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PREFACE

Preface Issue 4-2012

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"*Foundations of Discrete Optimization: in transition from linear to non-linear models and methods*." This is the title of the survey article by Jesús A. De Loera, Raymond Hemmecke, and Matthias Köppe in which the authors summarise a number of recent developments in discrete optimisation. These range from improving, extending, and discussing classical algorithms from linear optimisation to nonlinear transportation problems. A common feature of most real life optimisation problems is the huge number of variables and hence, the need for efficient, i.e. fast, algorithms. For a long time there has been an intimate relation between optimisation and convex geometry and in this respect it would be particularly interesting to find good bounds for the so called diameter of polytopes. The related prominent Hirsch conjecture was discussed in Issue 2-2010 of the Jahresbericht by Edward Kim and Francisco Santos. Although this particular conjecture was disproved by the latter author while their survey article was in press, the struggle to find good diameter bounds remains. Jesús De Loera, Raymond Hemmecke, and Matthias Köppe explain further how a more refined modelling changes linear problems into nonlinear ones and how algebraic, geometric, and topological techniques enter discrete optimisation in order to tackle the new challenges. For more detailed information and proofs the authors refer to their corresponding recent monograph.

Anybody, irrespective of the individual fields of interest, who has ever visited the Oberwolfach institute will know the name of Horst Tietz. The reason is that a former student of his established a trust named after his advisor to support the institute. Georg Schumacher reviews not only the mathematical and academic achievements of Horst Tietz, who died on 28th January 2012, but also focusses on some details

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from his life which are really worth thinking about. Due to his Jewish roots Horst Tietz was deported in 1943 first to the concentration camp Breitenau and later to Buchenwald. It is somehow a miracle that Horst Tietz survived the horrible years of this most degrading captivity. In spite of these traumatic experiences he stayed in Germany, finished his studies, and started an academic career just after the end of the Nazi-regime.

Recently released books on p-adic Lie groups, Feynman motives, and modern classical homotopy theory are extensively discussed and reviewed.

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SURVEY ARTICLE

Foundations of Discrete Optimization: In Transition from Linear to Non-linear Models and Methods

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Abstract Optimization is a vibrant growing area of Applied Mathematics. Its many successful applications depend on efficient algorithms and this has pushed the development of theory and software. In recent years there has been a resurgence of interest to use "non-standard" techniques to estimate the complexity of computation and to guide algorithm design. New interactions with fields like algebraic geometry, representation theory, number theory, combinatorial topology, algebraic combinatorics, and convex analysis have contributed non-trivially to the foundations of computational optimization. In this expository survey we give three example areas of optimization where "algebraic-geometric thinking" has been successful. One key point is that these new tools are suitable for studying models that use non-linear constraints together with combinatorial conditions.

Keywords Algorithms · Non-linear mixed-integer optimization · Linear optimization · Algebraic techniques in optimization · Graver bases · Generating functions · Complexity of the simplex method

Mathematics Subject Classification 90CXX · 90C05 · 90C11 · 90C27 · 90C29 · 90C60

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1 Introduction

Recently Germany hosted the International Symposium on Mathematical Programming, the biggest event in mathematical optimization in the world. It is thus timely to acquaint DMV members with a tiny corner of the advances highlighted in that remarkable mathematical festival. This survey aims to introduce students and mathematicians to some recent developments in discrete optimization. Clearly, the choice of topics presented here is driven by the authors own research; but we wish to point out that there are many more interesting facets of mathematical optimization that cannot be covered in this short survey. For a more thorough overview on recent developments in discrete optimization we recommend the wonderful survey books [38, 48].

Discrete Optimization seeks the best answers for problems where the set of solutions is finite or at least enumerable. Given a finite set, each of whose elements has an assigned cost or price, one aims to find the cheapest or optimal element. This could mean the shortest path on a network, the optimal assignment of jobs in a company, or the best distribution of fire stations in a city. Take for example the situation of a company that builds laptops in n factories in Germany, each with certain supply power a_i. At the same time m cities in Germany have laptop demands, say b_j for city j. There is a cost $c_{i,j}$ for transporting a laptop from factory i to city j. What is the best assignment of transports in order to minimize the cost? Although it makes theoretical sense, even for moderate values of m, n, running through all possible assignments of factory-to-city transportation to find the best assignment is a very bad idea. Still such problems can be solved very efficiently using various techniques. This problem is called the *transportation problem*, and we will use it as an example below.

The integer linear transportation problem is a very special example of a discrete optimization problem. It can be shown that one can drop the integrality requirements and solve the corresponding *continuous* linear transportation problem. If that problem is solvable, there is always an optimal solution that is integral. However, variables whose restriction to integer values is essential naturally occur in optimization problems, especially, when on/off- or yes/no-decisions have to be modeled. For example, a *start-up cost* may be incurred only if a machine is actually used in a production process (and thus, the machine has to be started and thus spends some energy/time/costs). To model this type of costs, it is necessary to introduce a binary variable encoding whether the machine is used or not.

Often discrete optimization can be modeled as maximizing or minimizing the value of a real function under the condition that the solution vector must also satisfy some constraining functions where some or all the variables are *integer* valued. In that case we are dealing with a (very general) *mixed-integer programming problem.*

$$\begin{aligned} \max/\min \quad & f(\mathbf{x}) \\ \text{subject to} \quad & g_i(\mathbf{x}) \le 0, \quad i = 1, 2, \ldots, k, \\ & h_j(\mathbf{x}) = 0, \quad j = 1, 2, \ldots, m, \\ & \mathbf{x} \in \mathbb{R}^{n_1} \times \mathbb{Z}^{n_2}. \end{aligned} \tag{1}$$

For example, for the transportation problem we can use $x_{i,j}$ as a variable indicating the number of laptops factory i provides to city j. There are a few constraints to describe the transportation problem: $x_{i,j}$ can only take non-negative integer values, thus $x_{i,j} \geq 0$. Since factory i produces a_i laptops we have $\sum_{j=1}^{n} x_{i,j} = a_i$, for all $i = 1, \ldots, n$; and since city j needs b_j laptops we have $\sum_{i=1}^{n} x_{i,j} = b_j$, for all $j = 1, \ldots, m$. Finally the objective function to minimize is $\sum c_{i,j} x_{i,j}$. In this case only *linear* equations and inequalities are used, so this is an example of an *integer linear program*.

Discrete optimization has many applications to diverse areas such as bioinformatics, industrial engineering, management, operations planning, finances, etc. The mathematical foundations of the subject began to develop steadily at the end of the second world war with contributions of mathematicians like George Dantzig, Ralph Gomory, John von Neumann, Richard Karp, Egon Balas, Ray Fulkerson, Alan Hoffman, Harold Kuhn, Jack Edmonds, and other pioneers. Today, after fifty plus years of development [38], this topic still thrives with fascinating mathematical questions. In this review we wish to recount how the use of new algebraic, geometric, and topological techniques have brought new advances to the theory and help to better understand application models that use non-linear constraints.

The late 1980s and early 1990s saw the use of convex geometry and the geometry of numbers for solving some important problems, from the use of the ellipsoid method, to give an efficient polynomial-time algorithm for linear programming [30], to the use of lattice reduction and diophantine approximation techniques that have given efficient algorithms for totally unimodular problems [28]. It was shown in [29] that many discrete problems with good convex polyhedral characterizations had efficient algorithmic solutions. Another great reference summarizing many developments in discrete optimization is [57–59].

But in just the past twenty years, there have been new developments on the understanding of the structure of polyhedra, convex sets, and their lattice points that have produced new algorithmic ideas to solve discrete optimization problems. The new techniques, from algebraic geometry and commutative algebra, combinatorial topology, representation theory, discrete geometry, number theory, and algebraic combinatorics, come to add a new set of useful tools for discrete optimizers and have already proved very suitable for the solution of a number of hard problems, specially for attempts to deal with non-linear objective functions and constraints in discrete optimization.

This survey is an invitation to researchers and students in mathematics to explore this interdisciplinary topic. For this purpose we present three case studies that hopefully capture the energy of the field and will attract others to propose new ideas. The reader can read all or just a few of the case studies independently to get a feeling of some of the algebraic, geometric, and topological methods used in optimization. We omit proofs entirely but most of the missing details and much more information can be found in the recent book [16].

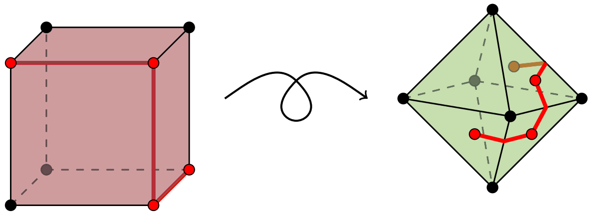
2 Case Study: Abstractions in the Theory of Linear Optimization

For years the most common version of the mixed integer programming problem involved only *linear* constraints. One typical strategy to solve or approximate solutions for general problems of the form (1) is to reduce them to subproblems where all constraints are linear and all variables are continuous. This is the so called *linear programming problem.* Linear programming is a workhorse of optimization, the set of feasible solutions for given linear constraints defines a convex polyhedron (see [56, 65]). One can already find useful new algebraic, geometric and topological insights in the theory of linear optimization and we begin by mentioning two recent ones.

Starting at an initial vertex the simplex method searches along the graph of the polytope, each time moving to a better-cost adjacent vertex using a shared 1-dimensional face or edge (the right edge to follow is selected according to a *pivoting rule*, which assures improvement). Although extremely powerful in practice, today many questions remain about the theoretical performance of the simplex method. The vertices and edges of the polyhedron define a graph, the one-skeleton of the polyhedron or *graph of the polyhedron*, which is composed of the zero- and one-dimensional faces of the feasible region (called *vertices* and *edges*). The distance between any pair of vertices is the length of the shortest path between that pair. The *diameter* of the graph of the polyhedron is simply the largest distance between any pair of nodes. Clearly the performance of the simplex method depends on the diameter. The primary question is whether there is a polynomial bound of the diameter in terms of the number of facets and the dimension?

The best general bounds known are exponential (see the wonderful Jahresbericht survey [45]), but for over fifty years the conjecture that the diameter is no more than the number of facets minus the dimension was the leading proposal for the true bound (there are indeed many examples that reach this bound). In 2010, a historic breakthrough was made by the Spanish mathematician Francisco Santos who constructed counterexamples for this conjecture [55]. Still, those counterexamples which are fully understood have again linear diameter; also in experiments and in many special cases we know linear bounds for the diameter (e.g., transportation problems). So, *could the correct diameter bound be linear?* Linear diameter for all polyhedra is certainly still a possibility but only very recently one avenue of attack was finally closed.

