Jahresber Dtsch Math-Ver (2013) 115:1–2
DOI 10.1365/s13291-013-0060-8

PREFACE

Preface Issue 1-2013

Hans-Christoph Grunau

Annette Werner provides an introduction to the research area of “Non-Archimedean analytic spaces”, i.e. analytic geometry over complete non-Archimedean fields. These fields arise in algebra and number theory and the corresponding geometry looks completely different from the usual analytic, projective or differential geometry. The author presents a new approach to this kind of arithmetic geometry with “better” topological properties.

Hans Grauert passed away in September 2011. Alan Huckleberry has written a very illustrative and informative obituary in which he reviews some of the most influential and fundamental works of “one of the greatest giants of the second half of the 20th century” in complex geometry. We learn about the—often quite classical—complex analytic background, the key achievements, the main implications and some of the really modern ideas of these works. While the precise formulation of the mathematics requires some abstract and deep concepts, the present article focusses on the underlying analytical intuition and geometrical imagination. Moreover, Hans Grauert’s interactions with his advisors, friends, coworkers, mathematical companions and students are outlined in some detail.

Recently released books on nonlinear potential theory on metric spaces, on statistical regression for specific kinds of data, and on random matrices in mathematical physics are extensively discussed and reviewed.

Every four years the (managing) editor of the Jahresbericht is elected by the members of the DMV. This is the beginning of my second term of office, which will last until the end of 2016. It is my hope that all the changes, the Jahresbericht had to undergo during the past four years, will help it to become internationally more visible as

H.-Ch. Grunau (✉)
Institut für Analysis und Numerik, Fakultät für Mathematik, Otto-von-Guericke-Universität,
Postfach 4120, 39016 Magdeburg, Germany
e-mail: hans-christoph.grunau@ovgu.de

a journal devoted to accessible articles on important and interesting issues of mathematics and to reviews of recent books of general interest. Also the editorial board changes every four years. I would like to take the opportunity to thank the members of the previous board for the very enjoyable and fruitful cooperation. The new editorial board consists of Felix Finster, Hansjörg Geiges, Martin Grothaus, Martin Hanke, Michael Hinze, Gabriele Nebe, Guido Schneider, Wolfgang Soergel, Thomas Bartsch, who is responsible for the book reviews, and myself. I am confident that we all together will collect a number of interesting contributions from many different vibrant areas of mathematics.

Jahresber Dtsch Math-Ver (2013) 115:3–20
DOI 10.1365/s13291-013-0056-4

SURVEY ARTICLE

Non-Archimedean Analytic Spaces

Annette Werner

Received: 5 October 2012 / Published online: 6 February 2013

Abstract This paper provides an elementary introduction to Vladimir Berkovich's theory of analytic spaces over non-Archimedean fields, focusing on topological aspects. We also discuss realizations of Bruhat-Tits buildings in non-Archimedean groups and flag varieties.

Keywords Non-Archimedean analytic spaces · Berkovich spaces · Bruhat-Tits buildings

Mathematics Subject Classification (2010) 14G20 · 11G25 · 32P05 · 20E42

1 Introduction

About two decades ago, Vladimir Berkovich introduced a new approach to analytic geometry over non-Archimedean fields. At this time, Tate's theory of rigid analytic spaces was quite well developed and established. Rigid analytic spaces, however, have rather poor topological properties. Berkovich's analytic spaces contain more points, which leads for example to better connectivity properties. Meanwhile, Berkovich's approach to non-Archimedean analytic geometry has become an active area of research with important applications to various branches of geometry.

This text aims at providing a very gentle introduction to the nice topological properties of Berkovich spaces. We hope to convey some of the fascination of non-Archimedean geometry to researchers from other areas. Due to the introductory nature of this text, we focus on topological aspects.

A. Werner (✉)
Institut für Mathematik, Goethe-Universität Frankfurt, Robert-Mayer-Straße 8, 60325 Frankfurt, Deutschland
e-mail: werner@math.uni-frankfurt.de

We start with a discussion of fields with non-Archimedean absolute values, pointing out features which are different from the probably better known Archimedean situation. Then we discuss in detail the structure of the Berkovich unit disc. Whereas the rigid analytic unit disc is the set of maximal ideals in the Tate algebra, the Berkovich unit disc is a set of suitable seminorms on the Tate algebra. This includes seminorms whose kernels are not maximal ideals. We show how these additional points lead to good connectivity properties. We also discuss the Berkovich projective line.

Then we introduce affinoid algebras as the building blocks of general analytic spaces and we outline some features of the algebraization functor. Starting from an embedding of the Bruhat-Tits tree of SL_2 we explain how buildings can be embedded in analytic spaces. This article concludes with a few remarks about recent research contributions.

2 Non-Archimedean Fields

Our ground field is a field K carrying a non-Archimedean absolute value, i.e. K is endowed with a function $| \; | : K \to \mathbb{R}_{\geq 0}$ such that for all $a, b \in K$ the following conditions hold:

$$\text{(i)} \quad |a| = 0 \quad \text{if and only if } a = 0.$$

$$\text{(ii)} \quad |ab| = |a||b|.$$

$$\text{(iii)} \quad |a+b| \leq \max\big\{|a|, |b|\big\}.$$

The last condition is the *ultrametric triangle inequality* which has very powerful consequences as we will see below. We endow K with the topology given by its absolute value. In the following we only consider complete fields, i.e. we assume that every Cauchy sequence has a limit in K. We can always place ourselves in this situation by embedding a field with a non-Archimedean absolute value in its completion, which is defined as the ring of Cauchy sequences modulo the ideal of zero sequences.

Example Here are some examples for fields which are complete with respect to a non-Archimedean absolute value.

(1) Let K be any field. Then the trivial absolute value

$$|a| = \begin{cases} 1, & \text{if } a \neq 0 \\ 0, & \text{if } a = 0 \end{cases}$$

is non-Archimedean and K is complete with respect to this absolute value.

(2) For every prime number p there is a p-adic absolute value $| \; |_p$ on the field of rational numbers $\mathbb{Q}$. It is defined by

$$|m/n|_p = p^{-v_p(m)+v_p(n)}.$$

Here m and n are non-zero integral numbers and v_p denotes the exponent of p in the prime factorization. The completion of $\mathbb{Q}$ with respect to this absolute value is denoted by $\mathbb{Q}_p$. It is a local field (as is any finite extension), i.e. it is locally compact in the topology given by the absolute value.

(3) If K is a non-Archimedean complete field and L/K is a finite field extension, then the absolute value $|\ |_K$ on K can be extended in a unique way to an absolute value $|\ |_L$ on L. Hence the absolute value on K extends uniquely to its algebraic closure. In particular, the field $\mathbb{C}_p$, which is defined as the completion of the algebraic closure of $\mathbb{Q}_p$, carries a non-Archimedean absolute value. $\mathbb{C}_p$ is an algebraically closed and complete field, which can be regarded as a p-adic analog of the complex numbers.

(4) Let k be any field and fix a real constant r strictly between 0 and 1. Then, the field of formal Laurent series

$$k\big((X)\big) = \left\{ \sum_{i \geq i_0} c_i X^i : c_i \in k \text{ and } i_0 \in \mathbb{Z} \right\}$$

is complete with respect to the X-adic absolute value

$$\left| \sum_{i \geq i_0} c_i X^i \right| = r^{i_0}, \quad \text{if } c_{i_0} \neq 0.$$

Let us now list some important consequences of the ultrametric triangle inequality which are fundamentally different from the more intuitive Archimedean situation.

Properties 1 *Let K be a field which is complete with respect to a non-Archimedean absolute value. Then*

(i) *The unit ball in K*

$$\mathcal{O}_K = \{a \in K : |a| \leq 1\}$$

is a ring, since by the ultrametric triangle inequality it is closed under addition. It is called the ring of integers in K. Since the negative logarithmic absolute value defines a valuation on $\mathcal{O}_K$, it is a local ring with maximal ideal $\mathfrak{m}_K = \{a \in K : |a| < 1\}$ (the open unit ball). The quotient field $\widetilde{K} = \mathcal{O}_K/\mathfrak{m}_K$ is called the residue field of K.

(ii) *If $|a| \neq |b|$, then $|a+b| = \max\{|a|, |b|\}$. Indeed, if $|b| < |a|$, the inequality $|a| \leq \max\{|a+b|, |-b|\}$ shows that $|a+b|$ cannot be strictly smaller than $|a|$. Hence in every ultrametric triangle at least two of the three sides have the same length!*

(iii) *For every a in K and $r \in \mathbb{R}_{\geq 0}$ we define the closed ball around a with radius r as*

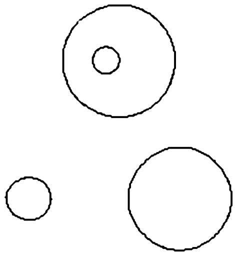
$$D(a,r) = \{x \in K : |x-a| \leq r\}$$

and the open ball around a with radius r as

$$D^0(a,r) = \{x \in K : |x-a| < r\}.$$

Contrary to the Archimedean situation, every closed ball is also open. A fortiori, every circle $\{x \in K : |x-a| = r\}$ is open in K, since it also contains the open ball $D^0(b,r)$ for each of its points b! This is an immediate consequence of (ii).

(iv) *A similar argument as in* (iii) *shows that two ultrametric balls are either nested or disjoint*



In other words, every point contained in a closed ball is its center.

(v) *A popular error in the Archimedean case becomes true in the non-Archimedean world: An infinite sum* $\sum_{n=0}^{\infty} a_n$ *in* K *converges if and only if* $a_n \to 0$ *as* $n \to \infty$.

(vi) *The topology on* K *is totally disconnected, i.e. it contains no non-trivial connected subset.*

For the rest of this paper we fix a field K which is complete with respect to a non-trivial non-Archimedean absolute value. Sometimes we make additional assumptions.

3 The Unit Disc

Over the field of complex numbers one can define analytic functions as functions which are locally given by convergent power series. However, a similar definition over non-Archimedean fields gives strange results as the next example shows.

Example 2 The function $f : D(0,1) \to \mathbb{R}$, which is equal to 0 on the open unit disc $D^0(0,1)$ and equal to 1 on the unit circle $\{x \in K : |x| = 1\}$ has a local expansion in convergent power series since by Properties 1(iii), $D^0(0,1)$ and the unit circle form an open covering of $D(0,1)$.

In order to exclude such pathological functions, Tate [20] defined his rigid analytic spaces by considering only so-called *admissible* open subsets and *admissible* open coverings. The technical tool here is to define a Grothendieck topology rather than look at the topology on K induced by the absolute value. Among the benefits of this approach is a good theory of sheaf cohomology.

In Tate's theory, the algebra of analytic functions on the closed unit ball $D(0,1)$ is the Tate algebra

$$T = \left\{ \sum_{n=0}^{\infty} c_n z^n : \sum_{n=0}^{\infty} c_n a^n \text{ converges for every } a \in D(0,1) \right\}.$$

Note that by Properties 1(iv), an infinite series $\sum_n c_n z^n$ converges at every point of the unit disc $D(0,1)$ if and only if $|c_n| \to 0$ for $n \to \infty$. Hence we can define a norm on T setting

$$\left\| \sum_n c_n z^n \right\| = \max_n |c_n|.$$

This norm is called the *Gauss norm*. It has the following properties which are easy to verify.

Lemma 3

(i) *The Gauss norm is multiplicative, i.e.* $\|fg\| = \|f\|\|g\|$ *for all* f, g *in* T.
(ii) *It satisfies the ultrametric triangle inequality.*
(iii) T *is complete with respect to* $\| \ \|$, *hence a Banach algebra.*

A very important feature of the Gauss norm is that it coincides with the supremum norm on the unit disc over the algebraic closure. The following result is called the Maximum Modulus Principle, see [6], Sect. 5.1.4.

Lemma 4 *Let* K^a *denote the algebraic closure of* K, *which we endow with the absolute value extending the one on* K, *and denote by* $D_{K^a}(0,1) = \{x \in K^a : |x| \leq 1\}$ *the unit disc in* K^a. *Let* $f = \sum_{n=0}^{\infty} c_n z^n$ *be an element in* T, *and let* $a \in D_{K^a}(0,1)$. *Then, the sequence* $\sum_{n=0}^{\infty} c_n a^n$ *converges, since* $|c_n a^n| \leq |c_n| \to 0$. *We write* $|f(a)|$ *for the absolute value of its limit. We can express the Gauss norm as follows*:

$$\|f\| = \sup_{a \in D_{K^a}(0,1)} \big|f(a)\big| = \max_{a \in D_{K^a}(0,1)} \big|f(a)\big|.$$

Proof Note that for every $a \in D_{K^a}(0,1)$ the inequality $|f(a)| \leq \max_n |c_n a^n| \leq \max_n |c_n| = \|f\|$ holds, so that the supremum of all $|f(a)|$ is also less than or equal to the Gauss norm.

It remains to show that for every element f in T there exists some $a \in D_{K^a}(0,1)$ such that $\|f\| = |f(a)|$. We may assume that $f \neq 0$. Since there is an element b in K with $|b| = \|f\|$ (namely, any coefficient c_n with maximal absolute value), we may replace f by $b^{-1} f$ and assume that $\|f\| = 1$. Hence all coefficients c_n of f lie in the ring of integers $\mathcal{O}_K$. We denote by $c \mapsto \tilde{c}$ the quotient map from $\mathcal{O}_K$ to the residue field $\widetilde{K} = \mathcal{O}_K / \mathfrak{m}_K$. Then $\tilde{f} = \sum_n \tilde{c}_n z^n$ is a non-zero polynomial over the residue field.

The residue field of the algebraic closure K^a of K is defined as $\widetilde{K^a} = \mathcal{O}_{K^a} / \mathfrak{m}_{K^a}$, where

$$\mathcal{O}_{K^a} = \left\{ x \in K^a : |x| \leq 1 \right\} \quad \text{and} \quad \mathfrak{m}_{K^a} = \left\{ x \in K^a : |x| < 1 \right\}.$$

Since $\widetilde{K^a}$ is an infinite field, it contains an element $\tilde{a}$ on which the polynomial $\tilde{f}$ does not vanish. Then any preimage $a \in \mathcal{O}_{K^a}$ of $\tilde{a}$ satisfies $|f(a)| = 1 = \|f\|$. □

Note that this lemma is in general not true if we only look at the supremum of $|f(a)|$ for points $a \in D(0,1)$, i.e. for K-rational points of the unit disc. We have to pass to the algebraic closure.

