \mathbb{C}^s$ be the closure (i.e., including points where $q_a = 0$ for some $a = 1, \dots, r$). Then the Frobenius structure on U extends to a logarithmic Frobenius structure on $(\overline{U}, D)$, $D = \bigcup_{a=1}^r D_a$, with $D_a = \{q_a = 0\}$.*

Our next task is to explain how one can construct a Frobenius structure (which will also acquire logarithmic poles along a normal crossing divisor) starting from a Landau-Ginzburg model, that is, from a family of Laurent polynomials $\widetilde{W} : (\mathbb{C}^*)^n \times (\mathbb{C}^*)^r \to \mathbb{C}$. We use the twisted de Rham cohomology constructed in Sect. 5, and the isomorphism (11) expressing it by hypergeometric differential equations. The main step towards the construction of Frobenius manifolds is contained in the following proposition. In order to keep notations simple, we restrict to the Fano case.

Proposition 7.4 ([27, Proposition 3.10]) *Suppose that X_Σ is smooth toric and Fano. There is a Zariski open subset $\overline{U} \subset \mathbb{C}^r$ including the limit point $\{\underline{q} = \underline{0}\} \in \mathbb{C}^r$ and an extension $\widehat{{}_0\mathcal{QM}_{\widetilde{A}}} \to \mathbb{P}^1_z \times \overline{U}$ of ${}_0\mathcal{QM}_{\widetilde{A}}$ which is a family of trivial $\mathbb{P}^1$-bundles and such that the connection extends with a logarithmic pole along $\{z = \infty\} \cup \bigcup_{a=1}^r \{q_a = 0\}$. Moreover, the restriction $(\widehat{{}_0\mathcal{QM}_{\widetilde{A}}})_{|\{z=0,q_a=0\}}$ is canonically equipped with a multiplication and is isomorphic as an algebra to the classical cohomology ring of X_Σ.*

Example We give here the simplest example for which the mirror correspondence can be established directly, namely, that of the projective spaces. For the Hirzebruch surfaces $\mathbb{F}_1$ and $\mathbb{F}_2$, a similar computation can be carried out. In fact, the representation of the twisted de Rham cohomology of the Landau-Ginzburg model of $\mathbb{P}^n$, i.e., formula (12) already gives the desired extension to $z = \infty$. More precisely, put $\omega_i := (zq\partial_q)^i$ for $i = 0, \dots, n$, then there is an isomorphism $H^n(\widetilde{W}_{\mathbb{P}^n}, pr) \cong \bigoplus_{i=0}^n \mathcal{O}_{\mathbb{C}_z \times (\mathbb{C}^*)^r} \omega_i$, and we have a connection

$$\nabla(\underline{\omega}) = \underline{\omega} \cdot \left[\left(A_0 \frac{1}{z} + A_\infty \right) \frac{dz}{z} - A_0 \cdot \frac{dq}{n \cdot z \cdot q} \right] \tag{13}$$

where $\underline{\omega} = (\omega_0, \dots, \omega_n)$, where

$$A_0 := \begin{pmatrix} 0 & 0 & \dots & 0 & c \cdot q \\ -1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & -1 & 0 \end{pmatrix},$$

with $c \in \mathbb{C}^*$ and where $A_\infty = \mathrm{diag}(0, 1, \dots, n)$.

Then we can simply define $\widehat{{}_0\mathcal{QM}_{\widetilde{A}}} := \bigoplus_{i=1}^n \mathcal{O}_{\mathbb{P}^1_z \times (\mathbb{C}^*)^r} \omega_i$, so that $U = (\mathbb{C}^*)^r$ in this case and it is easily seen from (13) that the connection ∇ on $\widehat{{}_0\mathcal{QM}_{\widetilde{A}}}$ has poles with the desired properties.

We now deduce from Givental's theorem two types of mirror statements. The first one concerns the small quantum $\mathcal{D}$-module F from Definition 3.2.

Theorem 7.5 ([19, Proposition 4.8], [27, Proposition 4.10]) *Let X_Σ be Fano. There is an isomorphism of bundles with connection on $\mathbb{P}^1 \times \overline{U}$*

$$\widehat{{}_0\mathcal{QM}}_{\widetilde{A}} \cong F.$$

The second mirror correspondence will be an isomorphism of Frobenius manifolds. There is a general strategy to construct Frobenius manifolds starting from families of trivial vector bundles on $\mathbb{P}^1$. Results of this kind are due to Malgrange, Dubrovin and Hertling-Manin (see [13] and the references therein). The version that we need here (taking into account logarithmic poles) can be found in [26], and this gives the following result.

Theorem 7.6 (Mirror symmetry for smooth toric Fano varieties) *Let X_Σ be smooth projective and Fano. Let $\widetilde{W}_{X_\Sigma}$ be its Landau-Ginzburg model. There is a germ $((M, 0), (\widetilde{D}, 0))$ of a canonical logarithmic Frobenius structure associated to $\widetilde{W}_{X_\Sigma}$. Here $M = \overline{U} \times \mathbb{C}^k$ and $\widetilde{D} = D \times \mathbb{C}^k$, where $k = n!\mathrm{vol}(\Delta(\underline{a}_1, \ldots, \underline{a}_m)) - r$. The Frobenius structure is defined by a family of $\mathbb{P}^1$-bundles on M which restricts to $\widehat{{}_0\mathcal{QM}}_{\widetilde{A}}$ on $\overline{U} \times \{0\}$.*

This Frobenius structure is isomorphic to the one from Theorem 7.2 (i.e. to the quantum cohomology of X_Σ) near the limit point $\underline{q} = 0$.

The very last statement can be considered as the final version of the mirror symmetry for smooth toric Fano varieties. The nef case can also be treated by these methods, and the result is basically the same, with some small technical modifications.

Finally, let us once again come back to the example of the projective spaces. Consider the twisted de Rham cohomology (i.e., either formula (12) or formula (13)), then we easily see (and this is of course a general fact) that the restriction $H^n(\widetilde{W}_{\mathbb{P}^n}, pr)_{|z=0}$ is a family of $\mathbb{C}[q^{\pm}]$-algebras. More precisely, write p for the class of $zq\partial_q$, then formula (12) gives that

$$H^n(\widetilde{W}_{\mathbb{P}^n}, pr)_{|z=0} \cong \frac{\mathbb{C}[p, q^{\pm}]}{(p^{n+1} - q)}$$

and we see from the isomorphism (5) that this is precisely the small quantum cohomology of $\mathbb{P}^n$. In the same way, it is easy to see that the quantum-$\mathcal{D}$-module of $\mathbb{P}^n$ is exactly given by the connection operator in formula (13). Hence, in this simple case there is an explicit identification of the differential systems on the two sides of the mirror correspondence.

We finish this survey by mentioning the following corollary, which follows directly from Theorem 7.5 and Theorem 6.8 above.

Corollary 7.7 *Let X_Σ be Fano. Then the (restriction to $\mathbb{C}_z \times (\mathbb{C}^*)^r$ of the) quantum $\mathcal{D}$-module F of X_Σ underlies a variation of a pure polarized non-commutative Hodge structures.*

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HISTORICAL ARTICLE

Wilhelm Klingenberg, 1924–2010

Jost-Hinrich Eschenburg

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Zusammenfassung Ein von persönlichen Erfahrungen geprägter Nachruf auf den Menschen, Lehrer und Wissenschaftler Wilhelm Klingenberg. Schlüsselworte seines wissenschaftlichen Werkes sind Geometrie, Krümmung und Topologie, Geodätische Linien.