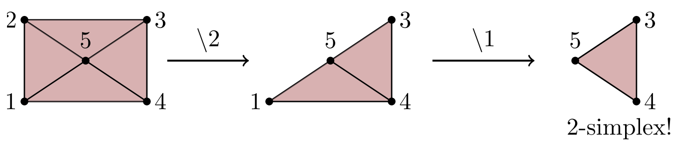
The *combinatorial-topological* approach to the diameter problem has a history (see e.g., [51]); Adler, Dantzig, and Murty [1, 2] and Kalai [39], Billera and Provan [9], and Klee and Kleinschmidt [42] to name a few. Recently the use of abstractions of the simplex method continues with the exciting paper of Eisenbrand, Hähnle, Razborov, and Rothvoss [26]. A key idea of Eisenbrand et al. is working with new topological abstractions called *base abstraction and connected layer families*. Eisenbrand et al. were also able to prove that their abstraction is a reasonable generalization because it satisfies some of the best known upper bounds on the diameter of polytopes. Larman proved in [46] that for a d-dimensional polytope P with n facets the diameter is no more than $2^{d-3}n$ (this bound was improved by Barnette [4]). This bound shows that in fixed dimension the diameter must be linear in the number of facets. The best general bound of $O(n^{1+\log d})$ was obtained by Kalai and Kleitman [43]. The authors of [26] proved that the Larman bound and the Kalai-Kleitman bounds hold again for their abstractions. But a great novelty in Eisenbrand et al.'s results is that there are abstraction graphs with diameter greater than $\Omega(n^2/\log n)$ (although they are far from being polyhedra). Thus, instead of thinking of polyhedra all these propose thinking of simplicial complexes. There is a direct translation of non-degenerate linear optimization problems (known to have the largest diameters anyway) to simplicial complexes by using the *polarity* operations (as illustrated in the following picture for a cube), where facets turn into vertices and vertices into facets.



Now, the distance between two facets of the simplicial complex, F_1, F_2, is the length s of the shortest simplicial path $F_1 = f_0, f_1, \ldots, f_s = F_2$. The diameter of a simplicial complex is the maximum over all distances between all pairs of facets. Scott Provan and Louis Billera [9] conceived a way to prove the linearity bound on the diameter which relies on decomposition properties of polyhedra and complexes. They introduced the notion of *weakly k-decomposable complex*. A d-dimensional simplicial complex Δ is weakly k-decomposable if:

1. All the maximal-dimension simplices are of the same dimension, and
2. either Δ is a d-simplex, or there exists a face τ of Δ (called a *shedding face*) such that $\dim(\tau) \leq k$ and $\Delta \setminus \tau$ is d-dimensional and weakly k-decomposable.

So one recursively "peels off" the simplicial complex using a sequence of faces so that finally we arrive at a (full-dimensional) simplex. In the next picture we show an example of a weakly 0-decomposable complex through a possible shedding order of vertices.



The reason this concept is so interesting for bounding diameters is the following theorem:

Theorem 2.1 (Billera, Provan, 1980) *If* Δ *is a weakly* k*-decomposable simplicial complex*, $0 \leq k \leq d$, *then*

$$\operatorname{diam}(\Delta) \leq 2 f_k(\Delta),$$

where $f_k(\Delta)$ *is the number of k-faces* Δ. *In the case of weakly* 0*-decomposable*, *we have the following* linear *bound* ($n = f_0(\Delta)$):

$$\mathrm{diam}(\Delta) \le 2 f_0(\Delta) = 2n.$$

Note that, if all simplicial polytopes are weakly 0-decomposable, then the diameter is linear, being no more than twice the number of facets of the polar simple polytope. One should also note that all simplicial d-dimensional polytopes are weakly d-decomposable because they are *shellable* (see e.g., [67] for an introduction). The question is then which simplicial polytopes are weakly 0-decomposable? More importantly, is there a fixed constant $k < d$ for which all simplicial polytopes are weakly k-decomposable? If this was true for $k = 0$ (weakly 0-decomposable), then the desired linear bound would be achieved.

Very recently these challenges have been settled. In [18] the authors construct the first ever examples of simplicial polytopes which are not weakly 0-decomposable disproving this method as an approach for a linear bound. For all $m \ge 2$, let Δ_{2m} be the simplicial polytope polar to the $2 \times (2m+1)$ transportation polytope $P(\mathbf{a}, \mathbf{b})$ with margins $\mathbf{a} = (2m+1, 2m+1)$ and $\mathbf{b} = (2, 2, \ldots, 2)$. Then Δ_{2m} is not weakly 0-decomposable. Shortly after, Hähnle, Klee, Pilaud have announced that the same family of complexes shows that there is no constant $0 < k < d$ such that every simplicial polytope is weakly k-decomposable.

Let us talk about another non-traditional analysis of a well-known method in linear optimization. Interior-point methods are also popular to solve linear optimization problems [54, 65]. Already in the early the 1980s, during the early excitement generated by Karmarkar's interior point methods, Bayer and Lagarias pioneered an algebraic analysis of the method [7, 8]. To explain this let us assume the linear program is given to us in the standard form $\mathrm{Maximize}_{\mathbf{x}\in\mathbb{R}^n} \mathbf{c}^\top \mathbf{x}$ s.t. $A\mathbf{x} = \mathbf{b}$ and $\mathbf{x} \ge \mathbf{0}$. The objective function is replaced by $\mathrm{Maximize}_{\mathbf{x}\in\mathbb{R}^n} f_\lambda(\mathbf{x})$ s.t. $A\mathbf{x} = \mathbf{b}$, where $\lambda \in \mathbb{R}_+$ and $f_\lambda(\mathbf{x}) := \mathbf{c}^\top \mathbf{x} + \lambda \sum_{i=1}^n \log|x_i|$. The maximum of the concave function f_λ is attained by a unique point $\mathbf{x}^*(\lambda)$ in the open polytope $\{\mathbf{x} \in (\mathbb{R}_{>0})^n : A\mathbf{x} = \mathbf{b}\}$. As we let λ change from ∞ to 0 we trace a curve, the *central path*, that traces a piece of a curve ending at the optimal solution (at infinity starts at the *analytic center*).

Bayer and Lagarias observed that the central path is actually a portion of an algebraic curve. They raised the question of finding explicit equations for the real algebraic curve containing the central path. In [19] the authors computed these equations and the degree of the algebraic curve. Surprisingly these invariants are expressed in terms of the matroid of the input matrix A. The relation with matroids and the simplex method is well documented (see, e.g., [6]), but this seems to be another evidence of the strong relation between linear programming and matroids. What happens is that (under some natural non-degeneracy assumption) the equations of the central curve can be read off from the circuits of the *linear matroid* associated to the matrix $\binom{A}{\mathbf{c}}$. Given a matrix B, whose columns are the vectors $\mathbf{x}^i$, $i = 1, \ldots, n$, the *circuits* of the matroid of B are the subsets of the index set that label subsets of the columns that are minimally linear dependent (i.e., removing one vector gives a linear independent set).

In practical computations, interior point methods follow only a piecewise-linear approximation to the central path. One way to estimate the number of Newton steps needed to reach the optimal solution is to bound the *total curvature* of the exact central path. The intuition is that curves with small curvature are easier to approximate with fewer line segments. This idea has been investigated by various authors (see, e.g., [50, 62, 66, 68]), and has yielded interesting results. In earlier work [19, 20] the authors obtained bounds for the total curvature in terms of the degree of the Gauss maps of the (algebraic) central curve. The paper [19] also improves on the given bound for the curvature and once again the bound is read from the matroid, this time from the broken circuit complex of the same matroid.

For practical applications the more relevant quantity is not the total curvature of the entire curve (a real algebraic curve extends beyond the LP feasible polyhedron) but rather the curvature in a specific feasible region. This has been investigated by A. Deza, T. Terlaky and Yu. Zinchenko in a series of papers [22–24]. They conjectured that the curvature of a polytope, defined as the largest possible total curvature of the associated central path with respect to the various cost vectors, is no more than $2\pi m$, where m is the number of facets of the polytope. They name their conjecture the *continuous Hirsch conjecture* to suggest the similarity with the well-known problem for the simplex method. Although the average value for the curvature for bounded cells is known to be linear, we still do not have a polynomial bound for the total curvature in a single cell.

3 Case Study: The Evolution of Integer Optimization in Fixed Dimension

The classical *integer linear optimization problem* (ILP) asks to find the solution of the following task:

Given $A \in \mathbb{Z}^{m\times n}$, $\mathbf{b} \in \mathbb{Z}^m$, and $\mathbf{c} \in \mathbb{Z}^n$, find $\mathbf{x} \in \mathbb{Z}^n$ that solves

$$\min\left\{\mathbf{c}^\top \mathbf{x} : A\mathbf{x} \leq \mathbf{b}, \mathbf{x} \in \mathbb{Z}^n\right\} \tag{ILP}$$

or report INFEASIBLE or UNBOUNDED.

It is well-known that this problem is in general NP-hard, but when the dimension is fixed, the integer linear optimization problem can be solved efficiently using Hendrik Lenstra's famous algorithm (see [47] and the excellent recent survey by F. Eisenbrand in [38]).

We want to stress that several geometric operations were necessary for this historic result. It was needed to approximate the shortest vector problem in a lattice $\mathcal{L}$, originally done using the LLL-reduced basis of $\mathcal{L}$ [49]. Lenstra's algorithm also relied on the possibility of decomposing the problem into a "small" family of lattice hyperplanes that pass through the feasible polyhedral region P. Khinchin's flatness theorem [41] guarantees there is a primitive lattice vector direction $\mathbf{d} \in \mathbb{Z}^n$ such that there are "very few" lattice hyperplanes of the form $\mathbf{d}^\top \mathbf{x} = \gamma$ (with $\gamma \in \mathbb{Z}$) that have a nonempty intersection with P. Finally, it is also a key idea to transform the feasible polyhedral region P to a more "rounded" shape (this requires finding concentric, inscribed and circumscribed ellipsoids to P that differ by some dilation factor β that only depends on the dimension n).