The rigid analytic version of the unit disc is defined as the space

$$Sp(T) = \{\mathfrak{m} : \mathfrak{m} \subset T \text{ maximal ideal}\}$$

together with a sheaf of analytic functions on Tate's Grothendieck topology. If K is algebraically closed, the maximal spectrum $Sp(T)$ coincides with the unit disc $D(0, 1)$.

In the approach to non-Archimedean analysis provided by rigid geometry one has to forfeit topological intuition to a certain extent. We will now see that Berkovich spaces provide topologically nice analytic spaces with good connectivity properties. We start by introducing the Berkovich version of the closed unit disc which we denote by $\mathcal{M}(T)$. Very roughly, Berkovich's idea is to add additional points to the classical unit disc $D(0, 1)$.

To simplify the exposition *we assume that K is algebraically closed and complete* with respect to a non-trivial non-Archimedean absolute value. Note that any field with a non-Archimedean absolute value can be embedded into an algebraically closed and complete non-Archimedean field.

Definition 5 We define the Berkovich spectrum $\mathcal{M}(T)$ of T as the set of all non-trivial multiplicative seminorms on T bounded by the Gauss norm, i.e. as the set of all maps γ satisfying the following conditions:

(i) $\gamma \neq 0$ is a map from T to $\mathbb{R}_{\geq 0}$.
(ii) γ is multiplicative, i.e. for all $f, g \in T$ we have $\gamma(fg) = \gamma(f)\gamma(g)$.
(iii) γ satisfies the strong triangle inequality

$$\gamma(f + g) \leq \max\{\gamma(f), \gamma(g)\}.$$

(iv) γ is bounded by the Gauss norm on T, i.e. for all f in T we have $\gamma(f) \leq \|f\|$.

Note that by (i) and (ii) $\gamma(1) = 1$, which implies together with (iv) that the restriction of γ to the field K (i.e. the constant functions) coincides with the absolute value on K.

Let us now show that the unit disc $D(0, 1)$ is contained in the Berkovich unit disc $\mathcal{M}(T)$. Let a be a point in $D(0, 1)$. We can associate to a the seminorm

$$\zeta_a : T \to \mathbb{R}_{\geq 0}$$
$$f \mapsto \big|f(a)\big|.$$

It is easy to check that ζ_a satisfies properties (i) to (iv) in Definition 5. Since for $a \neq b$ we find $\zeta_a(z-a) = 0$ and $\zeta_b(z-a) = |b-a| \neq 0$, the association $a \mapsto \zeta_a$ is injective. We use it tacitly to identify $D(0, 1)$ with a subset of $\mathcal{M}(T)$. Every point in the image, i.e. every seminorm of type ζ_a is called *a point of type* 1 in the Berkovich unit disc.

Note that $\mathcal{M}(T)$ is a subset of the set of all real valued functions on T. Hence, it can be endowed with a natural topology, namely the topology of pointwise convergence. This is the weakest topology such that for every $f \in T$ the evaluation map

$$\mathcal{M}(T) \longrightarrow \mathbb{R}$$
$$\gamma \mapsto \gamma(f)$$

is continuous. Its restriction to the subset of points of type 1, i.e. to $D(0,1) \subset \mathcal{M}(T)$ is the topology induced by the absolute value on K which is totally disconnected. We will now show that $\mathcal{M}(T)$ contains many additional points which "fill up the holes" in the classical unit disc.

We have seen in Lemma 3 that the Gauss norm on T is multiplicative, hence it is a point in $\mathcal{M}(T)$. Recall that by Lemma 4, the Gauss norm is the supremum norm on the unit disc $D(0,1)$. More generally, we can look at supremum norms on other discs.

Definition 6 Let $a \in D(0,1)$ and let r be a real number with $0 < r \leq 1$. For every $f \in T$ we define its supremum norm on $D(a,r)$ as

$$\zeta_{a,r}(f) = \sup_{x \in D(a,r)} \left|f(x)\right|.$$

Every seminorm $\zeta_{a,r}$ is a point in the Berkovich spectrum $\mathcal{M}(T)$. Properties (i), (iii) and (iv) of Definition 5 are obvious. In order to check multiplicativity it is useful to show that for $f = \sum_n c_n(z-a)^n \in T$ we have $\zeta_{a,r}(f) = \max_n(|c_n|r^n)$. If $r = |b|$ lies in the value group $K^\times$ this follows from Lemma 4 applied to $g(z) = f(bz+a)$. Otherwise, one can use a limit argument, since the value group $|K^\times|$ is dense in $\mathbb{R}_{>0}$.

If r is contained in $|K^\times|$, the point $\zeta_{a,r}$ is called a *point of type* 2. If r is not contained in the value group $|K^\times|$, then $\zeta_{a,r}$ is called a *point of type* 3.

Note that the Gauss norm is a point of type 2. It is equal to $\zeta_{0,1}$ in the notation of the previous definition. We can extend this notation by allowing the radius r to be zero, and define

$$\zeta_{a,0}(f) = \sup_{x \in D(a,0)} \left|f(x)\right| = \left|f(a)\right|.$$

Then $\zeta_{a,0} = \zeta_a$ is the point of type 1 associated to a we have previously studied.

The difference between points of type 2 and 3 can be seen in the branching behavior of paths in $\mathcal{M}(T)$. First of all, for every point ζ_a of type 1 there is a path $[\zeta_a, \zeta_{0,1}]$ from ζ_a to the Gauss point $\zeta_{0,1}$, which is given as the image of the map

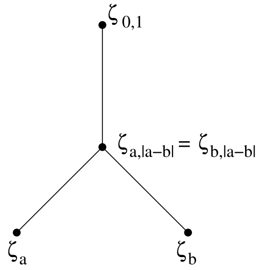
$$\begin{aligned} [0,1] &\longrightarrow \mathcal{M}(T) \\ r &\mapsto \zeta_{a,r}. \end{aligned}$$

Recall that by Properties 1(iv), the Gauss point $\zeta_{0,1}$ is equal to $\zeta_{a,1}$. Apart from the starting point ζ_a, this path consists only of points of type 2 or 3. Moreover, the map is continuous, since it is continuous after evaluation on all functions in T.

Now we look at a second point ζ_b of type 1. Then

$$\zeta_{a,r} = \zeta_{b,r} \quad \text{if and only if } |a-b| \leq r.$$

Hence on $[0, |a-b|)$ the two paths $[\zeta_a, \zeta_{0,1}]$ and $[\zeta_b, \zeta_{0,1}]$ are disjoint. They meet in $\zeta_{a,|a-b|} = \zeta_{b,|a-b|}$ and travel together to the Gauss point from there on.



This example shows the good connectivity properties of Berkovich spaces: Two points from the totally disconnected unit disc $D(0,1)$ are connected by a path in $\mathcal{M}(T)$ which hits the unit disc $D(0,1)$ only at the starting point and the terminal point.

Note that by definition, $\zeta_{a,|a-b|}$ is a point of type 2. We have already seen that some kind of branching occurs at this point, since the paths from $\zeta_{a,|a-b|}$ to the three points $\zeta_{0,1}$, ζ_a and ζ_b share only the starting point.

We will now investigate the branches meeting at the point $\zeta_{a,|a-b|}$. Recall that the residue field $\widetilde{K}$ is defined as

$$\widetilde{K} = \mathcal{O}_K/\mathfrak{m}_K = \big\{x \in K : |x| \leq 1\big\}/\big\{x \in K : |x| < 1\big\}.$$

For simplicity, set $r = |a-b|$. Consider the map

$$\begin{aligned} D(a,r) &\longrightarrow \widetilde{K} \\ c &\longmapsto \frac{a-c}{a-b} + \mathfrak{m}_K, \end{aligned}$$

where $\frac{a-c}{a-b} + \mathfrak{m}_K$ denotes the residue class in $\widetilde{K}$. Note that $\frac{a-c}{a-b}$ lies in $\mathcal{O}_K$, since $|a-c| \leq r = |a-b|$.

This map is obviously surjective, since for every $x \in \mathcal{O}_K$ the element $a - x(a-b)$ lies in $D(a,r)$ and maps to $x + \mathfrak{m}_K$. On the other hand, two points c and c' are mapped to the same residue class in $\widetilde{K}$ if and only if $|c - c'| < r$, which is equivalent to the fact that $\zeta_{c,s} = \zeta_{c',s}$ for all s in some interval $[t, |a-b|]$ of positive length. Hence c and c' are mapped to the same residue class in $\widetilde{K}$ if and only if the paths $[\zeta_{c,0}, \zeta_{0,1}]$ and $[\zeta_{c',0}, \zeta_{0,1}]$ meet in some point $\zeta_{c,t}$ for $t < r$ and travel together from there on, passing through $\zeta_{a,r}$ on their way to the Gauss point.

Hence we find a bijection between $\widetilde{K}$ and the set of equivalence classes of paths from $\zeta_{c,0}$ to $\zeta_{a,r}$ for $c \in D(a,r)$, where we call two paths equivalent if they coincide on an interval of non-zero length. These equivalences of paths are called branches.

If $\zeta_{a,r}$ is equal to the Gauss point, i.e. if $r = 1$, this gives a bijection between the set of branches in the Gauss point and the residue field $\widetilde{K}$. If $r < 1$, then there is one branch missing, which is the branch from $\zeta_{a,r}$ to the Gauss point. In this case we can identify the set of branches meeting in $\zeta_{a,r}$ with $\{\infty\} \cup \widetilde{K} = \mathbb{P}^1(\widetilde{K})$, i.e. with the projective line over the residue field.

Since we assumed that K is algebraically closed, the residue field $\widetilde{K}$ is infinite. Hence there is infinite branching in $\mathcal{M}(T)$ around every point of type 2.

We have not yet seen all the points in $\mathcal{M}(T)$. For every sequence of discs $D(a_n, r_n)$ in $D(0, 1)$ such that

$$D(a_{n+1}, r_{n+1}) \subset D(a_n, r_n) \quad \text{for all } n \geq 1$$

we define

$$\zeta_{(a_n,r_n)_n}(f) = \inf_n \sup_{x \in D(a_n,r_n)} \left| f(x) \right| = \inf_n \zeta_{a_n,r_n}(f).$$

This is also a point in $\mathcal{M}(T)$. If the intersection of the discs $D(a_n, r_n)$ is not empty, it is a point or a disc, and then this limit seminorm is nothing new, but a seminorm of type 1, 2 or 3. However, if the intersection of the discs $D(a_n, r_n)$ is empty, the corresponding point in $\mathcal{M}(T)$ is new, and we call it a *point of type* 4.

Note that since the sequence of discs is decreasing, the sequence of radii $(r_n)_n$ is a decreasing sequence of non-negative real numbers. If $\inf_n r_n$ is equal to zero, then the sequence $(a_n)_n$ of centers is a Cauchy sequence, which implies that the intersection of the discs $D(a_n, r_n)$ is equal to the limit of $(a_n)_n$. Hence, if the intersection of the discs $D(a_n, r_n)$ is empty, leading to a point of type 4, the infimum of the radii must be positive! Points of type 4 only exist if the field K is not spherically complete. Here, spherically complete means that every nested sequence of closed discs has non-empty intersection.

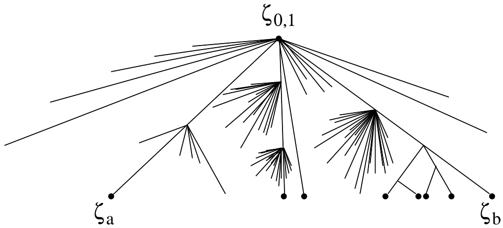
We can now describe the topological space $\mathcal{M}(T)$ as follows.

Theorem 7

(i) *Every point in* $\mathcal{M}(T)$ *is of type* 1, 2, 3 *or* 4. *If* K *is spherically complete only points of type* 1, 2 *or* 3 *occur.*
(ii) *The set of points of type* 1 *is dense in* $\mathcal{M}(T)$, *and the set of points of type* 2 *is dense.*
(iii) $\mathcal{M}(T)$ *is a compact Hausdorff space and uniquely path-connected.*
(iv) *We can visualize* $\mathcal{M}(T)$ *as a tree which has infinitely many branches growing out of every point contained in a dense subset of any line segment. Such a structure is an example of an* $\mathbb{R}$*-tree, not a combinatorial tree. More precisely, we can visualize the Gauss point as a root of the tree. The branches emanating from it are in bijection with the residue field* $\widetilde{K}$ *of* K. *The set of type* 2 *points is dense. At every point of type* 2, *the tree branches off again so that the set of branches in this point is in bijective correspondence with* $\mathbb{P}^1(\widetilde{K})$. *The leaves are the points of type* 1 *or of type* 4.

A detailed discussion of the Berkovich unit disc where these statements are proved can be found in [1], Chap. 1.

We conclude this section with a picture of a part of the Berkovich unit disc in the situation of a spherically complete field. It is slightly misleading in the sense that it does not capture the effect of infinite branching occurring in a dense set of branch points.



4 The Projective Line

We can define the Berkovich projective line $(\mathbb{P}^1)^{an}$ as the result of gluing two unit discs together along the automorphism $\gamma \mapsto \gamma^{-1}$ of the annulus $\{\gamma \in \mathcal{M}(T) : \gamma(z) = 1\}$. Note that a point ζ_a of type 1 is contained in the complement $\{\gamma \in \mathcal{M}(T) : \gamma(z) < 1\}$ of the annulus if and only if $|a| < 1$. Hence the paths $[\zeta_a, \zeta_{0,1}]$ of such points all lie on the same branch emanating from the Gauss point.

Therefore we can think of $(\mathbb{P}^1)^{an}$ as the unit disc $\mathcal{M}(T)$ with an additional branch attached to the Gauss point.

There is an alternative description which is reminiscent of the Proj construction in algebraic geometry.

Proposition 8 *Two seminorms γ and δ on the polynomial ring $K[X, Y]$ are called equivalent if there exists a constant $C \in \mathbb{R}_{>0}$ such that for every homogeneous polynomial f of degree d we have $\gamma(f) = C^d \delta(f)$.*

The analytic projective line $\mathbb{P}^{1an}$ can be identified with the set of all equivalence classes of multiplicative seminorms on $K[X, Y]$ which extend the absolute value on K and do not vanish on the maximal ideal (X, Y).