Schlüsselwörter Sphärensatz · Geodätischer Fluss · Geschlossene Geodätische · Geometrie

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> *Wenn der Knabe zu begreifen anfängt, dass einem sichtbaren Punkte ein unsichtbarer vorhergehen müsse, dass der nächste Weg zwischen zwei Punkten schon als Linie gedacht werde, ehe sie mit dem Bleistift aufs Papier gezogen wird, so fühlt er einen gewissen Stolz, ein Behagen. Und nicht mit Unrecht; denn ihm ist die Quelle alles Denkens aufgeschlossen, Idee und Verwirklichtes, potentia et actu, ist ihm klar geworden; der Philosoph entdeckt ihm nichts Neues, dem Geometer war von seiner Seite der Grund alles Denkens aufgegangen.*
> J.W. Goethe: Wilhelm Meisters Wanderjahre, Aus Makariens Archiv Nr. 40

> *It was Goethe who in the epigram no. 40 … most clearly gave expression to the essence of geometry.*
> W. Klingenberg, Vorwort zu [1]

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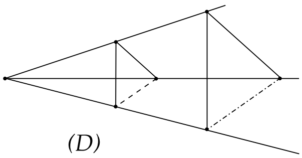
Abb. 1 Wilhelm Klingenberg, anlässlich seiner Ehrenpromotion 2001 in Leipzig

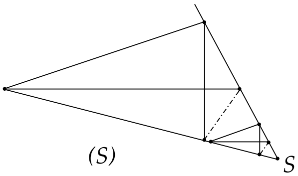
Wilhelm Klingenberg[1] hat für die Differentialgeometrie in Deutschland nach dem Zweiten Weltkrieg eine ähnliche Bedeutung wie Marcel Berger für Frankreich oder Manfredo Do Carmo für Brasilien: Fast alle, die heute in Deutschland mit diesem Fach zu tun haben, sind in irgendeiner Weise mit Klingenberg verbunden, als direkte oder indirekte Schüler von ihm selbst oder von Kollegen, die mit ihm eng verbunden waren. Unter den 27 Doktoranden, die in [1] aufgeführt werden, finden sich 14 spätere Professorinnen und Professoren, die die Differentialgeometrie in Deutschland und international vertreten. Diese außerordentliche Leistung war nicht allein sein Verdienst, es hatte auch sehr viel zu tun mit der ungewöhnlich anregenden und ambitionierten Atmosphäre an seiner hauptsächlichen Wirkungsstätte, an die er 1966 berufen wurde, dem Mathematischen Institut der Universität Bonn. Aber von entscheidender Bedeutung war seine ganz persönliche Art im Umgang mit seinen Schülern. Diese war sehr unkompliziert und weit entfernt von altem Ordinariengehabe. Noch bis ins hohe Alter blieb das Fahrrad in Bonn sein wichtigstes Transportmittel. Als Student wurde man sehr schnell geduzt und zu den zahlreichen mathematischen Tees und Partys eingeladen, die nicht selten im Klingenbergschen Haus in Bonn-Röttgen stattfanden, voll von ostasiatischer Kunst. Dort traf man informell mit den vielen Gästen am Institut und am Sonderforschungsbereich zusammen. Als Doktorand von Wilhelm Klingenberg konnte man von seiner Ermutigung, seinen Anregungen und vor allem seinen Verbindungen profitieren – er kannte einfach jeden in der Riemannschen Geometrie. Die Betreuung seiner Doktoranden war unterschiedlich intensiv. Als ich ihn nach meinem Diplom nach einem möglichen Thema für eine Doktorarbeit fragte, schob er mir einen Stapel neuerer Preprints zu und sagte, ich solle mir daraus etwas aussuchen. Es blieb nicht dabei; ich habe im Laufe meiner Arbeit viele Anregungen von ihm bekommen, Hinweise, die es zu überprüfen galt, keine fertigen Ideen oder gar Rezepte. Nur Leute, die schon eine gewisse Selbständigkeit in ihrer mathematischen Arbeit erworben hatten, konnten bei ihm Erfolg haben. Manche sind frühzeitig ausgeschieden, und auch in dieser Auswahl lag vermutlich ein Teil seines großen Erfolges als Lehrer.

Wilhelm Klingenberg hat sich in seinen frühen Jahren bei Karl-Heinrich Weise und Friedrich Bachmann in Kiel mit klassischer Geometrie beschäftigt; das hat seine

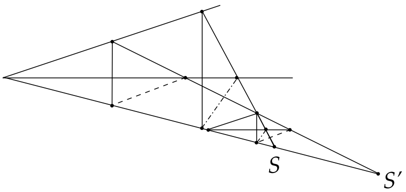
[1] Klingenbergs selbst verfasster Lebenslauf findet sich in [1]; weitere Biographien: http://de.wikipedia.org/wiki/Wilhelm_Klingenberg_(Mathematiker), www.gap-system.org/~history/Biographies/Klingenberg.html.

Denkweise geprägt. „Man darf einen Geometer als einen zur Gestaltwahrnehmung besonders begabten Menschen ansehen“, schreibt er 1997 in seinem Aufsatz [15] „Mathematik und Melancholie“ und fährt fort: „Aus der scheinbaren Wirrnis des Vexierbildes gliedert sich, vielleicht erst nach längerer Dauer, urplötzlich eine Figur heraus.“ Als Beispiel führt er einen Moment aus seiner Kieler Zeit an, in dem er blitzartig die Äquivalenz zweier Schließungssätze der affinen ebenen Geometrie erkannte, des Desargues-Satzes (D) und des Schmetterlingssatzes (S). Beide Sätze behaupten, dass das gestrichelte Geradenpaar parallel ist, wenn die durchgezogenen Geradenpaare parallel sind.





Es war klar, dass (S) aus (D) folgte, aber die Umkehrung war ein offenes Problem, mit dem sich bereits Ruth Moufang beschäftigt hatte. Wie Klingenberg in [15] ausführt, gelang ihm der Beweis allein durch die Vorstellung einer Figur, die zwei Versionen von (S) (mit Scheiteln S und S') mit der Kontraposition von (D) vereinigte: Aus (S) und (S') folgt die Nichtparallelität der gestrichelten und der Strichpunkt-Linie und damit (D) in der Kontraposition.[2]



Die Beschäftigung mit den Grundlagen der Geometrie prägte auch seine Habilitationsschrift an der Universität Hamburg (1954) mit dem Thema „Ebene Geometrien mit Nachbarelementen“, vgl. [2]. Es ging darin um eine Sorte von Ebenen, wo die Verbindungsgerade zweier Punkte und der Schnittpunkt zweier Geraden nicht mehr eindeutig sind, die aber eine geradentreue Projektion auf eine echte projektive Ebene gestatten; sie wurden später mit seinem Namen verbunden und sind bis heute Gegenstand zahlreicher mathematischer Arbeiten. In einer Rede anlässlich seiner Emeritierung 1989 in Bonn [14] sagte er dazu: „There even seems to be a Klingenberg plane. But whatever it is, it is not nearly as important as the Poincaré halfplane.“

Unter dem Einfluss vor allem von Marston Morse, den er durch Vermittlung von Wilhelm Blaschke in Hamburg kennen lernte, wechselte Wilhelm Klingenberg Ende

[2] Wenn das gestrichelte Geradenpaar nicht parallel ist, dann ist auch eins der durchgezogenen Geradenpaare nicht parallel.

der fünfziger Jahre sein Forschungsgebiet in Richtung Riemannsche Geometrie. Seine in Kiel erworbene Einstellung zur Geometrie lässt sich aber auch noch in seinen späteren Arbeiten erkennen. In [5] schreibt er über den Dreiecks-Vergleichssatz von Alexandrov-Toponogov auf krummen Flächen: „Ich glaube, dass ich kein Wort zu verlieren brauche über die Schönheit dieses Satzes, die jeder echte Geometer spüren wird.“ Er bleibt aber nicht bei dieser Feststellung stehen: „Alle vielleicht noch verbleibenden Zweifel an der Bedeutung dieses Satzes werden jedoch behoben, wenn ich nun zwei Beispiele gebe, wie dieser Dreiecksvergleichssatz den Schlüssel bildet für eine Reihe von Sätzen der Flächentheorie im Großen.“ Es folgen globale Sätze über Flächen, die danach auf beliebige Dimension erweitert werden und am Ende in dem berühmten Sphärensatz gipfeln, eine seiner folgenreichsten wissenschaftlichen Entdeckungen [4]:

> *Liegt die Schnittkrümmung einer einfach zusammenhängenden und vollständigen Riemannschen Mannigfaltigkeit zwischen* $\frac{1}{4}$ *und* 1 (*Grenzen ausgeschlossen*), *dann ist diese zu einer Sphäre homöomorph.*