But one can ask, *what happens if one aims to optimize a* non-linear *objective function?* In the last seven years, Lenstra's result has been extended in several *non-linear* ways, allowing the objective function to be much more complicated with various levels of non-linearity. Again non-standard techniques in algebraic combinatorics played a central role in the proofs of these new algorithms. Consider first the problem

$$\begin{aligned} \text{max/min} \quad & f(x_1, \ldots, x_n) \\ \text{subject to} \quad & A\mathbf{x} \le \mathbf{b} \\ & \mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{Z}^n. \end{aligned} \tag{2}$$

Here A is a rational matrix, $\mathbf{b}$ is a rational vector. The set of feasible integer solutions is a convex polyhedron, which we denote by P. The function f is a polynomial function of maximum total degree D with rational coefficients. We are interested in general polynomial objective functions f *without any convexity assumptions*.

It turns out that optimizing polynomials of degree 4 over problems with two integer variables is already an NP-hard problem (see [17]). Thus, even when we fix the dimension, we cannot expect to get a polynomial-time algorithm for solving the optimization problem. The best we can hope for as a generalization of Lenstra's algorithm, even when the number integer variables is fixed, is an *approximation algorithm*. This is precisely what new techniques (to be seen later) provide.

We say an algorithm $\mathcal{A}$ is an *ϵ-approximation algorithm* for a maximization problem with optimal cost $f_{\max}$, if $\mathcal{A}$ runs in polynomial time in the encoding length and returns a feasible solution with cost $f_{\mathcal{A}}$, such that

$$f_{\mathcal{A}} \ge (1-\epsilon) f_{\max}.$$

A family $\{\mathcal{A}_\epsilon\}_\epsilon$ of ϵ-approximation algorithms is a *fully polynomial time approximation scheme (or for short an FPTAS)* if the running time of $\mathcal{A}_\epsilon$ is polynomial in the encoding length and $1/\epsilon$. The polynomial dependence of the running time in $1/\epsilon$ is a very strong requirement, and for many NP-hard problems, efficient approximation algorithms of this type do not exist, unless P = NP. It was proved in the paper [17] that there is an FPTAS for the optimization problems generalizing Lenstra's result: There exists a fully polynomial time approximation scheme (FPTAS) for the maximization problem (2) for all polynomial functions $f(x_1, \ldots, x_n)$ with rational coefficients that are non-negative on the feasible polyhedral region P:

Theorem 3.1 *Let the dimension d be fixed. There exists an algorithm whose input data are*

- *a polytope $P \subset \mathbb{R}^d$, given by rational linear inequalities, and*
- *a polynomial $f \in \mathbb{Z}[x_1, \ldots, x_d]$ with integer coefficients and maximum total degree D that is non-negative on $P \cap \mathbb{Z}^d$*

with the following properties.

1. *For a given* k, *it computes in running time polynomial in* k, *the encoding size of* P *and* f, *and* D *lower and upper bounds* $L_k \leq f(\mathbf{x}^{\max}) \leq U_k$ *satisfying*

$$U_k - L_k \leq \left(\sqrt[k]{|P \cap \mathbb{Z}^d|} - 1\right) \cdot f\left(\mathbf{x}^{\max}\right).$$

2. *For* $k = (1 + 1/\epsilon)\log(|P \cap \mathbb{Z}^d|)$, *the bounds satisfy*

$$U_k - L_k \leq \epsilon f\left(\mathbf{x}^{\max}\right),$$

and they can be computed in time polynomial in the input size, *the total degree* D, *and* $1/\epsilon$.
3. *By iterated bisection of* $P \cap \mathbb{Z}^d$, *it constructs a feasible solution* $\mathbf{x}_\epsilon \in P \cap \mathbb{Z}^d$ *with*

$$\left|f(\mathbf{x}_\epsilon) - f\left(\mathbf{x}^{\max}\right)\right| \leq \epsilon f\left(\mathbf{x}^{\max}\right).$$

A key technique of the algorithm is to consider the *formal sum* of all lattice points inside the feasible polyhedral set $\sum_{\mathbf{u}\in\mathbb{Z}^d\cap P} \mathbf{x}^{\mathbf{u}}$. What happens is that properties of f evaluated at the lattice points of P can be compactly encoded in a generating function.

But before we explain the method let us state another generalization of Lenstra's theorem, this time dealing with *multi-objective optimization* problems where we have more than one linear objective function to be optimized. For general information about multi-objective discrete optimization problems we refer to the survey [25].

When more than one objective function is optimized, there is a different natural notion of optimum that we need to introduce. Let $A = (a_{ij})$ be an integral $m \times n$-matrix and $\mathbf{b} \in \mathbb{Z}^m$ such that the convex polyhedron $P = \{\mathbf{u} \in \mathbb{R}^n : A\mathbf{u} \leq \mathbf{b}\}$ is bounded. Given k linear functionals $f_1, f_2, \ldots, f_k \in \mathbb{Z}^n$, we consider the *multi-objective integer linear programming problem*

$$\begin{aligned} \text{Pareto-min} \quad & \left(f_1(\mathbf{u}), f_2(\mathbf{u}), \ldots, f_k(\mathbf{u})\right) \\ \text{subject to} \quad & A\mathbf{u} \leq \mathbf{b}, \\ & \mathbf{u} \in \mathbb{Z}^n, \end{aligned} \tag{3}$$

where Pareto-min is defined as the problem of finding all Pareto optima and a corresponding Pareto strategy: For a lattice point $\mathbf{u}$ the vector $\mathbf{f}(\mathbf{u}) = (f_1(\mathbf{u}), \ldots, f_k(\mathbf{u}))$ is called an *outcome vector*. Such an outcome vector is a *Pareto optimum* for the above problem if and only if there is no other point $\tilde{\mathbf{u}}$ in the feasible set such that $f_i(\tilde{\mathbf{u}}) \leq f_i(\mathbf{u})$ for all i and $f_j(\tilde{\mathbf{u}}) < f_j(\mathbf{u})$ for at least one index j. The corresponding feasible point $\mathbf{u}$ is called a *Pareto strategy*. Thus a feasible vector is a Pareto strategy if no feasible vector can decrease some objective function without causing a simultaneous increase in at least one other objective function.

In general the number of Pareto optimal solutions may be infinite, but in our situation the number of Pareto optima and strategies is finite. There are several well-known techniques to generate Pareto optima. Some popular methods used to solve such problems include weighting the objectives or using a so-called global criterion approach (see [25]). In particularly nice situations, such as multi-objective *linear* programs [37], one knows a way to generate all Pareto optima, but most techniques reach only some of the Pareto optima.

It turns out that, for fixed dimension, one can compute the sets of *all* Pareto optima and strategies of a multi-criterion integer linear program. Once more the math behind it involves the algebraic combinatorics of rational generating functions. The set of Pareto points can be described as the formal sum of monomials

$$\sum \big\{ \mathbf{z}^{\mathbf{v}} : \ \mathbf{u} \in P \cap \mathbb{Z}^n \text{ and } \mathbf{v} = \mathbf{f}(\mathbf{u}) \in \mathbb{Z}^k \text{ is a Pareto optimum} \big\}. \tag{4}$$

Under the assumption that the number of variables is fixed, we can compute in polynomial time a short expression for the huge polynomial above, thus *all* its Pareto optima can in fact be counted exactly. The same can be done for the corresponding Pareto strategies when written as a formal sum or generating function:

$$\sum \big\{ \mathbf{x}^{\mathbf{u}} : \ \mathbf{u} \in P \cap \mathbb{Z}^n \text{ and } \mathbf{f}(\mathbf{u}) \text{ is a Pareto optimum} \big\}. \tag{5}$$

Theorem 3.2 *Let $A \in \mathbb{Z}^{m\times n}$, an m-vector $\mathbf{b}$, and linear functions $f_1, \dots, f_k \in \mathbb{Z}^n$ be given. There are algorithms to perform the following tasks*:

(i) *Compute the generating function* (4) *of all the Pareto optima as a sum of rational functions. In particular, we can count how many Pareto optima are there. If we assume k and n are fixed, the algorithm runs in time polynomial in the size of the input data.*
(ii) *Compute the generating function* (5) *of all the Pareto strategies as a sum of rational functions. In particular, we can count how many Pareto strategies are there in P. If we assume k and n are fixed, the algorithm runs in time that is bounded polynomially by the size of the input data.*
(iii) *Generate the full sequence of Pareto optima ordered lexicographically. If we assume k and n are fixed, the algorithm runs in polynomial time on the input size and the number of Pareto optima.*

In contrast, it is known that for non-fixed dimension it is #P-hard to count Pareto optima and NP-hard to find them [27, 60].

So what is the mathematics behind the above algorithms for optimization with non-linear objective functions? Essentially one encodes lattice points in a special, compact data structure provided by a generating function which has been rewritten as a "short" sum of multivariate rational functions. The rational functions are recovered from the rays and vertices of unimodular cones associated to the polyhedral region. Given a convex polyhedron P (not necessarily a polytope anymore!), we compute the formal multivariate generating function

$$f(P; \mathbf{z}) = \sum_{\boldsymbol{\alpha} \in P \cap \mathbb{Z}^{\mathbf{d}}} \mathbf{z}^{\boldsymbol{\alpha}},$$

where $\mathbf{z}^{\boldsymbol{\alpha}} = z_1^{\alpha_1} z_2^{\alpha_2} \dots z_d^{\alpha_d}$. This is an infinite formal Laurent series if P is not bounded, but if P is a polytope it is a (Laurent) polynomial with one monomial per lattice point. For example, if we consider a rectangle with vertices $V_1 = (0, 0)$, $V_2 = (5000, 0)$, $V_3 = (0, 5000)$, and $V_4 = (5000, 5000)$ the generating function $f(P)$

has over 25,000,000 monomials,

$$f(P; z_1, z_2) = 1 + z_1 + z_2 + z_1 z_2^2 + z_1^2 z_2 + \cdots + z_1^{5000} z_2^{5000}.$$

The representation of $f(P; z_1, z_2)$ as monomials is clearly way too long to be of practical use. But Barvinok's method [5, 10] instead rewrites it as a compact sum of rational functions. For instance, only four rational functions suffice to represent the over 25 million monomials. Indeed $f(P, z_1, z_2)$ equals

$$\frac{1}{(1-z_1)(1-z_2)} + \frac{{z_1}^{5000}}{(1-{z_1}^{-1})(1-z_2)} + \frac{{z_2}^{5000}}{(1-{z_2}^{-1})(1-z_1)} + \frac{{z_1}^{5000}{z_2}^{5000}}{(1-{z_1}^{-1})(1-{z_2}^{-1})}.$$

Note that if we wish to know the number of lattice points in P, and we knew $f(P; \mathbf{z})$, we could compute it as the limit when the vector $(z_1, \ldots, z_n)$ goes to $(1, 1, \ldots, 1)$. Similarly the maximum of a linear functional over the lattice points equals the highest degree of the univariate polynomial one gets after doing a *monomial substitution* $z_i \to t^{c_i}$ (See [12]). These two calculations can be difficult because of the poles of the rational functions. One uses complex analysis (residue calculations) to find the answer.