Proof Every point in the Berkovich unit disc $\mathcal{M}(T)$, i.e. every non-zero and bounded multiplicative seminorm γ on T induces a multiplicative seminorm γ^* on $K[X, Y]$ by setting

$$\gamma^*\left(\sum_{m,n} c_{m,n} X^m Y^n\right) = \gamma\left(\sum_{m,n} c_{m,n} z^m\right).$$

This map is injective on $\mathcal{M}(T)$, and its image consists of all equivalence classes of seminorms γ^* with $\gamma^*(X) \leq \gamma^*(Y)$. If we take another copy of the Berkovich unit disc and glue it to the given one along the unit circle, we can write down an analogous map which also captures the seminorms satisfying $\gamma^*(X) \geq \gamma^*(Y)$. More details can be found in [1], Chap. 2. □

The analytic projective line $(\mathbb{P}^1)^{an}$ is compact and simply connected.

5 Berkovich Spaces

In this section the ground field K is complete with respect to a non-Archimedean non-trivial absolute value. We generalize the constructions of the previous sections from T to other Banach algebras. A commutative Banach K-algebra $(A, \| \,\|)$ is a commutative K-algebra A together with a submultiplicative norm $\| \,\|$ on the K-vector space A such that A is complete with respect to the induced metric. Hence the norm map $\| \,\| : A \to \mathbb{R}_{\geq 0}$ satisfies the following properties:

(i) $\|f\| = 0$ if and only if $f = 0$.
(ii) $\|af\| = |a|\|f\|$ for $a \in K$ and $f \in A$.
(iii) $\|f + g\| \leq \|f\| + \|g\|$ for all $f, g \in A$.
(iv) $\|fg\| \leq \|f\|\|g\|$ for all $f, g \in A$.

Definition 9 Let $(A, \| \,\|)$ be a commutative Banach K-algebra. The Berkovich spectrum $\mathcal{M}(A)$ is defined as the set of non-zero multiplicative seminorms on A bounded by the norm $\| \,\|$. It is endowed with the topology of pointwise convergence, i.e. with the weakest topology such that for every element $a \in A$ the evaluation map $\gamma \mapsto \gamma(a)$ on A is continuous.

In a natural way, every bounded morphism $A \to B$ between Banach algebras over K induces by composition a continuous map $\mathcal{M}(B) \to \mathcal{M}(A)$ of the associated spectra.

Note that for every multiplicative seminorm ρ on a commutative K-algebra over a non-Archimedean field K, the ordinary triangle inequality implies the ultrametric triangle inequality. Namely, if $\rho(x) \leq \rho(y)$, we deduce for every natural number n

$$\rho(x + y)^n \leq \sum_{\nu=0}^{n} \rho\left(\binom{n}{\nu}\right) \rho(x)^\nu \rho(y)^{n-\nu} \leq (n + 1)\rho(y)^n,$$

which implies our claim after taking the limit of the n-th roots for $n \to \infty$.

Theorem 10 ([2], Theorem 1.2.1) *If $(A, \| \,\|)$ is a non-zero Banach algebra, its Berkovich spectrum $\mathcal{M}(A)$ is a nonempty compact Hausdorff space.*

Definition 11 Fix $n \in \mathbb{N}$ and let $r = (r_1, \ldots, r_n)$ be a family of positive real numbers. Put $z = (z_1, \ldots, z_n)$, write $z^I = z_1^{i_1} \ldots z_n^{i_n}$ for any multi-index $I = (i_1, \ldots, i_n) \in \mathbb{N}_0^n$, and set $|I| = i_1 + \cdots + i_n$. We define the generalized Tate algebra as

$$K\left\{r_1^{-1} z_1, \ldots, r_n^{-1} z_n\right\} = \left\{f = \sum_{I=(i_1,\ldots,i_n)\in\mathbb{N}_0^n} c_I z^I : |c_I| r^I \to 0 \text{ as } |I| \to \infty\right\}.$$

We endow $K\{r_1^{-1} z_1, \ldots, r_n^{-1} z_n\}$ with the following variant of the Gauss norm:

$$\left\| \sum_I c_I z^I \right\| = \max_I |c_I| r^I.$$

The algebra $K\{r_1^{-1} z_1, \ldots, r_n^{-1} z_n\}$ is a K-Banach algebra with respect to the multiplicative Gauss norm. If all r_i are equal to 1, we write

$$T_n = K\{z_1, \ldots, z_n\}$$

and call this algebra *Tate algebra* over K. Then the Tate algebra T discussed in Sect. 2 agrees with T_1. The Berkovich spectrum $\mathcal{M}(T_n)$ is the Berkovich version of the unit polydisc in n-space. Note that in higher dimensions there is no explicit description of $\mathcal{M}(T_n)$ in terms of types as for $\mathcal{M}(T)$.

A Banach algebra A is called a *K-affinoid algebra* if there exists a surjective K-algebra homomorphism

$$\varphi : K\left\{r_1^{-1}z_1, \ldots, r_n^{-1}z_n\right\} \longrightarrow A$$

for some n and $(r_1, \ldots, r_n)$, such that the residue norm $\|f\|_A = \inf_{\varphi(g)=f} \|g\|$ on A is equivalent to its Banach algebra norm.

If we can take all $r_i = 1$, i.e. if A is a suitable quotient of a Tate algebra T_n, then A is called *strictly K-affinoid*. Berkovich spectra of affinoid algebras are the building blocks of Berkovich analytic spaces in a similar way as schemes are made up from spectra of rings.

Let us now define Berkovich affine space $(\mathbb{A}^k)^{an}$. If $r = (r_1, \ldots, r_k) \in \mathbb{R}^k_{>0}$ and $s = (s_1, \ldots, s_k) \in \mathbb{R}^k_{>0}$ satisfy $r_i < s_i$ for all $i = 1, \ldots, k$, the identity map

$$K\left\{s_1^{-1}z_1, \ldots, s_k^{-1}z_k\right\} \to K\left\{r_1^{-1}z_1, \ldots, r_k^{-1}z_k\right\}$$

is bounded, hence it defines a continuous map

$$\mathcal{M}\left(K\left\{r_1^{-1}z_1, \ldots, r_k^{-1}z_k\right\}\right) \to \mathcal{M}\left(K\left\{s_1^{-1}z_1, \ldots, s_k^{-1}z_k\right\}\right),$$

which is easily seen to be injective. The topological space $(\mathbb{A}^k)^{an}$ is then defined as the nested union of all $\mathcal{M}(K\{r_1^{-1}z_1, \ldots, r_k^{-1}z_k\})$.

Lemma 12 *The space $(\mathbb{A}^k)^{an}$ can be identified with the set of all multiplicative seminorms on $K[z_1, \ldots, z_k]$ extending the absolute value on K, which is endowed with the topology of pointwise convergence.*

Proof The restriction of a point in $\mathcal{M}(K\{r_1^{-1}z_1, \ldots, r_k^{-1}z_k\})$ to the polynomial ring is a multiplicative seminorm which extends the absolute value on K. Conversely, given any such seminorm γ, we put $r_i = \gamma(z_i)$. Then γ can be extended to a bounded multiplicative seminorm on $K\{r_1^{-1}z_1, \ldots, r_k^{-1}z_k\}$ in a natural way by writing an infinite series as a limit of polynomials. □

Similarly, the analytification X^{an} of every affine algebraic variety

$$X = \operatorname{Spec} K[z_1, \ldots, z_k]/\mathfrak{a}$$

can be identified with the set of multiplicative seminorms on the coordinate ring $K[z_1, \ldots, z_k]/\mathfrak{a}$ extending the absolute value on K.

Since every scheme Z of finite type over K is glued together from affine schemes X as above, we can define the analytification Z^{an} by gluing the spaces X^{an}. The resulting GAGA functor associates to every scheme Z of finite type over K a topological space Z^{an}. It has the following properties.

Theorem 13 ([2], Theorem 3.4.8)

(i) *Z is connected if and only if Z^{an} is path-connected.*
(ii) *Z is separated if and only if Z^{an} is Hausdorff.*
(iii) *Z is proper if and only if Z^{an} is (Hausdorff and) compact.*

Berkovich spaces also have a topological dimension which is compatible with the algebraic dimension under analytification.

The topological nature of Berkovich spaces is an important field of research. One-dimensional spaces are quite well understood (for details see [2], Sect. 4). In higher dimensions the situation is less clear. Unlike smooth complex analytic spaces, Berkovich spaces are in general not locally isomorphic to polydiscs. Nevertheless, Berkovich showed in [4] and [5] that smooth analytic spaces are locally contractible. Using tools from model theory, Hrushovski and Loeser [16] proved that for any quasi-projective algebraic variety Z over K the analytification Z^{an} is locally contractible and admits a strong deformation retraction onto a closed subset which is homeomorphic to a simplicial complex.

So far, we have only discussed Berkovich spaces as topological spaces, without really discussing analysis. Berkovich's analytic spaces are also equipped with a K-affinoid atlas which is used to define the structure sheaf of analytic functions. Details can be found in [3], in the Bourbaki talk [11] and in the introductory papers [21] and [10].

6 Embedding Buildings in Analytic Spaces

The goal of this section is to show how Bruhat-Tits buildings can be embedded in Berkovich spaces. In this section, we denote by K a *complete, discretely valued field with a perfect residue field* and a non-trivial absolute value. We fix a semisimple algebraic group G over K. For example, G could be a classical group like SL_n, PGL_n, Sp_{2n} or SO_n over K. Since K is not algebraically closed, there are also non-split groups to consider. These are algebraic groups such that the maximal torus over the algebraic closure is not defined over the field K.

We can associate to G its Bruhat-Tits building $\mathfrak{B}(G, K)$. This is a metric space which is a product of simplicial complexes. Moreover, it carries a continuous $G(K)$-action.

The space $\mathfrak{B}(G, K)$ can be defined by gluing a family of real vector spaces which are called apartments. These apartments are the real cocharacter spaces of the maximal tori in G. The gluing process is based on deep results by Bruhat and Tits [7, 8]. A nice introduction to Bruhat-Tits buildings and their application in representation theory can be found in Schneider's survey paper [18].

Let us first consider the group SL_2 over K. The Bruhat-Tits building $\mathfrak{B}(SL_2, K)$ has an explicit description as a set of equivalence classes of norms on the K-vector space K^2 satisfying the ultrametric triangle inequality. In this guise, $\mathfrak{B}(SL_2.K)$ was investigated by Goldman and Iwahori [12] before Bruhat and Tits developed their general theory.

Definition 14

(i) A map $\| \, \| : K^2 \to \mathbb{R}_{\geq 0}$ is a (non-Archimedean) norm on K^2 if the following three conditions hold:

- $\|v\| = 0$ if and only if $v = 0$.
- $\|av\| = |a| \|v\|$ for $v \in K^2$ and $a \in K$.
- $\|v + w\| \leq \max(\|v\|, \|w\|)$ for $v, w \in K^2$.

(ii) The norm $\| \, \|$ is called diagonalizable if there exists a basis v, w of K^2 such that for all $a, b \in K$ we have

$$\|av + bw\| = \max\big(|a| \|v\|, |b| \|w\|\big).$$

In this case we say that $\| \, \|$ is diagonalizable with respect to the basis $\{v, w\}$.

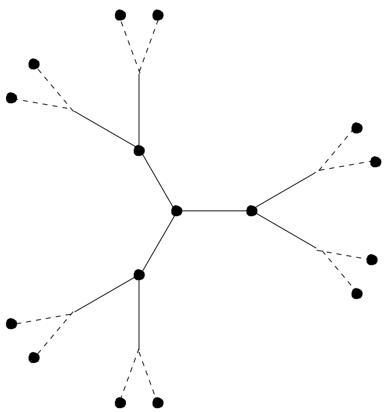
(iii) Two norms $\| \, \|$ and $\| \, \|'$ on K^2 are equivalent, if there exists a constant $c \in \mathbb{R}_{\geq 0}$ such that $\|v\| = c\|v\|'$ for all $v \in K^2$.

Note that if K is locally compact, then every norm on K^2 is diagonalizable with respect to a suitable basis.

Definition 15 Assume that K is locally compact. The Bruhat-Tits building $\mathfrak{B}(SL_2, K)$ is defined as the set of all equivalence classes of norms on K^2.

$\mathfrak{B}(SL_2, K)$ carries the topology of pointwise convergence on K^2 and a natural $SL_2(K)$-action given by $\| \, \| \mapsto \| \, \| \circ g^{-1}$ for all $g \in SL_2(K)$.

The building $\mathfrak{B}(SL_2, K)$ can be seen as a non-Archimedean analog of the complex upper half plane. It is a tree in the usual sense, i.e. a graph without cycles. Let q be the cardinality of the residue field of K. The tree $\mathfrak{B}(SL_2, K)$ is infinite and regular of valency $q + 1$, i.e. $q + 1$ edges meet in every vertex. If the residue field is $\mathbb{F}_2$, it looks like this (with branching infinitely continued):



The apartments of $\mathfrak{B}(SL_2, K)$ correspond to the doubly infinite geodesics in this tree. For each apartment A there exists a basis $\beta = \{v, w\}$ of K^2 such that $A = A(\beta)$

consists of all equivalence classes of norms which are diagonalizable with respect to β.

Using this description of the buildings in terms of norms, we can define an embedding of $\mathfrak{B} = \mathfrak{B}(SL_2, K)$ in the Berkovich projective line $(\mathbb{P}^1)^{an}$ over a non-Archimedean extension field L of K which is complete and algebraically closed.

Let $\beta = \{e_1, e_2\}$ be the canonical basis of K^2. Then we define a map

$$\vartheta_A : A(\beta) \longrightarrow \left(\mathbb{P}^1\right)^{an}$$

as follows: If $\| \ \|$ is a representative of the norm class $x \in A(\beta)$, then we define a seminorm on $L[X, Y]$ by

$$\sum_{m,n} c_{m,n} X^m Y^n \mapsto \max_{m,n} |c_{m,n}| \|e_1\|^m \|e_2\|^n .$$

The point $\vartheta_A(x) \in (\mathbb{P}^1)^{an}$ is defined as the equivalence class of this multiplicative seminorm. This class is independent of the choice of the representative of x. Moreover, we can recover $\| \ \|$ from the above formula by looking at the value of the induced seminorm on X and Y, hence ϑ_A is injective on $A(\beta)$.