Der Wert dieser Sätze liegt nicht allein in der Schönheit der Konstruktion, die vielleicht nur von einigen Mathematikern („echte Geometer“) so empfunden wird, sondern letztlich in dem Beitrag, den sie über ihren ursprünglichen Bereich hinaus für das Verständnis zentraler Teile der Mathematik leisten, hier für die Theorie der Mannigfaltigkeiten und das Verhältnis von Geometrie und Topologie. Die geometrische Intuition ist nicht Selbstzweck, sie wird in Dienst genommen zum Verstehen von tief liegender Mathematik weit jenseits des anschaulich Vorstellbaren. Die Effektivität dieses Dienstes beschreibt Klingenberg am Ende seines Artikels [5] mit den Worten: „Alle diese Sätze … leiten ihren Ursprung, ihre Motivation und auch die wesentlichen Ideen zu ihren Beweisen her aus jener Quelle, die ich nicht besser zu umschreiben vermag als mit dem mir hier viel zu abstrakten Begriff: Geometrische Intuition.“

Zum Sphärensatz[3] gab eine frühe Version von Harry Rauch (1951) mit nicht optimalen Krümmungsschranken, „somewhat mysterious and very difficult to understand“ [1]. Die optimalen Krümmungsschranken sind $\frac{1}{4}$ und 1, denn projektive Räume und Ebenen über den Divisionsalgebren $\mathbb{C}$, $\mathbb{H}$, $\mathbb{O}$ sind einfach zusammenhängend, aber nicht zu einer Sphäre homöomorph, und ihre Schnittkrümmung liegt zwischen $\frac{1}{4}$ und 1, wobei beide Grenzen angenommen werden. Klingenberg gelang ein entscheidender Schritt zu den optimalen Schranken: Abstandsbälle mit Radius π (die Menge der Punkte, die mit einem Punkt p durch eine Kurve von Länge $< \pi$ verbunden werden können), sind wirklich topologische Bälle, homöomorph (sogar diffeomorph) zum offenen Einheitsball im $\mathbb{R}^n$, wie auf der Sphäre mit Krümmung 1, wo π der sphärische Abstand von Pol zu Pol ist. Dies konnte Klingenberg 1958 [3] für gerade Dimension und 1961 [4] für ungerade Dimension zeigen; der Beweis ist im letzteren Fall ungleich schwieriger, die Behauptung wird sogar falsch ohne die untere Krümmungsschranke. Mit dieser Aussage und dem schon erwähnten Satz von Alexandrov und Toponogov zeigte Marcel Berger 1960, dass ein solcher Raum von zwei

[3] Zur Wirkungsgeschichte dieses Satzes und seiner erst kürzlich bewiesenen differenzierbaren Version siehe den Artikel von Simon Brendle: Der Sphärensatz in der Riemannschen Geometrie, Jahresber. Dtsch. Math.-Ver. 113 (2011), 123–138.

topologischen Bällen überdeckt wird und deshalb homöomorph zu einer Sphäre ist.[4] Der Beweis galt zunächst nur für gerade Dimensionen; erst Klingenberg [4] konnte ihn auch auf ungerade Dimensionen erweitern. Er hielt 1961 eine Gastvorlesung in Bonn über seine Ergebnisse. Zwei seiner Hörer, Detlef Gromoll und Wolfgang Meyer, arbeiteten diese Ideen aus und schufen damit „das" Lehrbuch über „Riemannsche Geometrie im Großen" [7]; selbst der Name des Gebietes war neu im Deutschen. Generationen von Studierenden haben nach diesem Buch Riemannsche Geometrie gelernt. Anfang der achziger Jahre gab Misha Gromov eine sehr einfache neue Beweisidee für den topologischen Sphärensatz: Krümmung misst die lokale Konvexität des Komplements von Abstandsbällen; je größer die Schnittkrümmung K, desto „konvexer" die Ball-Komplemente. Wegen $K < 1$ bilden alle von einem Punkt ausgehenden Geodäten der Länge π gemeinsam einen immersierten Ball, wie auf der stärker gekrümmten Sphäre mit Radius 1. Wegen $K > \frac{1}{4}$ ist das Komplement dieses lokalen Abstandsballs lokal konvex wie auf der schwächer gekrümmten Sphäre vom Radius 2 (Krümmung $\frac{1}{4}$), und wegen der strikten Ungleichung ist die Konvexität strikt. Weil sich strikt lokal konvexe Mengen bei $K \geq 0$ und Dimension ≥ 3 auf einen Punkt zusammenziehen lassen, wird der Raum von zwei am Rand diffeomorph zusammengeklebten Bällen überlagert, also von einer topologischen Sphäre.[5]

Klingenberg liebte die Weite, nicht nur die mathematische, sondern auch die räumliche. Schon in den frühen fünfziger Jahren bewarb er sich in Italien und verbrachte u.a. ein halbes Jahr in Rom bei Francesco Severi und Beniamino Segre. Bei mehrfachen Aufenthalten in den USA ab 1954 besuchte er Marston Morse in Princeton und Shiing Shen Chern in Berkeley, 1963 folgte er einer Einladung nach Recife, Brasilien. Nach Professuren in Göttingen und Mainz lehrte er ab 1966 an der Universität Bonn. Der dort gegründete Sonderforschungsbereich 40 „Theoretische Mathematik", die Keimzelle des späteren Max-Planck-Instituts, stellte die Mittel für ein umfangreiches auswärtiges Gästeprogramm zur Verfügung. Aus aller Welt, von Amerika bis Japan, kamen Mathematiker nach Bonn. Marcel Berger aus Paris und seine Studenten waren ohnehin oft und gern gesehene Gäste. Ich persönlich habe erst während meines Bonner Diplom- und Promotionsstudiums Anfang der siebziger Jahre ein erträgliches Englisch gelernt, die einzige Sprache, in der man mit allen Gästen kommunizieren konnte. So etwas war damals höchst ungewöhnlich; es gab kein zweites mathematisches Institut in Deutschland mit einem vergleichbaren Programm. Die japanischen Gäste, von denen einige mit Wilhelm Klingenberg eng zusammenarbeiteten, trugen auch zu seiner Ostasien-Begeisterung bei, die später durch seine Bücher über seine Tibet-Wanderungen und die Sammlung chinesischer Bronzen „Wilhelm und Christine Klingenberg" im Museum für Ostasiatische Kunst in Berlin-Dahlem einem großen Publikum bekannt wurde. Seine Tätigkeit in Bonn war allerdings nicht frei von Spannungen: „The number of students, guests and staff grew, and some of the intimate charm of a close-knit group went down the drain. Not without some pain and struggle, I finally accepted the change and concentrated my activities on my own

[4] M. Berger: Les variétés riemanniennes (1/4)-pincées, Annali della Scuola Normale Superiore di Pisa, Classe di Scienze (3) 14 (1960), 161–170

[5] Invent. Math. 84 (1986), 507–522

differential geometry group“ [1]. Es gereicht Klingenberg und den anderen Beteiligten zur Ehre, dass sie diese Spannungen nur unter sich austrugen; das kollegiale Verhältnis zu anderen Mitgliedern des Instituts und die kollegiale Atmosphäre am ganzen Institut blieben davon weitgehend verschont.