The first remarkable application of this representation was done by A. Barvinok who crafted an algorithm to *count* integer points inside polyhedra that runs in polynomial time for fixed dimension (see [5, 10]). Shortly after Barvinok's breakthrough, Dyer and Kannan [15] modified the original algorithm of Barvinok, which originally relied on Lenstra's result, giving a new proof that integer programming problems with a fixed number of variables can be solved in polynomial time. This gives a brand new proof of Lenstra's theorem (see Chap. 7 of [16]).

A beautiful theorem of M. Brion [11] says that to compute the rational function representation of $f(P; \mathbf{z})$ it is enough to do it for tangent cones at each vertex of P. Let P be a convex polytope and let $V(P)$ be the vertex set of P. Let $K_{\mathbf{v}}$ be the tangent cone at $\mathbf{v} \in V(P)$. This is the (possibly translated) cone defined by the facets touching vertex $\mathbf{v}$ (see Fig. 1). Then the following nice formula holds:

$$f(P; \mathbf{z}) = \sum_{\mathbf{v} \in V(P)} f(K_{\mathbf{v}}; \mathbf{z}).$$

Since it is enough to compute everything for cones, we first explain how to compute the rational function for the "smallest" cones, simple cones. A cone is said to be *simple* if its rays are linearly independent vectors. For instance all the tangent cones of the pentagon of Fig. 1 are simple. Obtaining the rational function representation of the lattice points of a simple cone $K \subset \mathbb{R}^d$, is easy (see for example Chap. IV of [63]; another exposition is given in Chaps. 5, 6, and 7 of [16], where also recently developed speedups are discussed). Here is the formula, which one can write directly from the coordinates of the rays of the cone and the lattice points in its *fundamental parallelepiped* Π:

$$f(K; \mathbf{z}) = \frac{\sum_{\mathbf{u} \in \Pi \cap \mathbb{Z}^d} \mathbf{z}^{\mathbf{u}}}{(1-\mathbf{z}^{\mathbf{c}_1})(1-\mathbf{z}^{\mathbf{c}_2}) \ldots (1-\mathbf{z}^{\mathbf{c}_d})}.$$

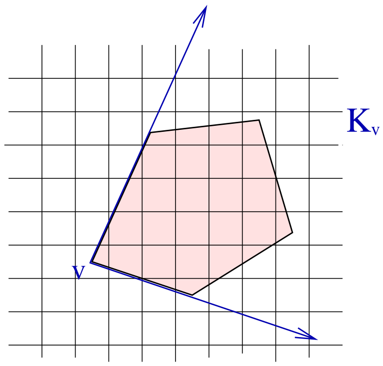


Fig. 1 The tangent cone at vertex **v**

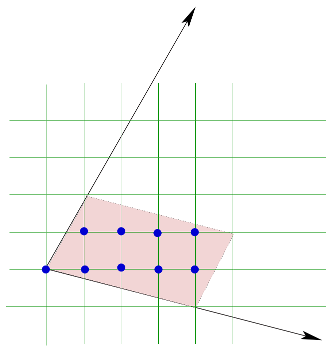


Fig. 2 A two dimensional cone (its vertex at the origin) and its fundamental parallelepiped

Here Π is the half open parallelepiped $\{\mathbf{x} : \mathbf{x} = \alpha_1 \mathbf{c}_1 + \cdots + \alpha_d \mathbf{c}_d, 0 \le \alpha_i < 1\}$. We can do a two-dimensional example shown in Fig. 2: we have $d = 2$ and $\mathbf{c}_1 = (1, 2)$, $\mathbf{c}_2 = (4, -1)$ and the vertex of the cone is the origin. We have:

$$f(K; \mathbf{z}) = \frac{z_1^4 z_2 + z_1^3 z_2 + z_1^2 z_2 + z_1 z_2 + z_1^4 + z_1^3 + z_1^2 + z_1 + 1}{(1 - z_1 z_2^2)(1 - z_1^4 z_2^{-1})}.$$

But what to do if the cone K is not simple? Break it into simple cones! The wonderful idea of A. Barvinok was noting that, although triangulating the cone K may be an inefficient way to subdivide the cone (i.e., exponentially many pieces may arise), if one is willing to add *and* substract cones, for fixed dimension d, there exists a polynomial time algorithm which decomposes a rational polyhedral cone $K \subset \mathbb{R}^d$ into simple unimodular cones K_i. A simple cone is *unimodular* if in its fundamental parallelepiped there is a single lattice point. See Fig. 3. In fact, via the decomposition, with numbers $\epsilon_i \in \{-1, 1\}$, we can write an expression

$$f(K; \mathbf{z}) = \sum_{i \in I} \epsilon_i f(K_i; \mathbf{z}), \quad |I| < \infty.$$

The index set I is of size polynomial in the input data as long as we work in fixed dimension.

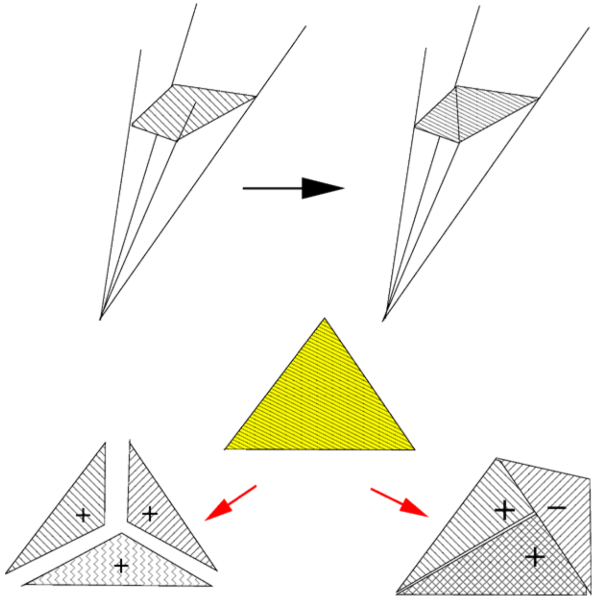


Fig. 3 The signed decomposition of a cone into unimodular cones. Two general steps

We have showcased the evolution of one famous result, Lenstra's theorem, from its linear version to some non-linear versions. In this particular case study, we have seen a new proof of the original Lenstra's result, as well as two non-linear generalizations, that were developed on top of the theory of rational generating functions, a topic traditionally used in algebraic and enumerative combinatorics and number theory.

The best complexity of a Lenstra-type algorithm today appears in [21]. In this paper, an "any-norm" shortest vector algorithm is developed. Other results in fixed dimension that generalize Lenstra's results include the first Lenstra-type algorithm for convex integer minimization, which was announced by Khachiyan [40]. The variant due to Khachiyan and Porkolab [44] is quite notable due to its generality, which includes the case of convex domains and arbitrary quantifier sequences. The case of quasi-convex polynomial functions appeared in [31] and was later improved by [32]. New related results appear in [53].

In the rest of this survey we outline another technique for problems in non-fixed arbitrary dimension.

4 Case Study Three: Non-linear Transportation Problems

In the previous case study the dimension of the problem was assumed to be fixed. Alas, we know that most of the interesting practical problems do not have this property and the curse of high-dimensionality comes to play a role. Can we still prove nice theoretical results in non-fixed dimension? The answer is yes, but we need to make assumptions about the structure of the problems. Some advances, originating in commutative algebra, have helped to approach well-structured discrete optimization problems.

We will use the transportation problem to explain the method. In the traditional transportation problem cost at an edge is a constant, thus we optimize a linear function. But there are natural situations when one may want to optimize a non-linear function. For instance, due to congestion or heavy traffic or heavy communication load the transportation cost on an edge is a non-linear function of the flow at each edge. For example, say the cost at each edge is $f_{ij}(x_{ij}) = c_{ij}|x_{ij}|^{a_{ij}}$ for a suitable constant a_{ij}. This results in a non-linear function $\sum f_{ij}(x_{ij})$, which is much harder to minimize. This problem is hard, but under some assumptions one can create an efficient algebraic algorithm to optimize a convex separable function over the set $\{\mathbf{x} \in \mathbb{Z}^n : A\mathbf{x} = \mathbf{b},\ \mathbf{x} \geq \mathbf{0}\}$. For the transportation problem A is the node-edge incidence matrix of the complete bipartite graph.

For this, consider the lattice $L(A) = \{\mathbf{x} \in \mathbb{Z}^n : A\mathbf{x} = \mathbf{0}\}$ and introduce a natural partial order on the lattice vectors as follows. For $\mathbf{u}, \mathbf{v} \in \mathbb{Z}^n$. We say that the vector $\mathbf{u}$ is *conformally smaller* than $\mathbf{v}$, denoted $\mathbf{u} \sqsubset \mathbf{v}$, if $|u_i| \leq |v_i|$ and $u_i v_i \geq 0$ for $i = 1, \ldots, n$. For instance $(3, -2, -8, 0, 8) \sqsubset (4, -3, -9, 0, 9)$, but it is incomparable to $(-4, -3, 9, 1, -8)$. Equivalently, for $\mathbf{u}$ to be conformally smaller than $\mathbf{v}$, they must lie in the same orthant of $\mathbb{R}^n$ and each component of $\mathbf{u}$ must be bounded by the corresponding component of $\mathbf{v}$ in absolute value.