Now we extend ϑ_A in an equivariant way to the whole tree $\mathfrak{B}$. For every basis $\beta' = \{v, w\}$ of K^2, there exists an element $g \in GL_2(K)$ such that $g(e_1) = v$ and $g(e_2) = w$. Hence, the action of g^{-1} on $\mathfrak{B}$, which is given by $\| \ \| \mapsto \| \ \| \circ g$, maps the apartment $A(\beta')$ to the apartment $A(\beta)$. On the other hand, g acts in a natural way on $(\mathbb{P}^1)^{an}$, sending the equivalence class of the seminorm γ on $L[X, Y]$ to the class of $\gamma \circ g^{-1}$, where g^{-1} is the algebra automorphism of $L[X, Y]$ given by the matrix g^{-1} in terms of the basis (X, Y) of the degree one part.

Then we define $\vartheta_{A(\beta')} : A(\beta') \to (\mathbb{P}^1)^{an}$ as the composition

$$A\left(\beta'\right) \xrightarrow{\| \ \| \circ g} A(\beta) \xrightarrow{\vartheta_A} \left(\mathbb{P}^1\right)^{an} \xrightarrow{\| \ \| \circ g^{-1}} \left(\mathbb{P}^1\right)^{an}.$$

This defines an $SL_2(K)$-equivariant injection of $\mathfrak{B}$ into $(\mathbb{P}^1)^{an}$.

Note that the image of the Bruhat-Tits tree only meets points of type 2 or 3, but none of the classical points of type 1.

The embedding in this example can be generalized. In fact, any Bruhat-Tits building $\mathfrak{B}(G, K)$ can be embedded in the analytic group G^{an} and also in suitable generalized analytic flag varieties. In the case of non-classical groups there is no explicit description of the building in terms of norms. Hence, a more intrinsic approach is necessary.

For split groups, it was shown by Berkovich in [2], Sect. 5 how to realize buildings in analytic group varieties and analytic flag spaces. The paper [17] contains a generalization to the non-split case. Let us outline the construction in the general case.

Let G be a semisimple algebraic group over the field K. Under the hypotheses on the ground field K stated at the beginning of this section, the Bruhat-Tits building of all base changes of G to non-Archimedean extension fields L exists. We denote it by $\mathfrak{B}(G, L)$. For every such extension field the group $G(L)$ acts continuously on $\mathfrak{B}(G, L)$.

Then, an embedding $\vartheta : \mathfrak{B}(G, K) \hookrightarrow G^{an}$ is defined as follows: First one shows [17], Theorem 2.1, that for every point $x \in \mathfrak{B}(G, K)$ there exists a unique K-affinoid subgroup G_x of G^{an} satisfying the following condition: for every non-Archimedean field extension L/K, the group $G_x(L)$ is the stabilizer in $G(L)$ of the image of x under the injection $\mathfrak{B}(G, K) \to \mathfrak{B}(G, L)$. Secondly, $\vartheta(x)$ is defined as the (unique) Shilov boundary point of G_x. Hence, if $G_x = \mathcal{M}(A_x)$ for a K-affinoid algebra A_x (see Definition 9), the point $\vartheta(x)$ is maximal with respect to evaluation on functions of A_x. The existence of a Shilov boundary consisting of finitely many points follows from general results by Berkovich (see [2], 2.4.5). It is a delicate fact that the Shilov boundary of the affinoid group G_x consists of one point only (this is proven in [17], Proposition 2.4).

The embedding $\vartheta : \mathfrak{B}(G, K) \to G^{\mathrm{an}}$ is useful to compactify the Bruhat-Tits building $\mathfrak{B}(G, K)$. For this purpose, we choose a parabolic subgroup P of G. Then the flag variety G/P is complete. By Theorem 13, the associated Berkovich space $(G/P)^{\mathrm{an}}$ is compact. Hence we can map the building to a compact space by the composition

$$\vartheta_P : \mathfrak{B}(G, K) \xrightarrow{\vartheta} G^{\mathrm{an}} \longrightarrow (G/P)^{\mathrm{an}}.$$

The map ϑ_P is by construction $G(K)$-equivariant and it depends only on the $G(K)$-conjugacy class of P: we have $\vartheta_{gPg^{-1}} = g\vartheta_P g^{-1}$ for any $g \in G(K)$.

However, ϑ_P may not be injective. By the structure theory of semisimple groups, there exists a finite family of normal closed subgroups G_i of G (each of them quasi-simple), such that the product morphism

$$\prod_i G_i \longrightarrow G$$

is a central isogeny. Then the building $\mathfrak{B}(G, K)$ can be identified with the product of all $\mathfrak{B}(G_i, K)$. If one of the factors G_i is contained in P, then the factor $\mathfrak{B}(G_i, K)$ is squashed down to a point in the analytic flag variety $(G/P)^{\mathrm{an}}$.

However, if we remove from $\mathfrak{B}(G, K)$ all factors $\mathfrak{B}(G_i, K)$ such that G_i is contained in P, then we obtain a product of buildings which is mapped injectively into $(G/P)^{\mathrm{an}}$ via ϑ_P.

Theorem 16 *Assume that the field K is locally compact, and that no almost simple factor G_i of G is contained in P. Then the closure $\overline{\mathfrak{B}}(G, K)$ of the image of $\mathfrak{B}(G, K)$ under ϑ_P is a compact space containing the building as an open dense subset. $\overline{\mathfrak{B}}(G, K)$ is a union of Bruhat-Tits buildings. The continuous $G(K)$-action on $\mathfrak{B}(G, K)$ extends in a natural way to a continuous $G(K)$-action on $\overline{\mathfrak{B}}(G, K)$.*

In [17], Theorem 4.1 we describe in detail which Bruhat-Tits buildings appear on the boundary.

This approach to compactifications has applications to the structure theory of the group $G(K)$, as the following Theorem shows, which is proven in [17], Proposition 4.20.

Theorem 17 *Fix an apartment A in $\mathfrak{B}(G,K)$ corresponding to the maximal split torus T of G. Let N be the normalizer of T in G. Moreover, denote by $\overline{A}$ the closure of A in $\overline{\mathfrak{B}}(G,K)$. For every $x \in \overline{A}$, we denote by P_x its stabilizer under the $G(K)$-action on $\overline{\mathfrak{B}}(G,K)$. Then for every choice of points x, y in the compactified apartment $\overline{A}$ we have the following generalized Bruhat decomposition of $G(K)$:*

$$G(K) = P_x N(K) P_y.$$

7 Some More Applications

During the last two decades, Berkovich analytic spaces have become an ubiquitous tool in non-Archimedean arithmetic geometry. We only name very few applications. In particular, he list of references we give here is far from complete.

Berkovich developed an étale cohomology theory for his analytic spaces and used it to prove a conjecture of Deligne on vanishing cycles. Harris and Taylor used étale cohomology of Berkovich spaces in their proof of the local Langlands conjecture for GL_n [14]. Berkovich spaces also play a vital role in non-Archimedean dynamics, see e.g. the book on the Berkovich projective line by Baker and Rumely [1]. Moreover, they can be used to develop a non-Archimedean substitute of the differential geometry at the infinite places in Arakelov theory. On curves, such a theory was developed by Thuillier [22], and there are results in higher dimensions by Chambert-Loir and Ducros [9]. Berkovich spaces are also useful for questions of diophantine geometry over function fields, as in Gubler's proof of the Bogomolov conjecture [13].

There exists also a related notion of analytic spaces incorporating all valuations on a Banach algebra, not only the ones of rank one. This was developed by Huber [15]. Recently, Peter Scholze's theory [19] of perfectoid spaces was formulated in the framework of Huber's analytic spaces. Perfectoid spaces provide a very useful framework for going back and forth between characteristic zero and positive characteristic. This leads to new results on the weight-monodromy conjecture.

Acknowledgements I thank Vladimir Berkovich, Maria Angelica Cueto, Gabriele Nebe, Nahid Shajari and Till Wagner for numerous helpful comments and corrections.

References

1. Baker, M., Rumely, R.: Potential Theory and Dynamics over the Berkovich Projective Line. Mathematical Surveys and Monographs, vol. 159. American Mathematical Society, Washington (2010)
2. Berkovich, V.G.: Spectral Theory and Analytic Geometry over Non-Archimedean Fields. American Mathematical Society, Washington (1990)
3. Berkovich, V.G.: Etale cohomology for non-Archimedean analytic spaces. Publ. Math. IHES **78**, 5–161 (1993)
4. Berkovich, V.G.: Smooth p-adic analytic spaces are locally contractible. Invent. Math. **137**, 1–84 (1999)
5. Berkovich, V.G.: Smooth p-adic analytic spaces are locally contractible II. In: Geometric Aspects of Dwork Theory, pp. 293–370. De Gruyter, Berlin (2004)
6. Bosch, S., Güntzer, U., Remmert, R.: Non-Archimedean Analysis. Springer, Berlin (1984)
7. Bruhat, F., Tits, J.: Groupes réductifs sur un corps local. I. Données radicielles valuées. Publ. Math. IHÉS **41**, 5–251 (1972)

8. Bruhat, F., Tits, J.: Groupes réductifs sur un corps local. II. Schémas en groupes. Existence d'une donnée radicielle valuée. Publ. Math. IHÉS **60**, 197–376 (1984)
9. Chambert-Loir, A., Ducros, A.: Formes différentielles réelles et courants sur les espaces de Berkovich. arXiv:1204.6277
10. Conrad, B.: Several approaches to non-Archimedean analytic geometry. In: P-Adic Geometry. Lectures from the 2007 Arizona Winter School. American Mathematical Society, Washington (2008)
11. Ducros, A.: Espaces analytiques p-adiques au sens de Berkovich. Exposé 958. In: Séminaire Bourbaki, Bd. 2005/2006. Société Mathématique de France (2007)
12. Goldman, O., Iwahori, N.: The space of p-adic norms. Acta Math. **109**, 137–177 (1963)
13. Gubler, W.: The Bogomolov conjecture for totally degenerate Abelian varieties. Invent. Math. **169**, 377–400 (2007)
14. Harris, M., Taylor, R.: The Geometry and Cohomology of Some Simple Shimura Varieties. Annals of Math. Studies, vol. 151. Princeton University Press, Princeton (2001)
15. Huber, R.: A generalization of formal schemes and rigid analytic varieties. Math. Z. **217**, 513–551 (1994)
16. Hrushovski, E., Loeser, F.: Non-Archimedean tame topology and stably dominated types. arXiv:1009.0252
17. Rémy, B., Thuillier, A., Werner, A.: Bruhat-Tits theory from Berkovich's point of view. I. Realizations and compactifications of buildings. Ann. Scient. ENS **43**, 461–554 (2010)
18. Schneider, P.: Gebäude in der Darstellungstheorie über lokalen Zahlkörpern. Jahresber. Dtsch. Math.-Ver. **98**, 135–145 (1996)
19. Scholze, P.: Perfectoid spaces. arXiv:1111.4914
20. Tate, J.: Rigid analytic spaces. Invent. Math. **12**, 257–289 (1971)
21. Temkin, M.: Introduction to Berkovich analytic spaces. arXiv:1010.2235
22. Thuillier, A.: Théorie du potentiel sur les courbes en géométrie analytique non archimédienne. Application à la théorie d' Arakelov. Thèse de l'Université de Rennes **1** (2005)

Annette Werner hat in Münster Mathematik studiert und im Jahr 1995 promoviert. Nach ihrer Habilitation im Jahr 2000 war sie wissenschaftliche Assistentin und Heisenberg-Stipendiatin. Zu Beginn des Sommersemesters 2004 wurde sie auf eine Professur an der Universität Siegen berufen, zum Wintersemester 2004/05 wechselte sie auf einen Lehrstuhl für Algebraische Geometrie und Algebra an der Universität Stuttgart. Seit dem Wintersemester 2007/08 ist sie Professorin an der Goethe-Universität Frankfurt. Ihr Arbeitsgebiet ist die arithmetische Geometrie, insbesondere die nicht-archimedische Geometrie.

Jahresber Dtsch Math-Ver (2013) 115:21–45
DOI 10.1365/s13291-013-0061-7

HISTORICAL ARTICLE

Hans Grauert (1930–2011)

Alan Huckleberry

Published online: 28 March 2013

Hans Grauert, 1983, private photo

AMS Subject Classification 32C15 · 32C55 · 32E40 · 32F10 · 32J25 · 32L99 · 32Q15 · 32Q15 · 01A61

Hans Grauert passed away at the age of 81 in September of 2011. His contributions to mathematics have and will be used with great frequency, and in particular for

A. Huckleberry (✉)
Ruhr Universität, Bochum, Germany
e-mail: ahuck@cplx.rub.de

A. Huckleberry
Jacobs University, Bremen, Germany

this reason will not be forgotten. All of us in mathematics stand on the shoulders of giants. For those of us who work in and around the area of *complex geometry* one of the greatest giants of the second half of the 20th century is Hans Grauert.

Specialists in the area know this, but even for them his collected works, annotated with the much appreciated help of Yum-Tong Siu, should at least be kept on the bedside table. An eloquent firsthand account of the *Sturm und Drang* period in Münster can be found in Remmert's talk (an English translation appears in [28]) on the occasion of Grauert receiving the von Staudt Preis in Erlangen. More recently, on the occassion of his receiving the Cantor Medallion, we presented a sketch of the man and his mathematics (see [21, 22]). In the AMS-memorial article [23] specialists in the area, some of whom were students of Grauert, give us a closer look. In the present article we attempt to give an in-depth view, written for non-specialists, of Grauert's life in mathematics and the remarkable mathematics he contributed.

Early Surroundings

Grauert was born in 1930 in Haren, a small town near the Netherlands in the northwestern part of Germany. Many of our friends who lived as children in this region recall their wartime fears, in particular of the bombings. Münster, which, together with the neighboring city Osnabrück, was the city of the signing of the Treaty of Westfalia ending the Thirty Years' War, was to a very large extent flattened. We never heard Grauert mention any of this; instead, he often told stories about having fun playing with unspent shells after the war, something that took the sight of another great complex analyst of the same generation, Anatole Vitushkin, far away in the Soviet Union. In [21, 22] we recalled Grauert's detailed remarks about these days at his retirement dinner. In particular, he wanted to explicitly thank one of his grade school teachers for not failing him for his lack of skill in computing with numbers, informing him that soon he would be thinking in symbols and more abstractly.