Während seines Aufenthaltes in Brasilien 1963 begann Klingenberg, sich mit einem neuen Thema zu beschäftigen: die Dynamik des geodätischen Flusses und insbesondere die Existenz geschlossener geodätischer Linien; Ausgangspunkt waren Arbeiten von A.S. Svarc und S.I. Al'ber über die Existenz von periodischen Lösungen in der Variationsrechnung, vgl. [6]. Einerseits kann der geodätische Fluss als ein spezielles Hamiltonsches Vektorfeld angesehen werden [9, 10], und es gehört wohl zu Klingenbergs Verdiensten, die Fruchtbarkeit dieser Verbindung zwischen Geometrie und Dynamik erkannt zu haben. Andererseits können geschlossene Geodätische als kritische Punkte des Energiefunktionals auf einem Raum geschlossener Wege betrachtet und mit Methoden der Morse-Theorie gefunden werden [8, 11, 12]. Während z. B. in John Milnors Buch über Morse-Theorie (1963) der Wegeraum bei beschränkter Weglänge durch geodätische Polygone endlich-dimensional approximiert wird, führt Klingenberg [6] die unendlich-dimensionale Hilbert-Mannigfaltigkeit der geschlossenen H^1-Kurven (geschlossene Wege mit quadratintegrierbarer Ableitung) ein, den maximalen Definitionsbereich des Energiefunktionals. Auf diesem Raum wirkt die Gruppe $O(2)$ durch Transformation der Parameter-Kreislinie, und es lassen sich Methoden der äquivarianten Morsetheorie auf das Energiefunktional auf diesem Raum anwenden. In seinem Lehrbuch über Riemannsche Geometrie [13] hat Klingenberg daher Riemannsche Mannigfaltigkeiten von Anfang an auf Hilberträumen modelliert. Seine Forschungen über geschlossene Geodätische gingen in zwei Richtungen: Existenz von kurzen und Anzahl von (beliebig langen) geschlossenen Geodätischen. Eins der schönsten von ihm angeregten Resultate stammt von Detlef Gromoll und Wolfgang Meyer: Wenn die Folge der Bettizahlen des freien Schleifenraums einer kompakten Mannigfaltigkeit M unbeschränkt ist, gibt es für jede Riemannsche Metrik auf M unendlich viele geschlossene Geodätische.[6] Klingenbergs Methoden wurden vielfach aufgegriffen und spielen sowohl in der Differentialgeometrie (Riemannsche, Finslersche und Lorentzsche Geometrie) als auch in der symplektischen Geometrie (Hamiltonsche Flüsse) eine Rolle.

Bei seiner Emeritierung [14] sagte er zu seinen Arbeiten über geschlossene Geodätische: „While I myself did not get the best results, I can pride myself with my students who got many important theorems in this long neglected and difficult field.“ In der Tat hat sich ein beträchtlicher Teil seiner Doktoranden mit diesem Fragenkomplex beschäftigt, viele von ihnen äußerst erfolgreich.

Nach seiner Emeritierung übernahm Wilhelm Klingenberg ab 1990 eine Gastprofessur an der Universität Leipzig und beteiligte sich an der Neugestaltung des mathematischen Fachbereiches nach der Wiedervereinigung. Am 4. Oktober 2001 verlieh ihm die Fakultät für Mathematik und Informatik der Universität Leipzig die Ehrendoktorwürde (s. Abb. 1)

[6] D. Gromoll, W. Meyer: Periodic geodesics on compact Riemannian manifolds, J. Diff. Geom. 3 (1969), 493–510.

in Würdigung seines richtungsweisenden wissenschaftlichen Werkes auf den Gebieten der Globalen Riemannschen Geometrie und der Theorie der Geschlossenen Geodätischen sowie in Anerkennung seines besonderen Engagements als Wissenschaftler und Hochschullehrer für die Mathematik in Leipzig.

Trotz dieser Tätigkeit blieb ihm genügend Zeit für zahlreiche Reisen und Wanderungen, besonders durch das Land, das für ihn „ein Stück Heimat“ [16] geworden war: Tibet. Einem seiner Reiseberichte [16] stellt er einige Verse aus dem „Cherubinischen Wandersmann“ von Angelus Silesius (1657) voran, die mir wie ein Motto seines ganzen bewegten Lebens erscheinen[7]:

Freund, so du etwas bist
so bleib doch ja nicht stehn:
Man muss von einem Licht
fort in das andere gehn.

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[7] Dieser Nachruf wäre nicht möglich gewesen ohne die Hilfe von vielen Kolleginnen und Kollegen, die mit Wilhelm Klingenberg verbunden waren. Dafür möchte ich mich sehr herzlich bedanken, ganz besonders bei Hans-Bert Rademacher, Hermann Karcher, Werner Ballmann, Victor Bangert, Gudlaugur Thorbergsson und Ernst Heintze.

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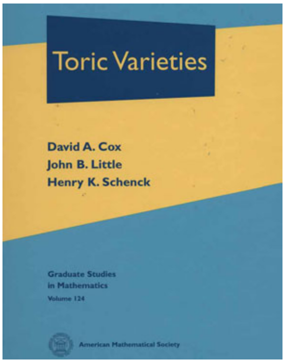
BOOK REVIEW

David A. Cox, John B. Little, Henry K. Schenck: "Toric Varieties"

American Mathematical Society, 2011, 841 pp.

Jürgen Hausen

Published online: 27 June 2012



1 Why Toric Varieties?

Toric varieties exist since the beginnings of algebraic geometry: the standard ambient spaces of classical algebraic geometry, the affine space $\mathbb{C}^n$ and the complex projective space $\mathbb{P}_n$ are toric varieties. Moreover, many of the very basic objects and constructions of algebraic geometry like the Hirzebruch surfaces, the Veronese (d-uple) embedding, the Segre embedding or blowing up the origin of $\mathbb{C}^n$ belong to toric geometry. However, it took a while until the common principle behind these objects and constructions got a name. The beginning was Demazure's work [2] on the Cremona group in the 1970s, where the concept of a toric variety and its close relations to combinatorics firstly showed up. It became clear quite rapidly that toric varieties allow a good and explicit understanding in terms of their describing combinatorial data, the so called fans. Meanwhile, toric varieties are established in algebraic geometry as a valuable testing class and they provide a useful, sufficiently flexible class of ambient spaces. The theory of toric varieties has many connections to other areas, like polyhedra, combinatorics, commutative algebra and symplectic geometry. Comparably important is the role of toric varieties in education: toric varieties allow an elementary example-oriented access to various advanced topics of algebraic geometry.

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2 Toric Geometry: A Micro Course

2.1 Algebraic Tori and Lattices

An n-dimensional *algebraic torus* is the n-fold product $\mathbb{T}^n = \mathbb{C}^* \times \cdots \times \mathbb{C}^*$ of the multiplicative group $\mathbb{C}^*$. Algebraic tori $\mathbb{T}^n$ and their (algebraic) homomorphisms $\mathbb{T}^n \to \mathbb{T}^m$ are in correspondence with lattices $\mathbb{Z}^n$ and their linear maps $\mathbb{Z}^n \to \mathbb{Z}^m$; here, *lattice* means free finitely generated Abelian group. For example, the *one-parameter subgroups* $\mathbb{C}^* \to \mathbb{T}^n$ are in bijection with the linear maps $\mathbb{Z} \to \mathbb{Z}^n$: such a linear map is determined by the image $v \in \mathbb{Z}^n$ of $1 \in \mathbb{Z}$ and the corresponding one-parameter subgroup can be written down explicitly as

$$\lambda_v\colon \mathbb{C}^* \to \mathbb{T}^n, \qquad t \mapsto \left(t^{v_1}, \dots, t^{v_n}\right).$$

The identification $v \leftrightarrow \lambda_v$ enables us to view the lattice $\mathbb{Z}^n$ associated to $\mathbb{T}^n$ as the *lattice of one-parameter subgroups of* $\mathbb{T}^n$. In general, a homomorphism $\varphi\colon \mathbb{T}^n \to \mathbb{T}^m$ of algebraic tori is nothing but a *monomial map* sending $t \in \mathbb{T}^n$ to $(t^{a_1}, \dots, t^{a_m}) \in \mathbb{T}^m$. The corresponding linear map $\mathbb{Z}^n \to \mathbb{Z}^m$ is given by the matrix A with the rows $a_1, \dots, a_m$ which is, by the way, just the Jacobian $J_\varphi(\mathbf{1})$ evaluated at the neutral element $\mathbf{1} \in \mathbb{T}^m$. What we have seen so far is, in more sophisticated words, a covariant equivalence of the category of algebraic tori with the category of lattices.