The *Graver basis* of an integer matrix A is the set of all conformally-minimal nonzero integer linear dependences on A. One can check, for example, that if the matrix $A = (1\ 2\ 1)$, then its Graver basis is given by the following vectors and their negatives: $(2, -1, 0), (0, -1, 2), (1, 0, -1), (1, -1, 1)$. For a fixed cost vector $\mathbf{c}$, we can visualize a Graver basis of an integer program by creating a graph over the lattice points $\{\mathbf{x} \in \mathbb{Z}^n : A\mathbf{x} = \mathbf{b},\ \mathbf{x} \geq \mathbf{0}\}$. The nodes are then $L(\mathbf{b}) := \{\mathbf{x} :\ A\mathbf{x} = \mathbf{b}, \mathbf{x} \geq \mathbf{0},\ \mathbf{x} \in \mathbb{Z}^n\}$ and we use the elements of the Graver basis as directed edges departing from each lattice point $\mathbf{u} \in L(\mathbf{b})$ when they connect to another member of $L(\mathbf{b})$.

Graver bases are interesting for integer optimization because they can be used as test sets, that is, as a finite collection of integral vectors with the property that every non-optimal feasible solution of an integer program can be improved by adding some suitable vector from the Graver basis. In the 1970s, Jack Graver showed that they can be used for linear integer optimization because they can solve the augmentation problem: Given $A \in \mathbb{Z}^{m\times n}$, $\mathbf{x} \in \mathbb{N}^n$ and $\mathbf{c} \in \mathbb{Z}^n$, either find an improving direction $\mathbf{g} \in \mathbb{Z}^n$, namely one with $\mathbf{x} - \mathbf{g} \in \{\mathbf{y} \in \mathbb{N}^n : A\mathbf{y} = A\mathbf{x}\}$ and $\mathbf{c}^\top \mathbf{g} > 0$, or assert that no such $\mathbf{g}$ exists. More recently, stronger augmentation regimes are possible for solving some convex optimization problems (both minimization or maximization), with respect to a convex function composed with linear functions, or convex separable functions. See Chaps. 3 and 4 of [16] and [34].

The main obstacle for using Graver bases was that they can be exponentially large even in fixed dimension and very hard to compute. In fact, for general matrices it is an NP-complete problem to decide whether a list of vectors is a complete Graver basis. The good news is that Graver bases do become very manageable and efficient for highly structured matrices, with regular block decompositions. This theory was developed by S. Onn and collaborators in a series of papers (see [52] or the references therein). The first example of such good structure came from *n-fold matrices*. Fix any pair of integer matrices A and D with the same number of columns, of dimensions $r \times q$ and $s \times q$, respectively. The n-fold matrix of the ordered pair A, D is the

$(s+nr)\times nq$ matrix constructed by stacking copies of A, D as shown below,

$$[A,D]^{(n)} := (\mathbf{1}_n \otimes D) \oplus (I_n \otimes A) \quad = \begin{pmatrix} D & D & \cdots & D \\ A & 0 & \cdots & 0 \\ 0 & A & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A \end{pmatrix} .$$

Naturally, n-fold matrices do appear in applications, our main example being the transportation problems with fixed number of factories on suppliers! For these kind of matrices there is a polynomial time algorithm that, given n, computes the Graver basis $G([A,D]^{(n)})$ of the n-fold matrix $[A,D]^{(n)}$. In particular, the cardinality and the bit size of $G([A,D]^{(n)})$ are bounded by a polynomial function of n (implying that separable convex integer minimization problems over n-fold matrices can be solved in polynomial time). The key idea to prove this goes back to work in combinatorial commutative algebra. A result by Santos and Sturmfels [61], and Hoşten and Sullivant [35] says that for every pair of fixed integer matrices $A \in \mathbb{Z}^{r\times q}$ and $D \in \mathbb{Z}^{s\times q}$, there exists a constant $g(A,D)$ such that for all n, the Graver basis of $[A,D]^{(n)}$ consists of vectors with at most $g(A,D)$ nonzero components. In [33], n-fold matrices were generalized to n-fold 4-block decomposable matrices, which are of the form

$$\begin{pmatrix} C & D & D & \cdots & D \\ B & A & 0 & \cdots & 0 \\ B & 0 & A & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ B & 0 & 0 & \cdots & A \end{pmatrix} .$$

Although, for fixed matrices A, B, C, and D, the Graver bases increase exponentially in size (with respect to n), the structure of the matrices implies structure of the Graver bases, which then can be exploited to find a best augmenting Graver basis direction for a non-optimal feasible solution efficiently. This implies that also separable convex integer minimization problems over n-fold 4-block matrices can be solved in polynomial time (see [33] and Chap. 4 of [16]).

It should be noted that Graver bases are special cases of Gröbner bases, which are quite popular in algebraic geometry. A *Gröbner basis* for an ideal I of the ring of polynomials is a set of generators with additional computational properties (see [3, 13]). Monomials can be naturally associated to lattice points and thus it was born a nice connection to integer optimization, where we care to optimize a function over the set of lattice points of a polyhedron (see the pioneering work of [14, 36, 64] and the exposition in Chap. 11 of [16]).

To conclude let us say that this is just a taste of the strong activity of research in the interaction between optimization and algebra and geometry. Much more is sure to come! We hope that you will be attracted to learn more in [16, 48].

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HISTORICAL ARTICLE

In Memoriam Horst Tietz (1921–2012)

Georg Schumacher

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Zusammenfassung Am 28. Januar 2012 starb *Horst Tietz*. In diesem Nachruf soll auf sein Leben und Werk und den historischen Zusammenhang eingegangen werden.

Schlüsselwörter Funktionentheorie · Riemannsche Flächen · Geometrie · Angewandte Mathematik

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Horst Tietz im Jahre 2004
(Foto Bildarchiv des Mathematischen Forschungsinstituts Oberwolfach, mit frdl. Genehmigung)

Am 28. Januar 2012 starb *Horst Tietz*, emeritierter ordentlicher Professor für Mathematik an der Leibniz-Universität Hannover. *Horst Tietz* war langjähriger Dekan des Fachbereichs Mathematik und Senator der Universität. Er war Mitglied der Braunschweigischen Wissenschaftlichen Gesellschaft und Ehrenmitglied der Mathematischen Gesellschaft in Hamburg. Im Jahre 2000 wurde sein Goldenes Doktorjubiläum mit einer akademischen Feier in Marburg begangen. Sein Schüler Peter Preuss aus La Jolla, Kalifornien, benannte seine Stiftung für das Mathematische Forschungsinstitut in Oberwolfach „Horst-Tietz-Fund“. Die wissenschaftlichen und menschlichen

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Leistungen von Horst Tietz spiegeln sich in seiner Ernennung zum „Chevalier dans l'Ordre des Palmes Académiques" und zum „Chevalier de la Légion d'Honneur" wider.

1 Leben und Werk

Horst Tietz wurde am 11. März 1921 in Hamburg geboren. Nach seinem Abitur 1939 studierte er zunächst Chemie in Berlin und von 1940 an Mathematik in Hamburg. Im Jahre 1950 wurde er in Marburg mit einer Dissertation über das Thema: „Fabersche Entwicklungen auf Riemannschen Flächen" promoviert. Der Hauptgutachter Professor Maximilian Krafft vergleicht in seinem Gutachten die Arbeit mit einem früheren Ansatz und kommt zu dem Schluss, dass „die Methoden von Behnke und Stein sehr viel komplizierter und weniger durchschlagskräftig sind. . . . Mangel der Arbeit ist, daß sie dem Auffassungsvermögen des Lesers sehr viel zumutet." Und abschließend heißt es „Die Beweise sind von einer vorbildlichen Eleganz und Knappheit."

Der Promotion vorausgegangen war das Staatsexamen in Marburg im Jahre 1947, und die erste Stelle am Physikalischen Institut bei Prof. Hückel und Prof. Walcher. Diese Zusammenarbeit mit Wissenschaftlern, die Mathematik anwenden, war vielleicht langfristig prägend. Horst Tietz bemerkte im Hinblick darauf einmal, dass ihm die „soziale Aufgabe der Mathematik" bewusst wurde, nämlich Mathematik Nichtmathematikern nahezubringen. Umgekehrt beeinflusste sie auch deutlich seine wissenschaftlichen Interessen, in dieser Zeit entstanden seine Arbeiten zur klassischen Mechanik und Transformationstheorie.

Nach dem endgültigen Abschluss des Promotionsverfahrens ging Horst Tietz auf eine Assistentenstelle nach Braunschweig. Trotz seiner Hinwendung zu anwendungsorientierter Mathematik blieb er Funktionentheoretiker.

Die Existenz nicht konstanter meromorpher Funktionen auf Riemannschen Flächen ist bekanntlich eine zentrale Aussage für die allgemeine Theorie bis hin zum Satz von Riemann-Roch – im Falle berandeter Riemannscher Flächen besitzt diese eine andere, ebenso grundsätzliche Qualität, nämlich die „Realisierbarkeit" berandeter Riemannscher Flächen als verzweigte Überlagerungen. Der Tietzsche Abbildungssatz klärte in diesem Sinne die Struktur: *Jede berandete Riemannsche Fläche kann realisiert werden durch eine solche, die aus einer gewissen Zahl von Vollebenen besteht und aus ebensovielen und kongruenten Kreisscheiben, wie die Anzahl der Randkontinuen beträgt; es können ebensoviele Nullstellen der Abbildungsfunktion vorgeschrieben werden. Die Randkreise verlaufen schlicht.* Weitere entscheidende Resultate beruhten auf Anwendungen von Faber-Polynomen.

Wohl auch unter dem Einfluss der Braunschweiger Umgebung widmete sich Horst Tietz weiterhin anwendungsbezogenen Problemen der reellen Analysis. Dazu gehörten Untersuchungen über verallgemeinerte vollständige elliptische Integrale. Eine Methode von Bartky wird aufgegriffen, ausgebaut und auf Konvergenzeigenschaften hin untersucht. Hervorzuheben ist der Erfolg mit durchaus anwendungsorientierten Methoden in einer eher theoretischen Disziplin. Die Ergebnisse von Horst Tietz auf einem Gebiet, welches er mit Ahlfors, Bergmann, Bochner und anderen teilte, hatten Behnke in Münster aufmerksam gemacht. Im Jahre 1956 wurde er dort Dozent.

Das Arbeitsgebiet war die Komplexe Analysis, und es entstanden hier wesentliche Arbeiten. In einem von Nevanlinna den Annales Academiae Scientiarum Fennicae vorgelegten Beitrag widmete er sich der Frage nach Funktionen mit Integraldarstellung auf nichtkompakten Gebieten Riemannscher Flächen. Hier ging es im einzelnen um den Zusammenhang von Laurent-Trennungen und Elementar-Differentialen. Es folgte eine größere Arbeit „Zur Klassifizierung meromorpher Funktionen auf Riemannschen Flächen“ in den Mathematischen Annalen.