Immediately after completing elementary school and his gymnasium education in nearby Meppen, Grauert began his studies in Sommersemester 1949 at the university in Mainz. In the fall of that year he transferred back to Münster, where he would go from schoolboy to one of the worldwide leading authorities in the area of several complex variables and holder of the Gauss Chair in Göttingen in a period of ten years.

Despite the destruction caused by the war (Germany was only beginning to *rise from the ashes*), Münster was one of the best places in the world to start out in complex analysis. At the leadership level Heinrich Behnke, Henri Cartan and Karl Stein were playing key roles. Among the students there were already the likes of Friedrich Hirzebruch, who began his studies in 1945, and Reinhold Remmert, who would become Grauert's lifelong friend and co-author of numerous fundamental research articles and expository monographs.

Behnke had come to Münster in 1927 as a proven specialist in the complex analysis in several variables of the time. Fundamental first results had already been discovered and proved. These include the remarkable facts about the location and nature of singularities of holomorphic functions proved by two giants of the early 20th Century, Eugenio Elie Levi and Friedrich Hartogs. To give a flavor of the times, let us summarize a bit of this mathematics.

Levi had understood that if the smooth boundary of a domain D in $\mathbb{C}^n$, $n \geq 2$, is locally defined as $\{\rho = 0\}$ with ρ being negative in the domain, then the complex Hessian $(\frac{\partial^2 \rho}{\partial z_i \partial \bar{z_j}})$ contains curvature information which determines whether or not a holomorphic function on the domain can be continued holomorphically across the boundary point in question. For example, the appropriate notion of concavity at a boundary point p with respect to this *Levi form* can be restated by requiring the existence of a holomorphic mapping F from the unit disk Δ in the complex plane to the closure of D with $F(0) = p$ and $F(\Delta \setminus \{p\}) \subset D$. If ∂D is concave at p, then every function holomomorphic on D extends holomorphically to a larger domain $\hat{D}$ which contains p in its interior.

Hartogs had understood related phenomena, the simplest example of which goes as follows. Consider a domain D, e.g., in $\mathbb{C}^2$ which can be viewed as a fiber space by the projection onto the unit disk Δ in the first variable. For some arbitrarily small neighborhood Δ' of the origin in Δ the fibers are assumed to be unit disks in the space of the second variable and otherwise are annuli with outer radius 1 and inner radius arbitrarily near 1. Then every holomorphic function on D extends holomorphically to the full polydisk $\hat{D} = \{(z_1, z_2) : |z_i| < 1,\ i = 1, 2\}$.

The so-called Cousin problems, formulated by Cousin in a special context in the late 19th century, which when positively answered are the higher-dimensional analogues of the Mittag-Leffler and Weierstrass product theorems of one complex variable, are also related to questions of analytic continuation. For example, Cousin I for a domain D in $\mathbb{C}^n$, asks for the existence of a globally defined meromorphic function on D with (locally) prescribed principal parts. This means that on an open covering $\{U_\alpha\}$ of D there are given meromorphic functions m_α which are compatible in the sense that $m_\beta - m_\alpha =: f_{\alpha\beta}$ is holomorphic on the intersection $U_{\alpha\beta}$. The question is whether or not there is a globally defined meromorphic function m on D with $m - m_\alpha$ holomorphic on U_α for every α.

The following is a connection of Cousin I to the study of analytic continuation. Let D be a domain in $\mathbb{C}^n$ with $p = 0$ in its smooth boundary and suppose that the set $\{z_1 = 0\}$ is locally in the complement of D. In order to show that some holomorphic function on D cannot be continued across p, one could try the following: Bump out D at 0 to obtain a slightly larger domain $\hat{D}$ which contains an open neighborhood U_0 of p. Define $U_1 = D$ and consider the Cousin I data for the covering $\{U_0, U_1\}$ of $\hat{D}$ of $m_1 \equiv 0$ and $m_0 = \frac{1}{z_1}$. If this has a "solution" m, then $m|D$ is an example of a holomorphic function that cannot be continued through p! We mention this here, because, as we explain below, Grauert brought this Ansatz to fruition and perfected it in its ultimate beauty.

Behnke certainly knew that several complex variables was an area ripe for development and set about building a research group for doing so. He was an active mathematician who understood where mathematics was going and where it should go. He was optimally connected to the world outside Münster. Caratheodory, Hopf, Severi and many others were his close friends. Perhaps above all, he was a remarkable organizer of all sides of our science! It must be emphasized, however, that he was fortunate! Even early on he had a group of magnificent students/assistants, three of whom we have had the honor of knowing: Peter Thullen, Friedrich Sommer and Karl

Stein. Secondly, for the seemingly innocuous reason that he had proved a small remark on circular domains which improved on an old result of Behnke, in 1931 Henri Cartan was invited by Behnke to give a few talks in Münster.

Thullen and Cartan became good friends and proved a basic result characterizing *domains of holomorphy* which possess holomorphic functions which cannot be continued across their boundaries. Thullen went on to prove a number of results, including important continuation theorems. Behnke and Thullen published their Ergebnisbericht which in particular outlines the key open problems of the time. In a series of papers which are essential for certain of Grauert's works, Kyoshi Oka solved many of these problems. Story has it that he wrote Behnke and Thullen a thank-you note for posing such interesting questions.

Friedrich Sommer, who was one of the founding fathers of the Ruhr University and who was responsible for continuing the tradition of complex analysis in Bochum, was one of the stalwarts of the Behnke group which Grauert joined.

Before the war, Behnke was still active in mathematics research, in particular with Stein who after the war became the mathematics guru of complex analysis in Münster. (See [20] for a detailed discussion of Stein's contributions.) The work of Behnke and Stein underlining approximation theorems of Runge type, and, for example, Stein's emphasis on implementing concepts from algebraic topology (he spent time in Heidelberg with Seifert) certainly influenced the young Grauert.

Not enough can be said about the importance of Henri Cartan for the *Münsteraner school of complex analysis*. The pre-war interaction indicated above was just the beginning. Despite the fact that the Nazi atrocities directly touched Cartan's family (his brother was assasinated in 1943), shortly after the war, in 1947, he accepted Behnke's invitation to visit Münster. For those of us who did not experience the horrific events of that time, it is difficult to imagine the magnitude of importance, maybe most importantly at the human level, of Cartan's reestablishing the Paris–Münster connection (see [24] for more on the importance of Cartan for postwar German mathematics). The importance for complex analysis, in particular for Hans Grauert, is discussed below.

Initial Conditions

When Grauert arrived in Münster, despite the fact that the worldly ammenities of the university were still at best minimal, Behnke had complex analysis up and running and, in a certain sense, the conditions for research were optimal. On the one hand there was Stein, a kind, modest man of enormous enthusiam and energy who had deep insight at the foundational level of, e.g., analytic sets, holomorphic mappings, etc. Certainly the Cousin problems and their relationships to domains of holomorphy had guided a big part of his thinking. As a result of research with Behnke before the war and published in 1948, he knew these were solvable on non-compact Riemann surfaces. To add a bit to a paper which he worried was otherwise too short he formulated three axioms for what are now known as *Stein manifolds* which he felt would be the correct general context for solving problems of Cousin type ([29]): Globally defined holomorphic functions separate points and give local coordinates, and given a divergent sequence $\{z_n\}$ there exists a holomorphic function f with $|f(z_n)|$ unbounded.

Stein was a hands-on craftsman and this certainly influenced the spirit of the Behnke seminar where there were lengthy naive (healthy!) discussions of examples such as $\sqrt{xy}$ (the cone singularity $z^2 - xy = 0$ which can be viewed as a 2:1 cover of $\mathbb{C}^2$ ramified only at the origin). On the other hand, the mathematics world outside Münster, particularly influenced by developments in France, had made a quantum leap in sophistication. However, Behnke made sure that Münster was not isolated.

Fritz Hirzebruch had begun his studies in Münster in 1945. He lived in Hamm where Stein also lived. We have heard that they traveled to Münster by train together often hanging on to the outer running boards with Stein making propaganda for the role of algebraic topology in complex geometry. During Hirzebruch's studies, Behnke sent him to his friend Hopf in Zurich. Hirzebruch happily reminisced about learning from Hopf about blowing up points and blowing down curves in surfaces. In fact his thesis (published in 1951) is a jewel about surfaces where this process plays a role. On another not unrelated topic, in a talk to a historical society on the Riemann-Roch theorem, Hirzebruch said that probably the most important new development for him in the early 1950's was understanding the notion of a line bundle! Just a few years later he fused a hefty portion of the new sophistication with his own ideas to prove his Riemann-Roch Theorem!! (published in 1954). For the young Münsteraner it must have been extremely motivating to see this remarkable development.

Cartan's early works, e.g., with Thullen and those on automorphism groups of domains, fit in the style of complex analysis at the time. However, Cartan not only made the leap from the classical to the post-war level of sophistication, he was one of the main figures who shaped it. Despite having formulated and proved Theorems A and B (published in 1951), which not only solve the Cousin problems on Stein spaces but put complex analysis in another world of abstraction (the distance from Münster to Paris could no longer be measured in kilometers), he remained in contact with and supported the members of the Behnke group. By the way, it was his idea to refer to these spaces as *Variété de Stein*.

Crescendo

At this point in the historical timeline Grauert entered the picture and, at certain points with the help of distinguished co-workers, took complex analysis to yet another level. Having set the stage above, we now turn to a description of representative aspects of his published works. We begin with an overview.

Grauert received his doctorate in Münster in mid-1954. His first publications appeared in 1955, the publication from his thesis in 1956. In the five or six years that followed, his contributions to mathematics were truly remarkable: a wealth of ideas, numerous basic results and simply quite a number of published pages. Disregarding research announcements (Comptes Rendus Notes), conference reports and expository articles, in this intense period he authored or co-authored (with Remmert, Andreotti and one with his student Docquier) 19 articles which covered a total of roughly 600 pages. In the three or four years after this period, when both he and Remmert were in Göttingen, they jointly wrote three basic research monographs in book form: *Analytische Stellenalgebren*, *Stein Theory* and *Coherent Analytic Sheaves*. The final versions of the latter two appeared much later. After settling in Göttingen, where he also devoted a great deal of time to his students (he guided more than 40 Ph.D. theses), Grauert continued to make important research contributions. Altogether he published more than 90 works, most of which were devoted to topics in the areas of several complex variables and complex algebraic geometry.

In a nutshell one can summarize Grauert's work as being fundamental for the foundations of the geometric side of complex analysis, particularly his early work with Remmert, and for our understanding of the multifaceted global phenomena related to Levi curvature. His solution of a certain Levi problem is just one of a number of results in this direction. There are two early works of Grauert that stand out as the peaks among many mountains: The Oka Principle (1957) and the Direct Image Theorem (1960). These and selected works in the areas indicated above are discussed in some detail in the next section.

In addition to those works which will be discussed in the next section, a number of important papers must be mentioned, e.g., that on the solution of the Mordell conjecture in the function field case. Weil mused that Manin, the algebraist, used analytic methods for this whereas Grauert, the analyst, approached it algebraically. In fact, if one looks at the paper, one immediately sees Grauert's geometric viewpoint. Other results which stand out are his construction of the versal deformation space for compact complex spaces (simultaneously with Douady) and that for deformations of isolated singularities, his basic cohomology vanishing theorem with Riemenschneider and results on conditions for the formal equivalence of neighborhoods of analytic subsets implying convergent equivalence. His work with Mülich on vector bundles on $\mathbb{P}_2$ has been extremely influential. Fundamental work on the analytic side, in particular solving $\bar{\partial}$-problems with bounded data, was carried out with his students, Ramirez and Lieb. He also wrote textbooks for basic real analysis with Lieb and for linear algebra with Grunau. In the area of several complex variables he wrote two textbooks with Fritzsche, together with Peternell and Remmert he edited and contributed several chapters to a volume of the Encyclopedia of Mathematical Sciences and wrote the three research monographs with Remmert which were mentioned above.

Grauert considered a wide range of topics. For example, one should not forget his ideas on hyperbolicity (see Demailly's comments in [23]) as well as his interests in vector bundles, deformation theory and in understanding analytic equivalence relations, a topic that had followed him since his early encounters with Karl Stein. He had a philosophical side as well which went along with his desire to understand certain kinds of physical (quantum mechanical) phenomena. It seems that he read Riemann's work having this in mind and, based on this, developed his own theory of discrete geometry. We recall his series of lectures at Notre Dame on his axiomatic approach and note that at the end of Volume II of his collected works he included several pages on this. Given that he obviously carefully polished these two volumes, it is clear that he took this subject very seriously and that it meant a great deal to him.

Comments on Selected Works

Under the given time and space constraints it is only possible to present a small sample of Grauert's works. Since he is perhaps best known for his results at the foundational level in complex geometry, those involving Levi geometry, his Oka principle and his proof of the direct image theorem, our remarks here will focus on these subjects.

Early Days

We begin with comments on certain aspects of Grauert's dissertation which underlined an important connection between complex differential geometry and complex analysis, published in [8]. This is followed by remarks on the paper where he vastly improved our foundational understanding of Stein spaces [7]. This early work emphasized the need for building the foundations of *complex spaces*. Much in this direction was accomplished in the basic paper *Komplexe Räume* of Grauert and Remmert [17] which is the third paper we review in this paragraph.

Charakterisierung der Holomorphiegebiete durch die vollständige Kählersche Metrik

When Grauert went to the ETH (1953), it was already quite fashionable to study Kähler manifolds. By definition such a manifold possesses a Hermitian metric whose imaginary part is a closed 2-form. Locally this form ω has a potential φ, i.e., $\omega = \frac{i}{2}\partial\bar{\partial}\varphi$, and the positivity of the metric translates to φ being strictly plurisubharmonic. Hodge's book, which appeared in 1941, was well known and Eckmann and Guggenheimer were busy in Zurich (Guggenheimer went to Israel in 1954) looking at more general manifolds. The fact that plurisubharmonic functions (Lelong, Oka 1942) are important in complex analysis was widely understood. Kähler himself had thought in terms of the potential function and had in fact proved that Kähler is, as mentioned above, the same thing as having a locally defined strictly plurisubharmonic potential. Much was in the air when Grauert started thinking in this direction.