2.2 Affine Toric Varieties and Polyhedral Cones

An *affine variety* is the solution set $X \subseteq \mathbb{C}^r$ of a system of polynomial equations, i.e. the locus $V(f_1, \dots, f_s)$ of common zeros of some polynomials $f_1, \dots, f_s$ in r variables. Obviously, $\mathbb{C}^r = V(0)$ is of this type. Two classical examples are the following quadrics, showing up in algebraic geometry as the "affine cones" over the images of the simplest non-trivial Veronese and Segre embeddings:

$$X_2 := V\left(z_1z_2 - z_3^2\right) \subseteq \mathbb{C}^3, \qquad X_3 := V(z_1z_2 - z_3z_4) \subseteq \mathbb{C}^4.$$

We say that an affine variety $X \subseteq \mathbb{C}^r$ is *toric* if it allows an (algebraic) action of a torus $\mathbb{T}^n$ such that we can identify $\mathbb{T}^n$ via the orbit map with some open dense orbit $\mathbb{T}^n \cdot x_0 \subseteq X$. For example, the two Veronese/Segre quadrics just mentioned are toric:

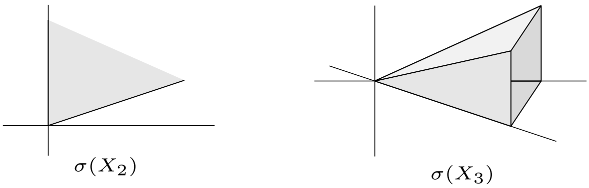
$$\mathbb{T}^2 \text{ acts on } X_2 \text{ by } t \cdot z = \left(t_1z_1, t_1^{-1}t_2^2z_2, t_2z_3\right),$$
$$\mathbb{T}^3 \text{ acts on } X_3 \text{ by } t \cdot z = \left(t_1z_1, t_2z_2, t_3z_3, t_1t_2t_3^{-1}z_4\right).$$

Here we find the acting tori $\mathbb{T}^2$ and $\mathbb{T}^3$ embedded into X_2 and X_3 as the orbits through the points $x_2 = (1, 1)$ and $x_3 = (1, 1, 1)$ respectively. Observe that both examples are defined by *binomial equations*—this is in fact a general feature: affine toric varieties are solution sets of systems of binomial equations.

The bridge from affine toric varieties to convex polyhedral cones is given by one-parameter subgroups. Let us look first at the example X_2. Take the one-parameter subgroups $\lambda_v\colon \mathbb{C}^* \to \mathbb{T}^2$, $t \mapsto (t^{v_1}, t^{v_2})$ and ask for those admitting a limit in the sense that $(t^{v_1}, t^{v_2}) = \lambda_v(t) \cdot x_2$ converges when t goes to zero. We obtain

$$\lim_{t\to 0} \lambda_v(t) \cdot x_2 \text{ exists} \quad \Longleftrightarrow \quad v_1 \geq 0,\ 2v_2 - v_1 \geq 0.$$

The linear inequalities on the right hand side describe a (rational convex) polyhedral cone $\sigma(X_2) \subseteq \mathbb{Q}^2$. The assignment $X \mapsto \sigma(X)$ works without changes in general, and we call $\sigma(X) \subseteq \mathbb{Q}^n$ the *cone of convergent one-parameter subgroups*. As one wants to keep the lattice $\mathbb{Z}^n$ in mind, one also refers to $\sigma(X)$ as a *lattice cone*.



Using a little algebraic geometry, one can almost reconstruct X from its cone $\sigma \subseteq \mathbb{Q}^n$ of convergent one parameter subgroups: take the dual cone $\sigma^\vee \subseteq \mathbb{Q}^n$, consider its additive monoid of integral points $S := \sigma^\vee \cap \mathbb{Z}^n$ and pass to the monoid algebra $\mathbb{C}[S]$. Then the spectrum $X(\sigma) := \operatorname{Spec} \mathbb{C}[S]$ is the *normalization* of X. If we require X to be normal, i.e. have not too bad singularities, then $X = X(\sigma)$ holds.

As before let us formulate a sophisticated summary for the observations just made: the assignments $X \mapsto \sigma(X)$ and $\sigma \mapsto X(\sigma)$ define essentially inverse covariant functors between the categories of normal affine toric varieties and polyhedral lattice cones. This correspondence enables toric geometers, for example, to read off immediately properties of the singularities of the Veronese/Segre quadrics X_2 and X_3 from the shape of their cones $\sigma(X_2)$ and $\sigma(X_3)$.

2.3 Toric Varieties and Fans

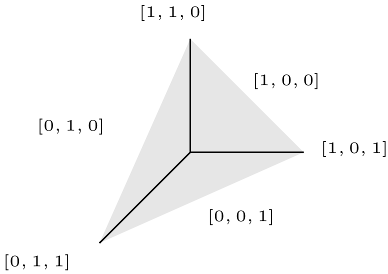
An *algebraic variety* is a space obtained by gluing (finitely many) affine varieties. A basic non-affine example is the projective plane $\mathbb{P}_2$. The points of $\mathbb{P}_2$ are the lines in $\mathbb{C}^3$; in this context one denotes the line through the origin 0 and a point $0 \neq z \in \mathbb{C}^3$ by $[z]$. The projective plane $\mathbb{P}_2$ comes with a natural action of the torus $\mathbb{T}^2$, given by

$$t \cdot [z] := [z_0, t_1 z_1, t_2 z_2].$$

Observe that the torus $\mathbb{T}^2$ is embedded into $\mathbb{P}_2$ as the open dense orbit $\mathbb{T}^2 \cdot [1, 1, 1]$. Thus $\mathbb{P}_2$ is an example of a *toric variety*. Let us take a look at one parameter subgroups $\lambda_v\colon \mathbb{C} \to \mathbb{T}^2$. The possible limits of $\lambda_v(t) \cdot [1, 1, 1]$ for $t \to 0$ are of the form

$$\lim_{t\to 0} \left[1, t^{v_1}, t^{v_2}\right] = [\varepsilon_1, \varepsilon_2, \varepsilon_3], \quad \text{where } \varepsilon_i \in \{0, 1\} \text{ but not all zero}$$

and, of course, depend on v. For example $v_1, v_2 > 0$ gives the limit $[1, 0, 0]$, or $v_1 = v_2 < 0$ gives the limit $[0, 1, 1]$. Drawing all these possibilities into a plane figure, we arrive at the following.



This is the *lattice fan* Σ describing the toric variety $\mathbb{P}_2$; as we see, Σ is a (finite) collection of lattice cones fitting nicely together. We can directly recover information on $\mathbb{P}_2$ from Σ. For example, the three big lattice cones stand for the three usual affine charts of $\mathbb{P}_2$, each of them isomorphic to $\mathbb{C}^2$. In this sense, the fan manages the gluing procedure and relieves us from—the sometimes cumbersome—taking care about the transition maps.

The picture just observed extends to the general case. If X is any toric variety, i.e. comes with a $\mathbb{T}^n$-action embedding $\mathbb{T}^n$ as an open dense orbit $\mathbb{T}^n \cdot x_0$, then a basic result on torus actions ensures that we obtain only finitely many points $\lim_{t \to 0} \lambda_v(t) \cdot x_0$ when varying v. If X is normal, then grabbing together the v with common limits gives rise to a collection of polyhedral cones forming a lattice fan $\Sigma(X)$ which now lives in $\mathbb{Q}^n$, the rational vector space associated to the lattice $\mathbb{Z}^n$ of one parameter subgroups of $\mathbb{T}^n$. From the differential point of view, the fan $\Sigma(X)$ is sort of a linear approximation of X: its cones store the tangent vectors of the one-parameter subgroups sharing a given limit.

As in the affine case, we can also go the other way round. Given a fan Σ we obtain for each of its cones σ a normal affine toric variety $X(\sigma)$. The compatibility of the cones $\sigma \in \Sigma$ ensures that we may glue the $X(\sigma)$ together to a variety. Again, if Σ was the fan of convergent one parameter subgroups of a normal toric variety X, then this procedure reconstructs X.

The routine sophisticated summary this time says that the assignments $X \mapsto \Sigma(X)$ and $\Sigma \mapsto X(\Sigma)$ define covariant essentially inverse functors between the categories of normal toric varieties and lattice fans. Toric geometry bases on this observation and now takes up its job: the big dictionary between algebraic geometry and combinatorics.

3 Cox, Little, Schenck: Toric Varieties

The attractive interplay between algebraic geometry and combinatorics, the rich connections to other areas as well as the new, elementary access to advanced questions in algebraic geometry made toric geometry popular. The number of textbooks, however, does not entirely reflect the recent parts of this development. The great classical texts by Danilov [1], Oda [4] and Fulton [3] provide an excellent base for the researcher. The modern book by Cox, Little, Schenck requires by far less background on algebraic geometry and thus opens in addition the subject to a greater readership.