Auch in Münster blieb Horst Tietz seinen anwendungsbezogenen Forschungsinteressen treu und arbeitete auf dem Gebiet der nichtpositiven Integralfunktionen und Anwendungen in der Theorie der Differentialgleichungen.

Ebenso besaßen Fragen der Geometrie für Horst Tietz stets einen Anwendungsbezug. In diesem Zusammenhang ist das Handbuch der Physik aus der Braunschweiger Zeit zu erwähnen. Aus der Münsteraner Zeit stammen Arbeiten über die Grundlagen der Geometrie und ein Lehrbuch über die Grundlagen der Linearen Geometrie, ferner Arbeiten aus dem Gebiet der Linearen und Differentialgeometrie. Einem größeren Leserkreis wurde er bekannt durch die beiden Bände des Fischer Taschenbuches Mathematik.

Im Jahre 1962 erhielt Horst Tietz einen Ruf auf eine ordentliche Professur in Hannover, die er bis zu seiner Emeritierung im Jahre 1990 innehatte. Seine Ausstrahlung als akademischer Lehrer wurde noch einmal sichtbar, als 700 Hörer zu seiner Abschiedsvorlesung kamen – wer keine Gelegenheit teilzunehmen hatte, konnte sein Manuskript in der Frankfurter Allgemeinen Zeitung nachlesen.

2 Verfolgung im Dritten Reich, spätere Ehrungen

Horst Tietz hat Marburg einmal als seine Schicksalsstadt bezeichnet. Bereits im Jahre 1933 hatte dieses Land den Kreis der zivilisierten Völker verlassen. Drei Wochen nach dem Reichstagsbrand, der Ende Januar gelegt worden war, hatte man in Dachau das erste Konzentrationslager errichtet und schließlich im Oktober den formalen Schritt mit dem Austritt aus dem Völkerbund getan. Nachdem Horst Tietz sein Studium in seiner Heimatstadt Hamburg (nach einem Trimester in Berlin) begonnen hatte, wurden die Studienbedingungen für ihn aufgrund seiner jüdischen Wurzeln immer schwieriger. Es erfolgte seine Zwangsexmatrikulation. Mit Zustimmung und Unterstützung seiner Lehrer Erich Hecke und Hans Zassenhaus wurde er zum Schwarzhörer, dies für eineinhalb Jahre, in denen er sich auch im Gebäude der Universität vor der Geheimen Staatspolizei verbergen musste. Diese Umstände, die in seinem Artikel im Jahrbuch der Philipps-Universität Marburg und den Beiträgen über seine akademischen Lehrer aus den Mitteilungen der Deutschen Mathematiker-Vereinigung und dem Mathematical Intelligencer beschrieben werden, sind unfassbar. Im Jahre 1943 drohte Denunziation, Hans Zassenhaus warnte ihn und lud ihn zu gemeinsamer Arbeit zu sich nach Hause ein. Schließlich wurden Horst Tietz und seine Eltern nach einem Bombenangriff obdachlos. Auf Empfehlung von Erich Hecke ging er nach Marburg zu dessen Schüler Kurt Reidemeister. Am Heiligen Abend 1943 wurde Horst Tietz zusammen mit seinen Eltern verhaftet und in das Konzentrationslager Breitenau und er selbst später nach Buchenwald gebracht. Seine Eltern kehrten nicht wieder zurück, Horst Tietz wurde aus Buchenwald befreit. Er nahm seine Studien in

Hamburg wieder auf, Hecke starb, und Zassenhaus wanderte aus, so dass Tietz im Sommersemester 1946 nach Marburg ging. Wesentlichen Einfluss in Marburg hatten Herbert Grötzsch und Maximilian Krafft, sein späterer Doktorvater.

Aufgrund seiner Verdienste um die Gruppe der französischen Mithäftlinge in Buchenwald wurde er später in den Orden der französischen Ehrenlegion aufgenommen, mit allen zustehenden Privilegien. Es wird berichtet, dass der französische Staat Dank und Anerkennung bei jeder Reise nach Frankreich durch seinen Vertreter an der Grenze Ausdruck verlieh.

Die Hamburgische Mathematische Gesellschaft ernannte Horst Tietz zum Ehrenmitglied. Ebenso wurde er in die Braunschweigische Mathematische Gesellschaft aufgenommen und war drei Jahre lang Vorsitzender der Mathematisch-Naturwissenschaftlichen Klasse.

3 Der Akademiker Horst Tietz

Von Anfang an waren akademische Belange für ihn wichtig: In Marburg war er Mitbegründer des ASTA, nachdem er bereits in Hamburg die Studentenschaft mitorganisiert hatte.

Die Verdienste des Akademikers Horst Tietz sind bedeutend. Bereits in Münster als Dozent in der Nähe von Heinrich Behnke hatte er sich besonders für die Ausbildung der Lehrer und Naturwissenschaftler engagiert. Achtzehn Jahre lang war er der Vorsitzende des Wissenschaftlichen Prüfungsamtes in Hannover, der Vertreter der Niedersächsischen Hochschulen im Landesschulbeirat und Mitglied des Ausschusses „Schule-Hochschule“ in der Westdeutschen Rektorenkonferenz, dem Verhandlungspartner des Schulausschusses der Kultusministerkonferenz.

Seine Stimme wurde gehört. So konnten er als federführender Herausgeber der Denkschrift der DMV zum „Mathematik-Unterricht der Schulen“ schließlich das Ende eines unsinnigen Mengenlehre-Unterrichts an Schulen herbeiführen.

Sehr früh war Horst Tietz als einer der Neubegründer der Deutschen Technion-Gesellschaft hervorgetreten. Einstmals war diese Hilfsorganisation für die Technische Universität in Haifa von Albert Einstein ins Leben gerufen worden. Schon am Anfang war er Mitglied des „Ständigen Büro GE-TH“ für die Verbindungen zwischen den französischen Grandes Ecoles und den Deutschen Technischen Hochschulen und war dessen langjähriger Präsident, zuletzt Ehrenpräsident. Dieser Einsatz für die deutsch-französische Zusammenarbeit wurde von französischer Seite mit der Aufnahme in den Orden „Palmes académiques“ honoriert.

Horst Tietz hat durch sein vielfältiges Wirken als Akademiker und Wissenschaftler bleibende Spuren hinterlassen.

Marburg im Oktober 2012, Georg Schumacher

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BOOK REVIEW

Peter Schneider: “*p*-Adic Lie Groups”
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Annette Huber

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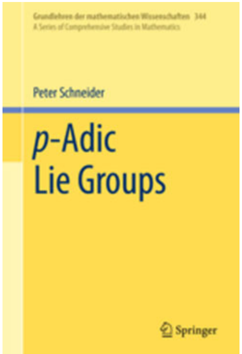

Mit diesem Text legt Peter Schneider das erste echte Lehrbuch über die Grundlagen der Theorie der p-adischen Lie-Gruppen und ihrer Lie-Algebren vor. Es füllt eine schmerzliche Lücke in der Literatur zu einem wichtigen Gegenstand, der bereits seit 50 Jahren studiert und intensiv benutzt wird.

Zu jeder Primzahl p gibt es auf $\mathbb{Q}$ einen Absolutbetrag. Er ordnet einer rationalen Zahl ap^v mit a teilerfremd zu p den Wert p^{-v} zu. Zahlen sind also klein, wenn sie durch hohe Potenzen von p teilbar sind. Zahlentheoretische Fragen nach Teilbarkeit werden so in analytische Sprache gefasst. So wie $\mathbb{R}$ durch die Komplettierung von $\mathbb{Q}$ bezüglich des gewöhnlichen Absolutbetrages entsteht, so definiert man die p-adischen Zahlen $\mathbb{Q}_p$ als Komplettierung von $\mathbb{Q}$ bezüglich des p-adischen Betrags. Da der p-adische Betrag genau das ist, was der Name sagt – ein Absolutbetrag – lassen sich viele Konzepte und Resultate aus der Analysis auch in dieser Situation anwenden.

Im Fall des vorliegenden Buches sind es Lie-Gruppen, d.h. Gruppen wie $\mathrm{Gl}_n(\mathbb{Q}_p)$, die gleichzeitig die Struktur einer Mannigfaltigkeit über $\mathbb{Q}_p$ (oder einer seiner Verallgemeinerungen) tragen. Sie sind von immenser Wichtigkeit in der Zahlentheorie, spielen aber z.B. auch in der algebraischen Topologie eine große Rolle. Zum einen ist ihre Darstellungstheorie Mitspieler im Langlands-Programm, einer weitreichenden vermuteten Verallgemeinerung der Klassenkörpertheorie. Eine andere Anwendung, die sich in der letzten Dekade entwickelte, ist nichtkommutative Iwasawa-Theorie, in der p-adische Lie-Gruppen als Galoisgruppen von algebraischen Erweiterungen

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von $\mathbb{Q}$ auftauchen. In der Topologie wird z.B. stetige Kohomologie von gewissen p-adischen Lie-Gruppen in Berechnungen der stabilen Homotopiegruppen von Sphären benutzt.

Das Buch behandelt in zwei Teilen zwei Aspekte der Theorie: Im ersten Teil der analytische Zugang mit Begriffen wie Mannigfaltigkeit und Tangentialbündel parallel zur reellen Analysis. Die Lie-Algebra zu einer Lie-Gruppe ist dann wie üblich der Tangentialraum am neutralen Element. Der erste entscheidende Unterschied zur reellen Situation ist topologischer Natur: $\mathbb{Q}_p$ ist total unzusammenhängend. Statt der differenzierbaren Funktionen benutzt man daher die Klasse der analytischen Funktionen, also derjenigen, die sich lokal als konvergente Potenzreihen schreiben lassen.

Der zweite Teil des Textes hat ein viel algebraischeres Flair, das zentrale Objekt ist der komplettierte Gruppenring. Auch dieser Zugang, der von Lazard in seiner großen Monographie [3] von 1965 entwickelt wurde, erlaubt die Definition einer Lie-Algebra. Lazards Arbeit ist ob des schieren Umfangs notorisch schwer zu lesen. Schneiders Darstellung konzentriert sich auf Gruppenringe und ihre Rolle in der Konstruktion der Lie-Algebra und auf den zentralen Fall von p-bewerteten pro-p-Gruppen. Er legt damit die erste Behandlung in Lehrbuchform vor.