At the beginning of his paper Grauert states that it is "naheliegend" to study the connection between complete Kähler metrics and domains of holomorphy D. In hindsight this is true, but he was the first to do this. For our purposes a *domain of holomorphy* is a domain in $\mathbb{C}^n$ for which there exists a function f holomorphic on D which cannot be extended to a function holomorphic on a larger manifold. In fact, given a sequence $\{z_n\}$ which converges to every boundary point, one can construct f with the property that $\lim |f(z_n)| = \infty$.

Grauert begins the article by pointing out that, given a complete Kähler metric on D and a closed analytic subset A of D, i.e., a set defined as the common 0-set of finitely many holomorphic functions, it is a simple matter to adjust the given Kähler metric appropriately to obtain a complete Kähler metric on $D \setminus A$. Since holomorphic functions extend across analytic sets which have codimension at least two, it is then immediate that there are domains with complete Kähler metrics which are not domains of holomorphy.

After making the above remark, Grauert then proves that the desired result holds if D has a $\mathbb{R}$-analytic boundary. In other words, such domains are domains of holomorphy if and only if they possess a complete Kähler metric. This result, which required substantial technical work, stimulated a great deal of further research, first of all due to the idea that the existence of a complete Kähler metric and pseudoconvexity are related. The regularity question also turned out to be of interest. For example, some years later Ohsawa showed that only a C^1-boundary is necessary ([27]).

This paper is Grauert's doctoral thesis. He profusely thanks Behnke and Eckmann. I would guess that Eckmann and Guggenheimer discussed Kählerian geometry with him, although their work exclusively dealt with compact manifolds and in particular had nothing to do with pseudoconvexity. Grauert received his degree in Münster in July of 1955, the first referee being Behnke and the second Friedrich Sommer.

Charakterisierung der holomorph vollständigen komplexen Räume

In this paper Grauert is clearly fascinated by the question of countability of the topology of complex spaces (Rado noticed this in the case of Riemann surfaces). Throughout the paper a complex space is an α-space and Grauert is thinking in terms of it locally being the graph of a multivalued holomorphic function. He does not yet have the result that every α-space is a β_n space (see our review of *Komplexe Räume* for the notation). As a result he proves his main results under assumption C which due to the later work of Grauert and Remmert just means that a complex space is locally the common 0-set of finitely many holomorphic functions on a domain in $\mathbb{C}^n$.

The *new* axiom here is that of K-Vollständigkeit, i.e., that global holomorphic functions define a map at a given point which is finite fibered near the point in question. Much work then shows that under this condition the topology is countable and finally that the n-dimensional space X is globally a ramified Riemann domain over $\mathbb{C}^n$, i.e., that there is a generically maximal rank holmorphic map $F : X \to \mathbb{C}^n$. Then, using nontrivial (but more or less classical) methods, Grauert shows that if X is holomorphically convex, then it is Stein. The main result then is an essential weakening of Stein's axioms: K-vollständig plus holomorphic convexity are equivalent to the following four axioms of Stein:

1. Countable topology.
2. Globally defined holomorphic functions give local embeddings.
3. Globally defined holomorphic functions separate points.
4. Holomorphic convexity, i.e., given a divergent sequence $\{x_n\}$ there exists a holomorphic function f with $\lim |f(z_n)| = \infty$.

Komplexe Räume

Here the authors work from the point of view of the definition of Behnke and Stein which is that a complex space X is locally the graph of a multivalued holomorphic function on $\mathbb{C}^n$. This was nothing new in the 1-dimensional case, because under the assumptions of Behnke and Stein the resulting space is smooth and locally just the graph of an algebraic function! However, in the higher dimensional case singularities arise. Members of the Behnke seminar in those years have told us that they spent great energy trying to understand $\pm\sqrt{xy}$ which is the cone defined by $z^2 - xy = 0$ over the xy-plane!

To be precise, a Behnke-Stein complex space is a Hausdorff space X which satisfies the following local condition: Every $x \in X$ is contained in an open neighborhood U which is equipped with a continous map $\varphi : U \to V$ onto an open set in $\mathbb{C}^n$ which contains a proper analytic subset A with the property that the restriction $\varphi : U \setminus \varphi^{-1}(A) \to V \setminus A$ is a proper finite covering map. In particular, φ is a local homoeomorphism and gives local holomorphic coordinates on $U \setminus \varphi^{-1}(A)$. The holomorphic functions on U are then defined to be the continuous functions which are holomorphic on $U \setminus \varphi^{-1}(A)$. One should mention in this context that one of the most quoted theorems of Grauert and Remmert is that if Y is, for example, a complex manifold which contains a proper analytic subset A and $F : X \to Y \setminus A$ is a proper unramified holomorphic map, then X can be (uniquely) realized as the complement of a proper analytic subset B in a larger complex space $\bar{X}$ so that the analytic cover $X \to Y \setminus A$ can be extended to a proper (finite) holomorphic map $\bar{X} \to Y$ with $\bar{X} \setminus B \to Y \setminus A$ being the original map.

The main goal of this paper is to show that the definition of Behnke and Stein is equivalent to that of Cartan and Serre of a *normal* complex space. To "clarify" matters Grauert and Remmert introduce a rather cumbersome notation. First, the Behnke-Stein spaces are called α-spaces. The spaces coming from Paris are called β- and β_n-spaces. The former is locally the common 0-set A of finitely many functions on a domain D in some $\mathbb{C}^n$ (of course depending on the point) and the sheaf of holomorphic functions is just the quotient $\mathcal{O}_D/\mathcal{I}_A$ of the sheaf of germs of holomorphic functions on D by the full ideal sheaf of functions which vanish on A. In modern terminology these are just called reduced complex spaces. The β_n spaces are those which are *normal* in the sense that if a meromorphic germ satisfies a monic polynomial equation with holomorphic coefficients, then it is itself holomorphic. Due to applications it was and still is important to understand the relations among these concepts. The Grauert-Remmert paper clears this up completely. Furthermore several basic results of independent interest are proved.

It is not terribly difficult to show that if the ramification on an α-space is given by a multivalued function (algebroid condition), then that space is a β_n-space. That, then, is the main theorem of the paper: α spaces are automatically algebroid. Since β_n-spaces are easily seen to be α-spaces, this now completes the circle: The Behnke-Stein spaces are exactly the normal complex spaces defined by Cartan!

Two basic results which we have not yet mentioned were proved along the way: (1) A β-space is normal if and only if the Riemann extension theorem holds. (2) The normalization of a β-space is constructed. By *Riemann extension* we mean that if A is a proper analytic subset of X and f is holomorphic on $X \setminus A$ and is locally bounded near A, then it extends to a holomorphic function on X. The normalization $\pi : \hat{X} \to X$ of a complex β-space, is a finite (proper, surjective) holomorphic map from a canonically determined normal complex space which is biholomorphic at least outside of the singular set of X. For $x \in X$ the number of points in $\pi^{-1}(x)$ is the number of local irreducible components of X at p. It is quite possible that the normalization is a homeomorphism with the only difference between $\hat{X}$ being that the structure on $\hat{X}$ is richer.

The reader should consult *Coherent Analytic Sheaves* for a more modern formulation of the above. On the other hand, this book was written in note form in Göttingen in the early to mid-1960s, i.e., not long after the original article.

Oka Principle

Grauert wrote three papers on what is now called Grauert's Oka principle [9–11]. The first two are full of deep ideas and fundamental work. The third is devoted to a statement of what was proved in the first two along with some simple applications. After giving some background on what was known when Grauert entered the picture, with the help of Cartan's formulation and streamlining we outline the statements and ingredients of proof of Grauert's results. These can be seen as proving that on a Stein space the only obstructions to solving problems of a complex analytic nature are topological.

Approximationssätze für holomorphe Funktionen mit Werten in komplexen Räumen
Holomorphe Funktionen mit Werten in komplexen Lieschen Gruppen

Analytische Faserungen über holomorph-vollständigen Räumen

The following is a simple but fundamental example of the Oka principle. Let G be a domain in $\mathbb{C}^n$ and D be a divisor on G, i.e., D is given locally on a covering $\{U_i\}$ by meromorphic functions m_i which satisfy the compatibility condition that $m_i = f_{ij} m_j$ on the intersection $U_i \cap U_j =: U_{ij}$ with the f_{ij} being nowhere vanishing holomorphic functions on the intersections $U_{ij} = U_i \cap U_j$. One might ask (the second Cousin problem) if there is a globally defined meromorphic function m on D with this divisor. In other words, m would be required to have the same poles and zeros (counting multiplicity) as the m_i in the sense that the functions $\frac{m}{m_i} =: f_i$ are holomorphic and nowhere vanishing on the U_i. This is of course the same as asking for the existence of such f_i with $f_i m_i$ being globally defined.

On the complex plane, i.e., for $D = \mathbb{C}$, the question of existence of the global function is answered in the positive by describing one such as the quotient of Weierstrass products. As a consequence of the Theorem of Behnke and Stein this even holds for every non-compact Riemann surface. On the other hand for compact Riemann surfaces, and of course for higher-dimensional compact complex manifolds, such a theorem does not hold and the fact that it does not guides many questions in the theory.

Returning to the non-compact case, if $D = \mathbb{C}^n$, the second Cousin problem can also be answered in the positive, even by using methods that are analogous to the Weierstrass products. However, it was realized early on that without further assumptions, even for domains D in $\mathbb{C}^n$, there would be no hope of solving this problem. The appropriate class of domains which appeared natural for solving this problem is the class of Stein domains or domains of holomorphy. Such is the natural domain of existence of some holomorphic function f in the sense that it cannot be extended holomorphically to any larger complex manifold. Although Stein domains are optimal from many points of view, as was realized by Oka, even on Stein domains there are obstructions to solving such problems. The point is that there might not even exist continuous functions f_i with this property. In our more modern language, the f_{ij} define a holomorphic line bundle $L(D)$ on D and the Cousin II problem has a positive solution if and only if $L(D)$ is holomorphically trivial. The problem has a continuous solution if and only if $L(D)$ is topologically trivial. Oka's basic theorem, proved in the pre-war years, states that a holomorphic line bundle on a Stein domain is holomorphically trivial if and only if it is topologically trivial. Cartan's Theorem B, or the more general theorem of Grauert (see above), immediately implies this statement on Stein spaces. It should be remarked that there are no topological obstructions to solving the additive (Cousin I) problem, i.e., that which asks for a globally defined meromorphic function with prescribed principle parts. On a Stein space it always has a positive solution.

For the reason sketched above, and for various other questions which arise in complex analysis, a vague Oka principle can be formulated: A problem on a Stein space which is formulated in complex analytic terms has a complex analytic solution if and only if it has a topological solution. This was more or less the state of the theory when Grauert entered the picture with his three papers which were published in 1957. Although it does no justice to his work, one simple-to-state consequence of Grauert's Oka principle is that on a Stein space the mapping which forgets complex structure defines an isomorphism between the categories of holomorphic and topological vector bundles. In other words, Oka's theorem holds for arbitrary rank.

Unlike the case of divisors where the transition matrices f_{ij} have values in an Abelian group and the usual cohomological technology on Stein spaces can be applied, in Grauert's non-Abelian setting classical results of the time cannot be applied. On the other hand it was certainly clear that in order to handle the Oka principle, e.g., some version of the classical Runge approximation theorem would be necessary. In fact an extremely deep version of this approximation theorem, one which involves homotopies of holomorphic and continuous maps, would be of essential importance in Grauert's theory.

Grauert opens his first paper as follows: *In the present paper functions $F(r)$ from a complex space R with values in an arbitrary complex space W will be studied.* Of course he has in mind Stein spaces; so let us assume that R is Stein and consider a Stein space $\check{R}$ which contains R. The main question is if such a "function" F can be approximated (uniformly on compact subsets) by a holomorphic function from $\check{R}$ to W. For usual functions with values in $\mathbb{C}$ the Behnke-Stein theorem was available: The approximation theorem holds if and only if R is *holomorph ausdehnbar* to $\check{R}$. This is a condition which is a bit complicated to formulate. It is a substantially weakend version of there being a continuous increasing family R_t, $0 \leq t \leq 1$, of Stein domains with $R_0 = R$ and $R_1 = \check{R}$. In any case, since this condition is already necessary and sufficient for usual functions, it was clear to Grauert that he should assume it for his more general question. This being clear, Grauert jumps to a suitable setting.

Following his notation, a Lie group bundle over a complex space R is a holomorphic fiber bundle $L^*(R, L)$ with fiber a complex Lie group L and structure group L^* contained in the group of Lie group automorphisms of L. Note that the case of $L = (\mathbb{C}^n, +)$ is that of a holomorphic vector bundle. Observe that such a bundle has the identity section, fiberwise multiplication makes sense, one has the associated Lie algebra bundle where the exponential is biholomorphic near its 0-section and cohomology concepts for the sheaf of sections make sense using the group multiplication. Note also that such a bundle is *not* a principal bundle.

Before going further we would like to simplify the notation and follow that of Cartan's paper ([3]) where he explains Grauert's work with great elegance. He simply denotes the Lie group bundle by $E \to X$ and then introduces the notion of an E-principal bundle defined by a cocycle $\{f_{ij}\}$ acting on L on the left. In this way one has the fiberwise action $F \times_X E \to F$ on the right. With this in mind one of Grauert's main theorems can be formulated as follows.

Let $\mathcal{E}_c$ be the sheaf of continuous sections and $\mathcal{E}_a$ be the sheaf of holomorphic sections of the Lie group bundle E.

Theorem *The inclusion $\mathcal{E}_a \hookrightarrow \mathcal{E}_c$ induces an isomorphism*

$$H^1(\mathcal{E}_a) \cong H^1(\mathcal{E}_c).$$

Of course one of the main issues to be handled is that of a Runge theorem. Here is Grauert's Runge theorem for beginners.

Theorem *Let R and $\check{R}$ be Stein spaces with R holomorph ausdehnbar to $\check{R}$ and let E be a Lie group bundle on $\check{R}$. Then a holomorphic section of E over R can be approximated by holomorphic sections of $\check{R}$ if and only if it can be approximated by continuous sections on $\check{R}$.*

This should be regarded as a mini-Runge theorem, because the final version must be proved in a context where homotopy is involved. Let us leave this in Grauert's language.

Theorem *Ist $\mathfrak{R}$ ein holomorph-vollständiger Raum, so gibt es zu jeder in $\mathfrak{R} \times \mathfrak{T}_1$ definierten (e, h)-Funktion mit Werten in einem Faserraum $L^*(\mathfrak{R}, L)$ eine (e, h, c)-homotope (e, h^0)-Funktion.*

Here is Cartan's formulation.