The first part of the book starts with a reader-friendly, thorough and detailed introduction to the basic constructions of toric geometry. The toric versions of fundamental topics of algebraic geometry follow: divisors and line bundles, projective and proper morphisms, canonical divisors and sheaf cohomology. Also among this, we find a chapter on (multi-)homogeneous coordinates. Meanwhile a standard instrument in toric geometry, homogeneous coordinates currently also serve to export toric techniques to larger classes of varieties.

The second part of the book starts with toric surfaces and the beautiful links to other subjects as continued fractions, the McKay correspondence and the appearance of the number twelve in counting integral points of lattice polygons. A chapter on toric resolutions of (e.g.) singularities follows; this is another topic where toric geometry plays an important role in the general context by providing ideas as well as ambient constructions. The topology of toric varieties fills a chapter, starting with basic subjects like the fundamental group and ending with the Chow ring and intersection cohomology. A subsequent advanced chapter treats the toric Hirzebruch-Riemann-Roch Theorem with its famous applications to convex polytopes. The last two chapters concern toric geometric invariant theory and its link to birational geometry; also here, the toric case perfectly illustrates how combinatorial principles show up in this context.

Besides presenting a very convincing choice of topics in a nice, reader-friendly style, the authors successfully manage to combine the advantages of an introductory text with those of a rich source for experts. Moreover, the book offers an enormous amount of very nice illustrative examples and highest quality exercises. Altogether, the book is highly recommendable to everybody: to those who want to get in touch with this beautiful area of Mathematics as well as to those who are already enthusiastic about it.

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DOI 10.1365/s13291-012-0044-0

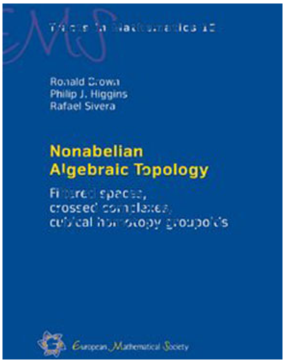
BOOK REVIEW

Ronald Brown, Philip J. Higgins, Rafael Sivera: "Nonabelian Algebraic Topology"

EMS, 2011, 703 pp.

Johannes Huebschmann

Published online: 27 June 2012



A result describing the fundamental group of a union of two connected subcomplexes of a simplicial complex goes back at least to Seifert [14], in the case when the intersection is connected. This is an early result on the fundamental group of the union of two spaces. For Seifert, this result was a technical tool for describing the fundamental groups of 3-manifolds he constructed in various ways, in particular in terms of a Heegard decomposition. In modern language, Seifert's result yields the fundamental group under discussion in terms of a pushout diagram of groups. Given an algebraic curve in the complex projective plane, using purely algebro-geometric methods, Zariski wrote down a presentation of the fundamental group of the complement and suggested to van Kampen to confirm the correctness of the presentation by purely topological methods, which he did in [17]. Thereafter van Kampen established a general theorem on the fundamental group of certain pathwise connected topological spaces [16]. This result underlies the contents of his previous paper; it is essentially the same as that established by Seifert for simplicial complexes.

The problem with the connectivity assumption of the intersection prevented the use of the theorem for deducing the result that the fundamental group of the circle is free cyclic. In [1], R. Brown could then overcome this obstacle by generalizing the statement of the theorem from the fundamental group on one base point to the

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fundamental groupoid on a system of base points chosen freely according to a given geometric situation, a key observation being that the notion of pushout diagram makes perfect sense for groupoids. The fundamental group of the circle drops out almost immediately.

In the 1920s, the definition of groupoid arose from Brandt's attempts to extend from the binary to the quaternary case Gauss's work on composition laws of quadratic forms. Groupoids appear in Reidemeister's book on topology [12] for handling the change of generators of the (combinatorially defined) fundamental group of a closed surface induced by the change of normal form of the surface, and for handling isomorphisms of a family of structures.

The book under review gives a newly organised exposition of work done over many years (since 1965), mainly of the first two authors. This subsumes some traditional algebraic topology and includes generalizations to higher dimensions of purely topological theorems of the type described above; such theorems are referred to in the book as *Seifert-Van Kampen theorems*.

The book deals with its topics in a thorough yet readable manner. It also touches on various foundational issues related to the perception and implementation of geometrical ideas, one such issue being the successful usage of the groupoid concept, as opposed to the mere notion of abstract group that is considered so central a concept of mathematics.

How do we encounter situations in mathematical nature that call for generalization of the Seifert-van Kampen theorem, what does such a generalization possibly look like and what might it signify?

Apart from the above situation where the usage of groupoids yields the solution, here are a few other such situations: Trying to unveil the structure of a second relative homotopy group, Whitehead isolated the notion of crossed module and in particular that of free crossed module [18, 19]. This structure was developed independently by Reidemeister and his student Peiffer [11]—the manuscript was submitted for publication in June 1944. Reidemeister and Peiffer explored *identities among relations* of a group presentation and thereby discovered free crossed modules. One of their aims was to develop normal forms for 3-manifolds. In a personal letter sent to me from R. Peiffer in the late 1970s I learnt that Reidemeister, as one of the founders of knot theory, was well aware of the apparent relations between the identities for crossed modules and those for knots and links. Crossed modules are typically nonabelian. For intelligibility, we recall that a crossed module $\partial\colon C \to G$ consists of two G-groups C and G where G is considered as a G-group via conjugation and a morphism ∂ of G-groups, subject to the rule

$$aba^{-1} = {}^{\partial a}b \in C, \quad a, b \in C. \tag{1}$$

Any element of C of the kind $aba^{-1}({}^{\partial a}b)^{-1}$ or the inverse of such an element is referred to in the literature as a *Peiffer element*.

Structural insight into the Cayley graph of a presentation of a group and its cousin, the geometric realization of the presentation, can be obtained in terms of the associated free crossed module. This yields, e.g., a description of the structure of the normal closure N of the relators R in the free group F on the generators as an F-group, relative to conjugation: The non-trivial *identities among relations*, that is, the identities modulo *Peiffer identities* (identities of the kind (1) above, independently of the particular presentation), yield the “correct” relations for N as an F-group generated by R. There are no non-trivial identities in this sense if and only if the second absolute homotopy group of the geometric realization is zero and hence N the free crossed F-module generated by R. A search in the MR database exhibits at present more than 50 papers that deal with “Peiffer commutators”, “Peiffer elements” and the like.

The method of diagrams has become a standard tool in combinatorial group theory to explore a Cayley graph; an exposition of this theory, with a special emphasis on identities among relations, can be found in [2]. This concept goes back at least to [15]; a version thereof appears in [11] under the name *Randwegaggregat*. Such a diagram represents an element of the associated free crossed module in a geometric manner.

Schreier’s approach to the extension problem for, in general, nonabelian groups, published in 1926, was reworked, clarified and completely settled by Eilenberg and Mac Lane [5]; indeed that extension problem was among the impetuses to the development of group cohomology. A key observation of Eilenberg and Mac Lane was to the effect that, given a discrete group G and a G-module A, the elements of $\mathrm{H}^3(G, A)$ correspond to classes of “abstract kernels”. Quoting from the historical note by S. Mac Lane, appended to [7], “Eilenberg, Mac Lane and Whitehead all knew that the elements of $\mathrm{H}^3(G, A)$ were closely connected with Whitehead’s ‘crossed modules’, but they missed the exact theorem, that there is a natural bijection from $\mathrm{H}^3(G, A)$ to equivalence classes of four term crossed sequences starting at A and ending at G”. The history of what Mac Lane refers to as an “exact theorem”, isolated only in the late 1970s, is described in that note in detail.

In [3], Brown and Higgins proved a Seifert-van Kampen theorem in dimension 2. Roughly speaking, the theorem says that, given a pair (X, A) of spaces and, furthermore, subspaces of X whose interiors cover X, under suitable connectivity assumptions, the resulting square of crossed modules is a pushout diagram. This implies Whitehead’s result on free crossed modules as well as the corresponding result of Reidemeister-Peiffer. The proof uses generalized groupoid techniques (double groupoids etc.) in an essential way. Also the proof does not assume the existence of pushouts of crossed modules; instead it verifies directly the required universal property for this case, so that the requisite pushout exists. This is an instance of what was meant above by “various foundational issues”.