Die beiden Teile stehen etwas unvermittelt nebeneinander. Ein Ergänzungsvorschlag für Folgeauflagen wäre der Vergleich der beiden Definitionen der Lie-Algebra, der mit den bereitgestellten Mitteln leicht zu führen ist.

Das Buch richtet sich an fortgeschrittene Studierende. Bei genügend Engagement kann diese Gruppe tatsächlich mit Gewinn mit dem Buch arbeiten. Seine Hauptleserschaft wird es unter dankbaren Doktoranden finden, sowie als verlässliche Referenz in der Forschung. Alle Grundlagen, beginnend bei den ersten Eigenschaften von ultrametrischen Beträgen werden sorgfältig und vollständig entwickelt. Nur gelegentlich wird auf die Lehrbuchliteratur verwiesen, dann aber mit vollständiger und sauberer Referenz. Beispiele treten bei der hier gegebenen Entwicklung der Theorie eher in den Hintergrund. Einzelne Übungsaufgaben sind eingestreut.

Im Folgenden soll der Inhalt des Buches etwas genauer beschrieben werden.

Die erste Hälfte ist wie bereits erwähnt analytischer Natur und setzt hierbei konsequent auf den p-adischen oder allgemeiner ultrametrischen Standpunkt. Es setzt sich hiermit klar ab von [1], das den klassischen und den ultrametischen Fall parallel entwickelt. Kenntnisse der reellen Analysis sind hilfreich beim Lesen, jedoch keine Voraussetzung. Kapitel I beginnt mit recht allgemein gehaltenen Grundlagen über Topologie, Differenzierbarkeit und Potenzreihen in ultrametrischen Räumen. Besondere Aufmerksamkeit wird jeweils den zugehörigen Vektorräumen von Funktionen geschenkt. Schneider stellt damit den Zusammenhang zu seinem Buch [4] über nicht-archimedische Funktionalanalysis her.

Kapitel II führt den Mannigfaltigkeitsbegriff und zugehörige Konzepte ein. Neben Standardaussagen, die aus der reellen Analysis wohlbekannt sind, finden sich auch ein harter Satz über die Darstellbarkeit von Derivationen durch Vektorfelder. Das Kapitel endet mit der Topologisierung des Raums $C^{\mathrm{an}}(M, E)$ der analytischen Funktionen auf einer parakompakten Mannigfaltigkeit mit Werten in einem ultrametrischen Banachraum E. Hier folgt Schneider der Doktorarbeit [2] von Féaux de Lacroix.

In Kapitel III treten dann die Hauptakteure, die Lie-Gruppen auf. Es wird die assoziierte Lie-Algebra und deren universell einhüllende Algebra eingeführt. Die Diskussion von formalen Gruppengesetzen wird benutzt, um zu zeigen, dass jede Lie-Gruppe parakompakt ist.

Mit Kapitel IV beginnt der eher algebraische zweite Teil des Buches. Die Konzepte stammen weitgehend aus Lazards Arbeit [3]. Schneiders klar strukturierte Darstellung konzentriert sich auf einen besonders wichtigen Spezialfall und eliminiert einige Begriffe wie den der Bewertung einer Lie-Algebra. Damit werden die Inhalte besser zugänglich. Leichte Kost ist es aber auch in dieser Fassung nicht.

Kapitel IV selbst ist eine schnelle – und dennoch vollständige – Einführung in die Theorie der komplettierten Gruppenringe von pro-endlichen Gruppen und deren topologische und algebraische Eigenschaften.

In Kapitel V wird der zentrale Begriff einer p-bewerteten (p-valué bei Lazard, p-valued in der englisch-sprachigen Literatur) Pro-p-Gruppe eingeführt. Die Bewertung induziert eine Filtrierung. Zentrales Werkzeug beim Studium dieser Gruppen ist das zugehörige graduierte Objekt in Charakteristik p. Der Zusammenhang zum ersten Teil wird dadurch hergestellt, dass jede Lie-Gruppe über $\mathbb{Q}_p$ eine offene Untergruppe enthält, die mit einer p-Bewertung versehen werden kann.

Kapitel VI bringt die beiden Konzepte zusammen. Aus der p-Bewertung der Gruppe wird eine p-Bewertung des komplettierten Gruppenrings. Hauptergebnis ist die Aussage, dass jede p-bewertete Pro-p-Gruppe eine natürliche Struktur als p-adische Lie-Gruppe trägt. Diese Struktur ist sogar unabhängig von der Wahl der Bewertung. Die zugrunde liegende Mannigfaltigkeit ist einfach $\mathbb{Z}_p^d$.

Im letzten Kapitel VII wird Lazards Definition der Lie-Algebra einer p-bewerteten Gruppe erklärt. Man findet sie als die Menge der primitiven Elemente im komplettierten Gruppenring. Exponential- und Logarithmus-Abbildung verbinden wie in der reellen Situation Lie-Gruppe und Lie-Algebra.

Die Hauptergebnisse Lazards neben der Definition der Lie-Algebra betreffen die Kohomologie der betrachteten p-adischen Lie-Gruppen. Im vorliegenden Buch werden diese Aspekte nicht angesprochen. Die Referentin hofft auf einen Fortsetzungsband.

Insgesamt füllt der von Schneider vorgelegte Band die eingangs erwähnte Lücke in der Lehrbuchliteratur in hervorragender Weise. Es ist jedem zu empfehlen, der einen Einstieg in die Theorie der p-adischen Lie-Gruppen sucht.

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BOOK REVIEW

Matilde Marcolli: Feynman Motives

World Scientific, 2009, 236 pp

Dirk Kreimer

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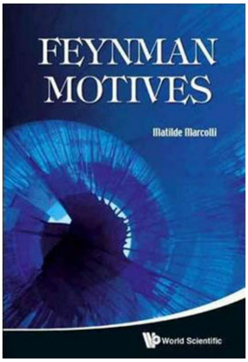

The study of fundamental interactions in physics is to a large extent based on the computation and study of Feynman graphs. By now, there is more than half a century of experience behind these computations. Amazingly, it is only in the last 15–20 years that the mathematical structure behind Feynman graphs started to reveal itself.

It turns out to relate to modern mathematical research in unexpected ways, which has promise for both disciplines, mathematics as well as physics.

To give an idea in which way such physics computations relate to mathematics, let us look at a probability amplitude describing a scattering experiment in physics. In very rough terms, one decides what the incoming particle flux is, and one wants to know what the probability is for a certain final state to be measured in an experiment. The determination of such probabilities, in accordance with the rules of special relativity and quantum physics, is the realm of quantum field theory (QFT).

Physicists have disentangled this problem in a graphical expansion, graded by the loop number (the first Betti number) of the graphs. In each graph, internal edges or vertices are to be integrated over all unobserved values, say momentum or position, in accordance with the rules of quantum physics.

Such a Feynman graph describes pointlike interactions of particles, which then freely propagate between spacetime points at which they interact. Those spacetime points are represented as vertices, the intermediate propagation as edges in a graph.

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The position of the vertices, or equivalently the momenta of internal propagation, are to be integrated out.

To each such Feynman graph Γ an integral $\Phi(\Gamma)$ is thus assigned.

These integrals are only well-defined though when the map Φ is replaced by its renormalized counterpart Φ^R, a transition which can be formulated using a Hopf algebra structure H on such graphs.

The most prominent map to define such a structure is a co-product

$$\Delta : H \to H \otimes H,$$

a map which is dual to the product in an algebra. For Feynman graphs, it reads

$$\Delta(\Gamma) = \sum_{\gamma \subseteq \Gamma} \gamma \otimes \Gamma/\gamma,$$

which uses a set-theoretic decomposition of graphs Γ into subgraphs γ and co-graphs Γ/γ.

The 'renormalized contribution'—which is the contribution eventually measured in an experiment—

$$\Phi^R(\Gamma) = m_{\mathbb{C}}\big(S_\Phi^R \otimes \Phi\big)\Delta(\Gamma) \tag{1}$$

is obtained using a 'counterterm'

$$S_\Phi^R = -R\bar{\Phi} \tag{2}$$

defined through 'Bogoliubov's preparation map'

$$\bar{\Phi} = m_{\mathbb{C}}\big(S_\Phi^R \otimes \Phi P\big)\Delta(\Gamma), \tag{3}$$

where R is a map specifying renormalization conditions on Feynman graphs, and P projects into the augmentation ideal.

Φ^R is a polynomial when applied to a Feynman graph:

$$\Phi^R(\Gamma) = \sum_{j=1}^{\mathrm{cor}(\Gamma)} c_j^\Gamma\big(\{\Theta\}\big) \ln^j \frac{S}{S_0}.$$

One finds that Φ^R evaluates graphs to periods c_j (for suitably fixed angles in the scattering amplitude) or functions of such angles more generally.

It is here where connections to modern mathematics and algebraic and arithmetic geometry emerge: the periods are at sufficiently low loop order periods of mixed Tate motives, the functions polylogarithmic at least at first loop order.

This appearance of periods, a countable set of numbers which organize our understanding of nature at its purest, is a most fascinating research topic.

The questions then are: what are the periods, and hence motives, assigned to a Feynman graph? Is there a dedicated class of periods for a chosen renormalizable QFT, or is the situation generic? And what happens when we sum over graphs?

Answers as of today are sparse, we just list a few striking facts:

- At rather low loop order, periods are periods of mixed Tate motives, but not in general. There are well-defined graph-theoretic criteria though to decide when a Feynman graph is Tate. The general case is open [4].

- The transcendental weight $|p|$ of a period p provided by a graph Γ never exceeds $|p| \leq 2|\Gamma| - 3$, where $|\Gamma|$ is the loop number of the graph [3, 5, 7].
- Theories with internal symmetries like gauge symmetry and/or super-symmetry provide—by computational evidence, hence conjecturally only—for 'weight drop' [2]: summing over all graphs contributing to a given scattering amplitude at a given loop number n, the total answer only involves periods such that $|p| \leq 2n - 4$. The question how restrictive internal symmetries are in confining the space of periods is one of the most-debated research topics in QFT these days.

This book reviews these questions and the underlying physics and mathematics in a rather superficial manner, aiming at no more than a first and short introduction to the subject.