Notation (The (N, H, K)-sheaf $\mathcal{F}$):

- K is an auxilliary compact parameter space with $N \subset H \subset K$ such that N is a deformation retract of K.
- $\mathcal{F}(U)$ is the topological group of continuous sections $s(x, t) : U \times K \to E(U)$ which are the identity section for $t \in N$ and holomorphic for $t \in H$.

Theorem *If X is Stein, then*

1. *$H^0(X, \mathcal{F})$ is arcwise connected.*
2. *If U is holomorphically convex in X, the image of the restriction $H^0(X, \mathcal{F}) \to H^0(U, \mathcal{F})$ is dense.*
3. *$H^1(X, \mathcal{F}) = 0$.*

Of course we have swept a great deal of work under the table. Implementing in particular the refined Runge theorem, Grauert spends a great deal of time solving the non-Abelian Cousin problems.

Here are some of the consequences of Grauert's remarkable work.

- Every continuous section $s : X \to F$ is homotopic to a holomorphic section.
- If two holomorphic sections are homotopic in the space of continuous sections, then they are homotopic in the space of holomorphic sections.
- Every continuous isomorphism between E-principal bundles F and G is homotopic to a holomorphic isomorphism.

Note that for the last result it is important that the quotient $E_g := (G \times_X G)/E$ is a Lie group bundle and the bundle of isomorphisms from G to F is the E_g-principal bundle $(F \times_X G)/E$.

There are numerous consequences of these results, and even recently there has been a big explosion of further developments (see [6]). A big additional step, both technically and conceptually, was Gromov's h-principal ([19]).

Levi Convexity and Concavity

Grauert made numerous fundamental contributions to understanding the role of Levi-curvature in complex analysis. If for example a domain D with smooth boundary in a complex manifold is defined by a smooth function, $D = \{\rho < 0\}$, it has been known since the beginning of the 20th century that signature invariants of the complex Hessian of ρ along ∂D play an essential role in determining the complex analytic nature of D. Grauert solved the version of the *Levi problem* which states that if the appropriate curvature form is positive-definite, i.e., D is strongly pseudoconvex, then D is essentially a Stein manifold [12]. We review this basic paper here along with four other works where convexity, concavity or both are essential ingredients.

In his paper with Docquier [4] it is shown that if D is contained in a Stein manifold and is weakly pseudoconvex (in any of a variety of ways) at the boundary, then it is Stein. (Oka and others had shown this for domains in $\mathbb{C}^n$.) Andreotti and Grauert wrote two very interesting papers where concavity is involved. One can be regarded as a mixed signature version of Grauert's previously handled positive-definite case where higher cohomology spaces replace function spaces [2]. In the other they describe the structure of the field of meromorphic functions on a pseudoconcave space and show how to apply their methods to situations where the space at hand is a discrete group quotient, e.g., where the meromorphic functions arise as quotients of modular forms [1].

Finally, we review Grauert's beautiful paper *Über Modifikationen und exzeptionelle analytische Mengen* [14]. In brief, here he shows us how to use strong pseudoconvexity in the theory of compact complex spaces, in particular to settings of algebraic geometric interest. The title indicates one of the themes in the paper where he proves that a compact complex subvariety in a complex space can be blown down if and only if its normal bundle satisfies a natural curvature condition. This is just one of many other results which are proved, e.g., new ampleness criteria, Kodaira type embedding theorem, etc.

On Levi's Problem and the Imbedding of Real-Analytic Manifolds

Let us begin here with a classical situation where D is a domain with smooth boundary in $\mathbb{C}^n$. Every point $p \in \partial D$ has an open neighborhood U which is equipped with a smooth function ρ with nowhere vanishing differential so that $U \cap D = \{\rho < 0\}$. In particular, $\partial D \cap U = M$ is the smooth hypersurface $\{\rho = 0\}$. Note that the full (real) tangent space T_pM contains a unique maximal complex subspace $T_p^{CR}M$, the *Cauchy-Riemann tangent space* to M at p, which is 1-codimensional over $\mathbb{R}$. Let us say that the Levi-form $L_p(\rho)$ of the defining function ρ at p is the restriction to the CR-tangent space of the complex Hessian

$$Hess_p(\rho) = \left(\frac{\partial^2}{\partial z_i \partial \bar{z}_j}\right).$$

One can show that the signature of L is a biholomorphic invariant. If for example L is positve-definite, then supposing that $p = 0$ and that the CR-tangent space is given by $z_1 = 0$ one can introduce holomorphic coordinates (z_1, z') so that the restriction of ρ to $\{z_1 = 0\}$ is

$$\rho(z) = \left\|z'\right\|^2 + O(3).$$

Thus locally near p the CR-tangent space lies outside D and except at p is contained in the complement of the closure of D. One can think of it as a (local) supporting complex hypersurface outside D at the point p. If the Levi-form is positive-definite at every point of ∂D, one says that the domain (or its boundary) is strongly pseudoconvex. If at each point $p \in \partial D$ the Levi-form is only positive semidefinite, one just says that D is pseudoconvex. One can imagine that there is a huge difference between these two concepts, particularly if the rank of L_p is allowed to vary wildly with the point p.

In the early part of the 20th century E.E. Levi realized that if L_p is not positive-semidefinite, then every function defined and holomorphic on D near p extends holomorphically across ∂D at p. On the other hand, if it is positive-definite, locally in the coordinates used for the above normal form, the function $\frac{1}{z_1}$ is holomorphic on D near p and does not continue across the boundary. Therefore one asks if the same holds at the global level. Levi himself showed that if at some $p \in \partial D$ the Levi-form is not positive semidefinite, then every function holomorphic on D continues across ∂D. The Levi-Problem can be stated as follows:

$$D \text{ pseudoconvex} \overset{?}{\Rightarrow} D \text{is a domain of holomorphy.}$$

In other words, if D is pseudoconvex, given a divergent sequence $\{z_n\}$ one would like to prove that there exists a holomorphic function f on D with $\lim |f(z_n)| = \infty$. Recall that this property is the only one of Stein's axioms that is not automatically fulfilled for a domain in $\mathbb{C}^n$. Hence, for domains one is really asking if pseudoconvex domains are Stein with the above property being called *holomorphic convexity*.

In 1942 Oka solved the problem for pseudoconvex domains unramified over $\mathbb{C}^2$, and in 1953/54 this was extended to arbitrary dimensions independently by Oka, Bremermann and Norguet. Grauert points out that by using the Behnke-Stein theorem (limits of domains of holomorphy are domains of holomorphy) in $\mathbb{C}^n$, one needs only to prove the result for strongly pseudoconvex domains. This is not true for domains in arbitrary manifolds. For example, it is a simple matter to construct a domain D in a torus with Levi-flat boundary, i.e., the Levi-form of an appropriate boundary defining function vanishes identically (nevertheless D is pseudoconvex!), with the property that every holomorphic function on D is identically constant. Grauert has constructed much more sophisticated examples in [15], where except for a small set ∂D is strongly pseudoconvex.

Grauert states his theorem for bounded domains D with smooth boundaries in arbitrary complex manifolds:

$$\text{strongly pseudoconvex} \;\Rightarrow\; \text{holomorphically convex.}$$

Actually he proves much more: *If D is strongly pseudoconvex, then it contains finitely many pairwise disjoint maximal compact analytic subvarietes which can be blown down so that the resulting complex space is Stein*. For details see our discussion of his paper *Über Modifikationen und exzeptionelle analytische Mengen*.

Grauert's elegant proof begins by implementing his now famous bumping technique where he constructs a (finite) increasing sequence $\{D_k\}$ of domains containing D such that at each step the restriction mapping $H^\nu(D_{j+1}, \mathcal{O}) \to H^\nu(D_j, \mathcal{O})$ is surjective for all $\nu \geq 1$. The largest domain contains D as a relatively compact subset. Hence he proves that for D' sufficiently near D with $D \subset\subset D'$ the same surjectivity result holds. Actually in the final step of his proof he uses this for the sheaf of a holomorphic line bundle where the surjectivity is proved in exactly the same way.

Having achieved the above indicated surjectivity Grauert applies L. Schwartz' Fredholm theorem which states that if $\varphi : E \to F$ is a surjective, continuous linear map of Fréchet spaces and $\psi : E \to F$ is compact, then $\varphi + \psi$ has closed image of finite codimension. To apply this Grauert organizes a (finite) Leray covering $\mathcal{U}$ of D' which is refined to a Leray covering $\mathcal{V}$ of D and considers the mapping

$$\varphi : Z^q(\mathcal{U}, \mathcal{O}) \oplus C^{q-1}(\mathcal{V}, \mathcal{O}) \to Z^q(\mathcal{V}, \mathcal{O})$$

which is the sum of the restriction map R and the Čech boundary map δ. The cohomological surjectivity implies that this map is surjective. Hence, the Schwartz Theorem implies that the image of $\delta = \varphi - R$ has finite codimension. This is exactly the desired finiteness theorem.

Grauert uses this finite dimensionality to prove a result that is actually stronger than the holomorphic convexity of D: Given a boundary point x_0, he enlarges D as above so that in addition D' contains a 1-codimensional complex submanifold S which contains x_0 but is entirely contained in the complement of D. Using the finite dimensionality of the cohomology of powers F^k of the line bundle defined by S as well as its restriction to S, for k sufficiently large he finds a section s of F^k which does not vanish at x_0. Thus if h is the defining section of S in F, then $\mathfrak{s}h^k$ is a meromorphic function on D' which is holomorphic on D with a pole at x_0.

As the title of the article indicates, Grauert applies his theorem to prove that paracompact real analytic manifolds M can be embedded in Euclidean spaces of the expected dimension. For M compact this result was proved by Morrey using PDE methods slightly earlier. At the time Bruhat and Whitney had shown that M can be regarded as the set of real points of a complex manifold X, and Grauert then constructed a non-negative strictly plurisubharmonic function ρ on a neighborhood of M in X which vanishes exactly on M and otherwise has non-vanishing differential. If M is compact, then Grauert's solution to the Levi problem immediately implies that $T = \{\rho < \varepsilon\}$ is Stein and it is immediate that there is an everywhere maximal rank, injective holomorphic map of T (shrunk a bit) to some $\mathbb{C}^N$. If M is not compact, more sophisticated arguments must be used. In particular, in order to prove that there is a Stein tube one must use a generalized version of the Behnke-Stein Runge theorem which is proved in [4] (see our discussion of *Levisches Problem und Rungescher Satz*). Then Remmert's theorem for Stein manifolds yields the desired embedding.

In closing we should add that for weakly pseudoconvex domains in Stein manifolds, the Docquier-Grauert results are optimal. Strongly pseudoconvex domains in Stein spaces have been handled by Narasimhan [26] with the analogous results to those discussed above. There are numerous partial results for weakly pseudoconvex domains, also in the case where singularities play a role. However, the general situation is far from being understood.

Levisches Problem und Rungescher Satz für Teilgebiete Steinscher Mannifaltigkeiten

Here Grauert and Docquier begin by discussing nine conditions which are relevant for the study of the pseudoconvexity of a complex manifold. These are denoted by $(h, p_1, \dots, p_7, p_7^*)$ and are interrelated by a graph of implications with the condition h of holomorphic convexity being the strongest and p_7^* being the weakest. The latter condition is a weak version of the condition that a Hartogs figure cannot be mapped biholomorphically into the manifold so that its image is not relatively compact but the image of its Shilov boundary is compact. Riemann domains G which are unramified over a Stein manifold M are considered, and it is shown that if such a domain satisfies p_7^*, then it is Stein. It is therefore holomorphically convex, i.e., h is satisfied and consequently all of the conditions are fulfilled.

The basic idea of the proof is to apply Remmert's embedding theorem to embed M in some $\mathbb{C}^n$. Then, using the normal bundle of M in this embedding the domain G is thickened to an unramified domain $\hat{G}$ over $\mathbb{C}^n$ which also satisfies p_7^*. In that situation Oka's methods can be applied to $\hat{G}$ to achieve the desired result.

Again using the idea of thickening a Remmert embedding, Runge approximation theorems are proved via Oka-Weil approximation for domains unramified over $\mathbb{C}^n$. A condition for a Stein domain M to be Runge in a complex manifold $\check{M}$ (strongly simplified for our presentation) is that there is a continuous increasing family of Stein domains M_t starting at M and ending at $\check{M}$. For $t_1 < t_2$ it is proved that M_{t_1} is Runge in M_{t_2} and then by the classical Runge Theorem of Behnke and Stein it follows that $\check{M}$ is Stein.

This last mentioned result is really just a corollary of results proved in much greater generality. However, we wanted to particularly underline it, because it is exactly what is needed in [12] for proving that Grauert's tube around a *non-compact* real analytic manifold is Stein.

Théorèmes de Finitude pour la Cohomologie des Espaces Complexes

Recall that a smooth function on, e.g., a domain in $\mathbb{C}^n$ is strictly plurisubharmonic if its Levi form (complex Hessian) is positive-definite. Stein manifolds are those complex manifolds X which possess a strictly plurisubharmonic exhaustion. This is, in a certain sense, the solution of the Levi problem. In the work of Andreotti-Grauert, which we will now review, a q-Levi problem is solved. Just as the Levi problem for strongly pseudoconvex domains was handled by Narasimhan in the case of complex spaces, the context here is also for complex spaces. However, in order to explain the essential ideas, it is enough to consider the smooth case.

Let B be a bounded domain with smooth boundary ∂B in a complex manifold X. Andreotti and Grauert say that ∂B (or B) is q-pseudoconvex if the Levi form of a defining function has at least $n - q + 1$ positive eigenvalues, the case of $q = 0$ being reserved for compact manifolds. The main goal of the paper is to prove the finite dimensionality of $H^k(B, \mathcal{F})$ for $k \geq q$ where $\mathcal{F}$ is a coherent sheaf on X. For example, if $\rho : X \to \mathbb{R}^{\geq 0}$ is an exhaustion which is q-pseudoconvex outside of a compact set K which is contained in a ρ-sublevel set B, then the restriction map $H^k(X, \mathcal{F}) \to H^k(B, \mathcal{F})$ is an isomorphism. Thus the finite dimensionality for X follows and if K is empty, then $H^k(X, \mathcal{F}) = 0$ for $k \geq q$.