The idea of szyzygy, applied to crossed modules, leads in an obvious manner to the more general notion of crossed complex. The classification problem of homotopy n-types was raised by J.H.C. Whitehead [19]. Crossed complexes were used implicitly by Eilenberg-Mac Lane to determine the n-type (old terminology: $(n+1)$-type) of an n-complex with non-trivial fundamental group π_1 and higher homotopy groups π_j zero for $2 \leq j < n$ (empty assumption when $n = 2$) [6]; the requisite additional invariant developed by Eilenberg-Mac Lane is the k-invariant in $\mathrm{H}^{n+1}(\pi_1, \pi_{n+1})$, identified in 1951 by Postnikov as the first of an entire family of invariants which, together with the homotopy groups, completely characterize the homotopy type of a CW complex. Crossed complexes yield an interpretation of group cohomology in arbitrary dimension; again the history thereof is described in Mac Lane’s historical note in detail. In this interpretation, it is the crossed complex itself which represents the k-invariant [8].

We already pointed out that the successful cure to what might be considered an anomaly prompted the development of what the book is about: a systematic approach from the abstract topology point of view whose aim is to generalize to all dimensions the notion of fundamental group, including relaxing the connectivity assumption in dimension zero. The generalization proceeds in various ways and involves, among other items, filtered spaces, crossed complexes, ω-groupoids, cubical sets, etc. The idea of a filtered space arises somewhat naturally out of generalizing the idea of fundamental groupoid on a system of base points chosen freely according to a given geometric situation. A filtered space can be seen as an instance of a structured space in the sense of Sect. 5 of Grothendieck's *Esquisse d'un programme*, published in [13]. The idea of structured space was developed further in the work of R. Brown with Loday which involves n-cubes of spaces [4]; this links with classical work in homotopy theory on the homotopy groups of n-ads.

In Part I (Chapters 1–6), the book covers all the items mentioned so far (except the algebro-geometric origin of van Kampen's result). In particular, the 2-dimensional Seifert-van Kampen theorem is given as Theorem 2.3.1. Part II of the book (Chapters 7–12) introduces and explores a higher homotopy Seifert-van Kampen theorem. This theorem includes the 1- and 2-dimensional theorems so far explained and, in a sense, generalizes them to arbitrary dimension. The method of proof is analogous to the 2-dimensional theorem but technically more complicated and deferred to Part III (Chapters 13–16).

The higher homotopy Seifert-van Kampen theorem (Theorem 8.1.5, p. 262) is phrased as the statement that a certain diagram is the coequalizer in the category of crossed complexes. This theorem gives a mode of calculation of the fundamental crossed complex functor Π from filtered topological spaces to crossed complexes. This functor is defined homotopically, that is, in terms of suitably defined homotopy classes of certain maps. A consequence of the definition is that Π preserves coproducts; this is one of the advantages of the groupoid approach. More subtle is the application to gluing spaces, and the authors approach this concept, as for the 1- and 2-dimensional version of the theorem, through the notion of coequalizer; here again, a connectivity condition in all dimensions is needed.

The authors then show how the higher homotopy Seifert-van Kampen theorem gives some computations of homotopy groups of pairs of spaces and, as a consequence, some classical results such as the suspension theorem, the Brouwer degree theorem, and the relative Hurewicz theorem. These are basic theorems in homotopy theory but are obtained here without homology theory machinery. At the present stage, the authors' concern with foundational issues shows up clearly: A major aspect of the book is to tie in the fundamental group and higher homotopy groups without passing to the universal covering space, as is done in the more conventional approach. It is also unclear how to obtain the 1- and 2-dimensional versions of the Seifert-van Kampen theorem by covering space methods. The applications culminate in a homotopy classification theorem (Theorem 11.4.19, p. 391). To describe it, let X denote a CW complex, C a crossed complex, ΠX_* the crossed complex associated with X, and BC the cubical classifying space C. The classification theorem employs the cubical theory in an essential way to give, in terms of crossed complexes, a description of the weak homotopy type of the mapping space $(BC)^X$. This description, in turn, entails a bijection from the homotopy classes $[\Pi X_*, C]$ of crossed complex morphisms to the homotopy classes $[X, BC]$ of ordinary maps of spaces. Thereafter applications to nonabelian cohomology, using fibrations, i.e. model category kind of structures and arguments, are worked out.

In Part III the topics covered, including ω-groupoids, cubical sets, connections and compositions in cubical sets, etc., are motivated by the need to develop the requisite technology so that the authors can eventually craft a proof of the higher homotopy Seifert-van Kampen theorem and develop the key monoidal closed structures required for the homotopy classification theorem. Working through this technology may be seen as a challenge looming over the reader. But, amice lector, if you find this rather special, keep in mind that the authors need, e.g., an algebra of cubes which enables them to handle composition in all directions, and the underlying nonabelian algebra is still in the process of intense development, comparable, perhaps, to the early stage of what are now standard algebraic topology or algebraic geometry notions.

Groupoids nowadays play a significant role in various areas distinct from topology, e.g., in the Lie theory of symplectic groupoids and in the related issue of quantization, in differential Galois theory as what is known as Malgrange groupoid, in algebraic geometry, see below, etc. Crossed modules and notions related to them arise under various circumstances where, at first, one would not expect to see them. One such situation is the positive answer to a question posed by Atiyah whether there is a finite-dimensional construction of the Chern-Simons function in dimension 3; the answer in [9] involves identities among relations in an essential way. There is an intimate relation between braids and crossed modules; this relation is already present in Whitehead's original proof of the freeness of the crossed module discussed earlier. Recently I have reworked and somewhat extended that relationship [10]. Crossed modules show up in the theory of gerbes and in string theory. There are so many names attached to this activity that I prefer not to mention any of them. What is known as the Teichmüller cocycle in Galois theory admits its natural interpretation in terms of crossed modules. Lie crossed modules are nowadays studied in differential geometry.

In classical algebraic geometry, points are characterized in terms of functions, a point being an algebra map from a coordinate ring to the base field. In topology points belong to a space which is usually a continuum, and ordinary commutative algebra, so well suited to algebraic geometry, is not strong enough to recover the nonabelian phenomena that are attacked in the book. Nonabelian phenomena play as well a major role in algebraic geometry (Brauer-Severi varieties, Teichmüller groupoid, etc., to list a few instances). It may well be that, in the future, the ideas presented in the book contribute to some of the many open questions in these areas. Van Kampen is well known in algebraic geometry circles for his result on the fundamental group of the complement of a plane curve but hardly any algebraic geometer is aware of the more general topological result that underlies it and was proved two years earlier by Seifert. Likewise hardly any topologist knows that a special case of the Seifert-van Kampen theorem established, at the time, an important result in algebraic geometry. In this sense, the book will, perhaps, also contribute to the unity of mathematics.

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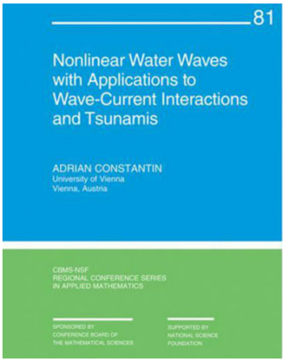
BOOK REVIEW

Adrian Constantin: "Nonlinear Water Waves with Applications to Wave-Current Interactions and Tsunamis"

SIAM, 2011, 331 pp.

Joachim Escher

Online publiziert: 26. Juni 2012



Die mathematische Strömungsmechanik gehört zweifelsohne zu den wichtigsten Triebfedern der weit gefächerten Entwicklung der neueren Angewandten Mathematik. Zentrale Gebiete der Analysis, wie die Theorie der partiellen Differentialgleichungen, die Funktionalanalysis, die harmonische Analysis, die Theorie der dynamischen Systeme, die topologische Abbildungsgradtheorie und die unendlich-dimensionale Differentialgeometrie wurden und werden durch das Wechselspiel zwischen Theorie und strömungsmechanischen Anwendungen befruchtet. In ganz besonderem Maße gilt dies für Untersuchungen zum klassischen Wasserwellenproblem. Das vorliegende Buch von Adrian Constantin bietet eine Einführung in aktuelle Entwicklungen auf dem Gebiet der Analysis zur Theorie nichtlinearer Wasserwellen. Es ist die ausgearbeitete Version der Vortragsreihe, die der Autor anläßlich der NSF-CBMS Konferenz *Nonlinear Water Waves with Applications to Wave-Current Interactions and Tsunamis* an der Texas-Pan America University, Edinburgh, Texas, im Mai 2010 gehalten hat.