As such, it is useful, but the reader will have to consult the original literature for precise and up-to-date information.

Let us review the book in question now chapter by chapter.

Chapter 1 gives a short introduction to pQFT and Feyman graphs. For the newcomer, this will be the most useful chapter, as it transpires the fascination of the author with the nexus of questions described above most clearly. As a first point of contact of a mathematician with the relevant physics notion this is quite useful.

Already here, a bias of Marcolli for the technique of dimensional regularization, invented by physicists, is prominent. This should be taken with a grain of salt by the reader: QFT often becomes most clear upon analysing Feynman graphs in the sober light of configuration polynomials and parametric representations.

Chapter 2 reviews, on 27 pages, the theory of motives and periods. The experts will clearly find that this chapter does no justice to the topic, as the author herself indeed happily admits. For the non-expert, the whole chapter is yet another baffling account of the notion of motives in the literature.

The reader can take away a superficial idea of pure motives, with some motivation for mixed Tate motives. Here, an account of the recent work of Brown, Schnetz, Yeats and others [4, 5], emerging at the time when the book was written, would be a wonderful addition. By now, Brown and Schnetz showed in particular that mixed Tate motives are too narrow a universe in which to frame Feynman graphs.

Similarly missing is an account of the fundamental methods based on point-counting over $\mathbb{F}_q$ developed recently, which provide for many detailed results of the nature of periods in the expansion above [5]. In this respect, the book in review unfortunately misses to prepare the reader for the most interesting results obtained in recent years [4].

Chapter 3 gives an account of QFT when approached through parametric representations for Feynman graphs. Parametric representations very elegantly combine combinatorics and graph theory to turn rather convoluted analytic expressions of physics into rather well-structured polynomials suitable to be analysed by methods of algebraic geometry. This chapter is useful as an introduction to these polynomials, which are mathematical gems in their own right. The account given here only scratches at the surface though of a wide ranging analysis of Feynman graphs in such terms which started in [1] and is still ongoing [5]. What the reviewer misses here is a more detailed account of Dodgson polynomials which underly almost any of the progress of recent years.

Also, a more prominent exhibition of the work of Belkale and Brosnan would have been welcome here, emphasizing the fact indeed that at higher loop orders, graphs have no reason to restrict themselves to the mixed Tate universe.

The census by Schnetz on Feynman graphs and periods known so far [7] is unfortunately missing here as well. Given that any relation between algebraic geometry, periods and motives relies on such computational data, this is really a painful omission.

Chapter 4 promotes an idea of the author. At this point, it reflects an isolated viewpoint which is only useful in the rather speculative later parts of the book.

Chapter 5 reviews renormalization from a Hopf algebra viewpoint. This is by now an active area of study in its own right. The account given is again useful as a first introduction, explaining Eqs. (1), (2), (3) in three devoted subsections, but far from self-contained, and with some lapses in accuracy in the technical details. The original literature will have to fill this gap [6].

Chapters 6–9 are very speculative, and mix an account of the literature with a very personal viewpoint which is somewhat disconnected from the progress made in recent years. These chapters often deviate in directions whose promise is rather unclear to the reviewer.

In summary, the research covered in this book is rather recent. The non-specialist literature available to a novice in the area is sparse. If this book has motivated and enabled the reader to study the original mathematics and physics literature, it has fulfilled a very useful purpose.

This book will appeal to mathematicians who seek an entry point into this area, as well as to physicists who enquire about the most basic mathematical notions involved. As always, any serious interest will necessitate a study of the original research literature in the field.

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BOOK REVIEW

Jeffrey Strom: "Modern Classical Homotopy Theory"
AMS, 2011, xxii+835 pp.

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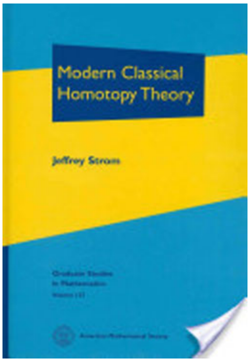

Homotopy theory is a very broad subject. The basic idea is easy to describe: the objects of study are (nicely behaved) topological spaces but the crucial point is that continuous maps between spaces are considered up to homotopy, that is, up to continuous deformation. The concept of homotopy allows for classifications that are, in general, much coarser than for instance the classification of spaces up to homeomorphism or classifications in differential or complex geometry.

The subject has a long history starting around 1950 (with some important earlier contributions) and homotopy theory is thriving today. Concepts and constructions from homotopy theory influence many areas of mathematics, for instance motivic homotopy theory plays an important role in algebraic geometry, Hopkins' spectrum of topological modular forms has connections with the classical theory of modular forms, with elliptic curves and with conformal field theories, and algebraic K-theory connects homotopy theory and algebraic number theory. In the foundations of mathematics, homotopy theory is used in homotopy type theory and in geometric topology methods and results from homotopy theory are used to gain genuine geometric information. Simplicial methods and Quillen model category structures by now belong to the standard toolkit of many mathematicians. An important recent result is the solution of the Kervaire invariant problem by Hill, Hopkins and Ravenel. The original problem is a question about smooth framed manifolds but the solution is given in terms of stable homotopy theory.

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Homotopy theory is a very diverse subject. The Mathematics Subject Classification mentions topics such as cofibrations and fibrations, homotopy equivalences, classification of homotopy types, Eilenberg-MacLane spaces, Spanier-Whitehead and Eckman-Hilton duality, (infinite) loop spaces and suspensions, stable homotopy theory and spectra, spectra with extra structure, operads, localization and completion, string topology, rational homotopy theory, shape theory and proper homotopy theory and equivariant homotopy theory. This list is not complete at all and it does not include technical tools, such as spectral sequences or cohomology operations, or calculational aspects. One of the features of homotopy theory is a mix of methods, ranging from genuinely homotopy theoretic ones to geometric and algebraic methods.

The long list of topics above indicates that even a book on homotopy theory like Jeffrey Strom's with more than 800 pages cannot give a comprehensive introduction into the subject. Any author of such a volume has to choose which topics to discuss and of course this choice depends on personal taste: Strom focuses on the concept of homotopy limits and homotopy colimits and the model category of topological spaces with a thorough discussion of cofibrations and fibrations. Most of the standard topics of unstable homotopy theory are covered by the book.

The book starts with a short introduction to category theoretical concepts, in particular it contains a chapter on limits and colimits. Part two contains the basic concepts of homotopy theory, starting with a discussion what properties a nice category of topological spaces should have, introducing the concept of homotopy, the notions of (co)fibrations, introducing homotopy (co)limits, discussing (co-)H-spaces and Lusternik-Schnirelmann category and finally treating Quillen model category structures. Connectivity, n-equivalences, the Seifert-van-Kampen theorem and cellular approximation are dealt with in the part three. In this part Strom also explains what one can say about pullbacks of cofibrations. As cofibrations behave well with respect to colimits, but in general not with respect to limits, this is a non-standard topic. The result, namely that cofibrations are preserved under pullbacks along fibrations, is useful to know. Notions like coverings and bundles, Serre fibrations and quasifibrations are subsumed under the chapter "Related Topics"; this does not quite reflect their importance.

Part four features some of the main results in classical homotopy theory: This part starts with skeleta of spaces, Postnikov towers and classifying spaces, and then deals with loop spaces and suspensions. The Freudenthal suspension theorem and the Blakers-Massey theorem are proven and some consequences for homotopy groups and Eilenberg-MacLane and Moore spaces are discussed. Part four closes with a chapter on "Further Topics" which collects themes ranging from Lusternik-Schnirelmann category to infinite symmetric products.

Cohomology and homology show up in Part five. Strom introduces cohomology in the represented form, then gives the general definition of a cohomology theory before discussing concrete examples. He states Brown representability and then describes basic properties of homology theories. Cohomology operations, the structure of the Steenrod algebra and cohomology and homology via the cellular and singular (co)chain complexes are other topics of this part.

Spectral sequences are the main topic of Part six. An extensive discussion of filtrations is the starting point and the spectral sequence associated to a filtration and the

Leray-Serre spectral sequence are described in some detail with applications, ranging from some classical cohomology calculations to Bott periodicity.

The book closes with Part seven, which is called 'Vistas'. Four main topics are presented: Localizations and completions of spaces, exponents for homotopy groups, classes of spaces and a theme and variations on Miller's theorem of the triviality of the space of pointed maps from the classifying space of a cyclic group of prime order to a finite-dimensional CW complex.

Strom clearly states his preferences in the introduction: "I have generally used topological or homotopy-theoretical arguments rather than algebraic ones." If readers think that this preference results in a plenitude of geometrical arguments, they will be disappointed. There are a lot of diagrams in the book, but the only figures are of a schematic nature, explaining homotopy extensions, for instance. Homotopy theory often transfers topological questions into algebraic ones, thus a certain amount of algebraic arguments is intrinsic to the subject and cannot be avoided. Cohomological methods and spectral sequences appear rather late in the book (in Chapters 21 and 30 out of a total of 37 chapters). Many theorems that are typically proved using these methods are treated in the chapters before Chapter 21, and are proved using homotopy-theoretic arguments on space level.

The book is not a classical textbook whose content is structured as a sequence of results followed by proofs with some remarks and examples. Rather it encourages learning-by-doing. Strom says that "theorems are followed by multi-part problems that guide the readers to find the proofs for themselves". In these problems, proofs are broken down into smaller portions. Only a reader who works on the problems and exercises will gain something from this book.

Having no outright proofs at all in a book might have its drawbacks: Some things are difficult to learn if you never see them done. For instance, finding cartoon proofs in homotopy theory that you might translate to a full proof later (or you don't because you are happy with them as they are), is something you learn from role models; otherwise this aspect of homotopy theory might just be lost on you. As Strom does not provide detailed references to the literature, a reader who does not manage to solve the problems might find it hard to fill in gaps.

According to Strom, the intended readership of the book consists of people who had an introductory course in topology, but have not necessarily seen the fundamental group. While no knowledge in topology is required that goes beyond that, some experience with arguments and proofs in topology is necessary, in order to solve the problems and do the exercises.

Strom's book is certainly different from the existing literature on homotopy theory. His book is *not* suited for someone who just wants to apply homotopy theory, get a quick impression what the subject is about and how the proofs work. But for someone with a serious interest in the deeper aspects of those topics in homotopy theory that are presented in the book, the book can help to learn them in an active way.