Dem Autor gelingt es in beeindruckender Weise, wichtige neuere Entwicklungen auf diesem Gebiet in umfassender Weise aufzuarbeiten. Dabei scheut er sich nicht, sowohl in der Modellierung als auch in der mathematischen Analyse soweit auszuholen, um ein gehaltvolles und in sich stimmiges Werk schaffen zu können.

Zum Inhalt: Das Buch ist neben der Einleitung in sechs Kapitel gegliedert. Das zweite Kapitel führt in die wesentliche strömungsmechanische Modellierung ein. Es

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werden zunächst die Eulerschen Gleichungen samt dynamischer und kinematischer Randbedingungen hergeleitet. Mit besonders viel Sorgfalt behandelt der Autor danach die Vortizität und die Helmholtzschen Sätze über die Wirbelverteilung in einer Strömung. Es werden hier die Grundlagen gelegt, die es dem Autor im weiteren Verlauf der Abhandlung erlauben, die in der Literatur oft verwendete Annahme der Wirbelfreiheit zu diskutieren und zu beurteilen.

Im darauf folgenden Kapitel werden zentrale Resultate aus aktuellen Studien zu zweidimensionalen symmetrischen Wanderwellen (engl. *travelling waves*) mit allgemeiner Wirbelverteilung besprochen. Es wird zunächst eine Stromfunktion eingeführt und dann das klassische Wasserwellenproblem als nichtlineares freies Randwertproblem für die Stromfunktion formuliert. Mit Hilfe lokaler und globaler Bifurkationstheorie wird die Existenz von Lösungen dieses Problems bewiesen, wobei besonderes Augenmerk darauf gelegt wird, die Amplituden der Lösungen a priori nicht zu beschränken, sondern auch Lösungen mit großen Amplituden zu konstruieren. Dazu ist anzumerken, dass lineare Theorien zur Beschreibung von Wellen mit großen Amplituden eine zu weit gehende Vereinfachung darstellen, da die entsprechenden Ergebnisse weder einer experimentellen noch einer numerischen Validierung standhalten. Aus diesem Grund bezeichnet man die Beschreibung von Wellen mit großen Amplituden auch als *nichtlineare Wellentheorie*. Neben der Existenz von Wellen mit großen Amplituden werden auch qualitative a-priori–Eigenschaften solcher Wellen studiert. Dazu gehören Symmetrie- und Regularitätseigenschaften sowie die Druckverteilung unterhalb einer Welle. Im Appendix zu diesem Kapitel werden die zuvor verwendeten mathematischen Werkzeuge zusammengestellt: der lokale Bifurkationssatz von Crandall-Rabinowitz, der Leray-Schaudersche Abbildungsgrad und globale Bifurkation, analytische globale Bifurkation, Existenztheorie und Maximumsprinzipien für nichtlineare elliptische Randwertprobleme in zwei Raumdimensionen.

In Kapitel 4 werden Aspekte der Kinematik von Stokesschen Wellen beleuchtet. Mit Hilfe Lagrangescher Koordinaten werden qualitative Eigenschaften von Partikelbahnen untersucht. Auch hier legt der Autor großen Wert darauf, die Ergebnisse der klassischen linearen Theorie den neueren Erkenntnissen über nichtlineare Wellen gegenüberzustellen und einzuordnen. Im Appendix zu diesem Kapitel wird die Gerstensche Welle aus dem Jahr 1802 vorgestellt. Es handelt sich dabei um die bis heute einzige bekannte explizite Lösung eines Wellenzuges mit nicht-konstanter Oberfläche.

Im folgenden Kapitel werden Soliton-Lösungen des klassischen Wasserwellenproblems behandelt. Hierbei handelt es sich um zweidimensionale Wellen mit einem lokalisierten Profil, welche sich mit konstanter Geschwindigkeit und ohne Formänderung des Profils fortpflanzen. Im Text werden Partikelbahnen und Druckverteilung unterhalb eines Solitons untersucht. Im ausführlichen Appendix zu diesem fünften Kapitel stellt der Autor am Beispiel der Korteweg-de Vriesschen Gleichung (KdV) in kompetenter Weise die mannigfaltigen mathematischen Aspekte integrabler Systeme zusammen. Nach allgemeinen Betrachtungen zu Hamiltonschen Systemen wird die bi-Hamiltonsche Struktur der KdV-Gleichung dargelegt. Anschliessend werden die direkte und inverse Streutheorie für die KdV-Gleichung ausführlich erörtert. Schließlich ist wohlbekannt, dass Soliton-Lösungen der räumlich periodischen KdV-Gleichung durch die Weierstaßsche $\wp$-Funktion ausgedrückt werden können. Damit stehen Methoden der algebraischen Geometrie zur Verfügung, um die periodische KdV-Gleichung zu untersuchen. Ausführungen zu diesem Zugang beschließen das Kapitel 5.

Die KdV-Gleichung ist auf sehr großen Phasenräumen global wohlgestellt. Dies bedeutet, dass die Lösungen der KdV-Gleichung keine Singularitäten entwickeln, selbst wenn die Anfangswerte wenig räumliche Regularität aufweisen (z.B. nicht stetig sind) und keinen Kleinheitsbedingungen unterliegen. Andererseits erwartet man von einem Wasserwellenmodell, dass neben zeitlich global existierenden Wellen (z.B. Solitonen) auch Lösungen beobachtet werden können, die in endlicher Zeit eine Singularität in Form einer Wellenbrechung aufweisen. Vor diesem Hintergrund ist FRS (Fellow of the Royal Society) G.B. Withams Forderung zu verstehen, wenn er schreibt: *Although both breaking and peaking, as well as criteria for the occurrence of each, are without doubt contained in the equations of the exact potential theory, it is intriguing to know what kind of simpler mathematical equation could include all these phenomena.*[1] In Kapitel 6 widmet sich der Autor dem Phänomen der Wellenbrechung. Es wird die so genannte Johnson-Gleichung hergeleitet, die die KdV- und die Camassa-Holm-Gleichung verallgemeinert, und es wird gezeigt, dass diese Gleichung klassische Lösungen besitzt, die in endlicher Zeit Singularitäten in Form von Wellenbrechungen hervorbringen.

Im letzten Kapitel werden Aspekte der Modellierung von Tsunami-Wellen besprochen. Dazu unterscheidet der Autor die drei Bereiche Wellenentstehung, Wellenausbreitung auf offener See und Wellenbrechung in Küstenbereichen. Das Hauptaugenmerk wird dann auf die Wellenausbreitung gelegt. Es wird verdeutlicht, dass Tsunamis auf offener See (über einem als nahezu flach angenommenen Meeresboden) mit Hilfe linearer Flachwasserwellentheorie beschrieben werden können. Außerdem wird anhand der verfügbaren Messdaten argumentiert, dass weder der Tsunami von 2004 im Indischen Ozean noch der Tsunami von 1960 im Pazifischen Ozean vor Chile auf offener See als Solitonen betrachtet werden können: die Messdaten schließen eine dazu notwendige Balance zwischen Dispersion und nichtlinearer Solitoneninteraktion aus.

Insgesamt hat der Autor ein gehaltvolles Werk vorgelegt und den aktuellen Stand der Theorie nichtlinearer Wasserwellen und ihre Anwendungen eindrucksvoll dokumentiert. Meines Erachtens stellt das Buch somit eine unverzichtbare Quelle dar, wenn man sich ernsthaft mit diesem Bereich der mathematischen Strömungsmechanik befassen will.

[1] G.B. Witham, *Linear and Nonlinear Waves*, J. Wiley and Sons, New York, 1980.