A domain B is strictly q-pseudoconcave if ρ is a defining function as above and $-\rho$ is strictly q-pseudoconvex. With the addition of some technicalities in the case of singular spaces and coherent sheaves which are not locally free, the finiteness and vanishing theorems hold in the concave case for $0 \leq k \leq n - q$.

It is interesting that, in order to prove these global results, the main new work needed is of a local nature! This point actually comes up in Grauert's previous paper, but since it is handled by classical methods, one tends to forget it. In that paper ∂B is strongly pseudoconvex. In a Stein coordinate chart U containing a given boundary point, Grauert constructs a bump on B to obtain a domain B_1. It is of fundamental importance that the cohomology of $U \cap B_1$ vanishes, i.e., Cartan's Theorem B for this domain! Here, in the q-pseudoconvex case, the analogous "Lemma", along with Grauert's proof idea of bumping and then using the Fredholm Theorem of L. Schwartz, yields the proof. Of course there are substantial technical preparations which must be carried out.

Roughly speaking the above mentioned Lemma amounts to doing the following. In a local coordinate chart U at a boundary point ξ_0 one chooses a transversal polydisk of dimension $n - q + 1$ so that the restriction to it of the boundary defining function is strictly plurisubharmonic. Then one thickens it to obtain a holomorphic family of such polydisks parameterized by, e.g., a polydisk of complementary dimension. Then, as in the strongly pseudoconvex case, one creates a bumped region B_1 which intersects each transversal polydisk in a strongly pseudoconvex region and which only changes B in a compact region in U. It is now necessary to prove a cohomology vanishing theorem for $V := U \cap B$ for $k \geq q$, and the authors do exactly this by viewing V as a family of $(n - q + 1)$-dimensional Stein domains. One of the main difficulties for this is proving the appropriate Runge theorem.

Algebraische Körper von automorphen Funktionen

Although the discussion here is in fact quite general, applying to any pseudoconcave complex space, the work in this paper is carried out in the special situation where modular and associated automorphic functions are playing an essential role. In fact only one example is considered, the quotient of the Siegel upper halfplane H by the modular group Γ, but as Borel later pointed out, there is a wide class of examples where the Andreotti-Grauert method applies.

A (connected) complex space X is said to be pseudoconcave if it contains a relatively compact open subset Y with the property that for every $x \in \partial Y$ there is a map $\varphi : \mathrm{cl}(\Delta) \to \mathrm{cl}(Y)$ which is holomorphic in a neighborhood of the closure of a 1-dimensional disk with image in the closure of Y with the properties that $\varphi(0) = x$ and $\varphi(\partial\Delta) \subset Y$. By thickening such "disks" Andreotti and Grauert obtain a double covering of $\mathrm{cl}(Y)$ by images of polydisks (one relatively compact in the other) so that their Shilov boundaries are contained in Y. Here the Shilov boundary of a polydisk $\Delta = \{|z_i| < 1,\ i = 1, \ldots, n\}$ is the set where $|z_i| = 1$ for all i. They then apply Siegel's method using the classical Schwarz Lemma to prove the following fact: The field $\mathbb{C}(X)$ of meromorphic functions on X is a finite algebraic extension $\mathbb{C}(f_1, \ldots, f_k)[g]$ of the field of rational functions in k-algebraically independent meromorphic functions where $k \leq \dim(X)$. It should be mentioned that Andreotti went on to develop this theory in several ensuing works.

If, for example, X arises as the quotient $\widehat{X}/\Gamma$ of some other space by the proper action of a discrete group, then the notion of pseudoconcavity can be formulated at the level of $\widehat{X}$. Andreotti and Grauert do this and then restrict their attention to the case where $\widehat{X} = H$ is the Siegel upper halfplane of complex $n \times n$-matrices $Z = X + iY$ which are symmetric and where $Y > 0$. The discrete group which is of interest here is $\Gamma = \mathrm{Sp}_{2n}(\mathbb{Z})$. It is acting properly and discontinuously so that the quotient H/Γ has the natural structure of a complex space. It is well known that Γ-periodic meromorphic functions, i.e., functions on the quotient, are important in more than one area of mathematics.

Using a well-known fundamental region Ω_0 for the Γ-action along with the strictly plurisubharmonic function $k(z) = -\log|Y|$, Andreotti and Grauert determine a region in H that descends to the quotient to show that it is pseudoconcave. An essential part of the proof is devoted to achieving the periodicity of $k(z)$ by minimizing it over Γ. This can be done, because the minima are taken on in the fundamental region.

As a consequence, the result on function fields can be applied in this case. This was known already, but was proved by using vastly more complicated methods. Furthermore the possibility of using pseudoconcavity in this area of mathematics was a totally new, extremely useful idea. It should be remarked that the same type of method can be used to prove that natural spaces of modular forms, e.g., for the canonical bundle, are finite dimensional. Furthermore, pseudoconcavity implies that the quotients H/Γ close up in projective embeddings to compact complex spaces to which all meromorphic functions extend.

Über Modifikationen und exzeptionelle analytische Mengen

Given a complex space X and a compact subvariety A one is interested in understanding when there is a complex space Y with a distinguished point $y \in Y$ and a surjective holomorphic mapping $\pi : X \to Y$ which is biholomorphic from $X \setminus A$ to $Y \setminus \{y\}$ and with $\pi(A) = \{y\}$. In other words, one would like to have sufficient conditions for A to be blown down to a point. In the projective algebraic setting in the case where X is a surface certain results were already known, e.g., for blowing down a smooth rational curve. In this beautiful paper Grauert answers this question in a general analytic setting in terms of the neighborhood geometry of A and its normal bundle. Underway he proves a number of results that can be considered as preparatory but which are also extremely useful in many areas of global complex geometry. As is often the case for Grauert, the guiding light is given by the notion of strong pseudoconvexity.

Here Grauert begins by noting that his solution to the Levi problem for relatively compact domains G in complex manifolds X had just been extended to the case where X is singular ([26]). Using Remmert's reduction theorem, he observes that the result can be stated as follows: If G has strictly pseudoconvex boundary, then it contains a maximal compact analytic subset A which can be blown down to a finite number of points (corresponding to its connected components) by a map $\pi : G \to Y$ where Y is a Stein space. Conversely, if a connected compact analytic set can be blown down to a point, then it has a strongly pseudoconvex neighborhood. So it is natural to study the relation of this type of question to the pseudoconvexity of neighborhoods of the 0-section of the normal bundle of A or more generally for any bundle.

For line bundles F over a compact complex manifold X the importance of the notion of the positivity of a Hermitian bundle metric was known. One says that F is ample if some power F^k defines an embedding $X \hookrightarrow \mathbb{P}(\Gamma(X, F^k)^*)$ by mapping a point $x \in X$ to the hyperplane of sections which vanish at x. Kodaira's basic theorem states that F is ample if and only if it possesses a positive bundle metric. Since the region defined by $\|\cdot\| > 1$ can be regarded as a tubular neighborhood of the 0-section of F^*, Grauert reformulates positivity in terms of the strong pseudoconvexity of the 0-section of the dual bundle. It is important to emphasize that this also makes sense in the case where X is singular. He calls this property *schwach negativ*. Thus the embedding theorem can be stated as F is ample if and only if F^* is schwach negativ. This is then equivalent to the 0-section of F^* being the maximal compact subset of F^*. It can be blown down to a Stein space which Grauert shows to be affine. Grauert's notion for vector bundles of higher rank is defined analogously: A vector bundle V over a compact complex space is said to be Grauert-positive if and only if the dual bundle V^* is schwach negativ in the above sense, i.e., its 0-section can be blown down. It should be remarked that in the vector bundle case the relation of Grauert's positivity condition to Griffiths-positivity is still not understood.

One of the main results of the paper is the embedding theorem: If a complex compact space X possesses vector bundle which is the schwach negativ, then it is projective algebraic. The key is the Stein property for blown down bundle space. Even in the case of line bundles $F \to X$ the result is new, because here singular spaces are allowed. This also gives a proof of the embedding theorem for Hodge spaces, another result of Kodaira in the smooth case.

The following is a more general version of the fact mentioned above, i.e., that the blown down dual bundle space is affine: Let F be the bundle of a divisor which has support A. Suppose $F|A$ is positive and that $X \setminus A$ contains no positive dimensional compact analytic subsets. Then $X \setminus A$ is affine and F is a positive bundle on X. The key ingredient for the proof, called a Hilfslemma by Grauert, is probably even more useful: A line bundle F over a compact complex space X is positive if and only if for every analytic subset A there exists $k > 0$ so that $F^k|A$ has a section which vanishes at some point of A but does not vanish identically.

Returning to the main theme of the paper, Grauert considers the notion of the normal bundle of a compact complex subvariety A of a complex space X. Due to the possible singular nature of these spaces, this must initially be regarded as a sheaf corresponding to the ideal sheaf $\mathfrak{m}$ of A or more generally any coherent ideal sheaf $\mathcal{I}$ which defines A. In typical Grauert fashion he is not phased by this difficulty but rather introduces the (quite natural) notion of a linear fiber space associated to a coherent sheaf. Locally over a trivializing neighborhood U this is a subvariety of $U \times \mathbb{C}^n$ where the fibers are subvector spaces of $\mathbb{C}^n$ so that addition and scalar multiplication are well defined. Thus, given A and the ideal sheaf $\mathcal{I}$ as above one has its normal linear fiber space $N_{\mathcal{I}}$ with its 0-section and the notion of schwach negativ has the obvious meaning. In elegant fashion Grauert transfers the pseudoconvexity of a neighborhood of the 0-section to that of a neighborhood of A in X and proves the desired result: A can be blown down if for suitable $\mathcal{I}$ the normal linear fiber space $N_{\mathcal{I}}$ is schwach negativ.

Of course the results in this paper have numerous applications. Even in the case of surfaces one needs Grauert's results to show that an irreducible curve C has negative self-intersection number if and only if it can be blown down to a point. Grauert's most general result in this direction is that a 1-dimensional subvariety in a surface can be blown down if and only if its self-intersection matrix is negative definite.

Direct Image Theorem

Ein Theorem der analytischen Garbentheorie und die Modulräume komplexer Strukturen

The proof of the Direct Image Theorem (*Bildgarbensatz*) is one of Hans Grauert's greatest accomplishments [13]. We will state it here, say a bit about the proof and give an application mentioned by Grauert in the paper. As we wrote in ([21]), the applications are so far reaching in complex analytic geometry that it would be unimaginable to work in the area without having it available.

Let us turn to the setting of complex analysis at the time (the late 1950's). A great deal was known, at least compared to ten years before. The notion of a complex space had been clarified, Grauert already had a huge experience as *Handarbeiter* in dealing with problems of cohomology, e.g., the Oka Theorems and his proof of Theorems A and B were behind him, and he understood very well how to deal with refining covers and using the relevant Fréchet spaces and compact operators between them. Given all of this he was in a position to consider the problem of the coherence of direct images of coherent sheaves.

The initial geometric context of this theorem is quite simple. One begins with a holomorphic map $F : X \to Y$ between complex spaces which is defined in the most naive way, e.g., locally it is given by holomorphic functions. Associated to an open subset U in Y one has the algebra $\mathcal{O}_X(f^{-1}(U))$ on its preimage. This has the structure of an $\mathcal{O}_Y(U)$-module which is given by multiplication by lifted functions $F^*(f)$. This presheaf defines a sheaf $F_*(\mathcal{O}_X)$ on Y. It contains a great deal of information about the map F, in particular about its singularities. It would clearly be of interest to know whether or not it is coherent.

If indeed the direct image $F_*(\mathcal{O}_X)$ is coherent, then its support is a closed analytic subset of Y. Hence, the correct condition for the direct image theorem to hold must be something that guarantees that images of analytic subsets are analytic. At the time, Remmert's theorem, which guarantees that this is the case for F being a proper (holomorphic) map, had been proved. It should be underlined that even the notion *proper*, i.e., inverse images of compact sets are compact, was rather new. Cartan had introduced this in the 1930's while discussing the fact that the action of groups of automorphisms on a bounded domain is proper and the notion was explicitly described in Bourbaki. Kuhlmann had pointed out that there is a weaker notion (semi-proper) and had proved that Remmert's theorem holds for this kind of map. Stein and Grauert were always interested in understanding holomorphic equivalence relations and finding a good condition which would insure that the quotient is analytic. In fact, one of Grauert's last papers was devoted to a situation where a sort of semi-properness was built into the assumptions.

In any case, at the time when Grauert considered the problem of the coherence of direct images of coherent sheaves, much was known, but even at the set-theoretic level (Remmert's theorem) things had not settled in. There were also a huge number of foundational issues. For one, even the notion of an analytic morphism had to be improved. One reason for this, at least from Grauert's point of view, was that the entire project had to be carried out in the context of complex spaces where the structure sheaf is allowed to have *nilpotent* elements. This means that the local model is as before an analytic set A in some domain D in $\mathbb{C}^n$, but the sheaf of germs of holomorphic functions is $\mathcal{O}_D/\mathcal{I}_A$ where $\mathcal{I}_A$ is any coherent ideal sheaf which defines A as its 0-set. Since the structure sheaf is not necessarily a subsheaf of the sheaf of continuous functions, the classical definition of a map being holomorphic, i.e., pullbacks of holomorphic germs are required to be holomorphic, is not sufficient. A holomorphic map is then a pair (F_0, F_1) where $F_0 : X \to Y$ is a usual map of sets and F_1 is a map of structure that encodes the notion of pullback, a continuous homomorphism of sheaves of algebras $F_1 : Y \times_{F_0} \mathcal{O}_Y \to \mathcal{O}_X$. Grauert begins his paper with a rather long discourse on how to deal with these *new* complex spaces where he proved the key theorems such as Theorems A and B in this more general setting.

Given a morphism $F : X \to Y$ and sheaf $\mathcal{S}$ of $\mathcal{O}_X$-modules, F_1 is applied to equip the direct image sheaf $\pi_*(\mathcal{S})$ with the structure of a sheaf of $\mathcal{O}_Y$-modules. One can go an important step further: For U open in Y and every $q \geq 0$ the cohomology space $H^q(F^{-1}(U), \mathcal{S})$ is equipped by means of F_1 with the structure of a $\mathcal{O}_X(U)$-module. Hence, for every q we have the direct image sheaf $R^q F_*(\mathcal{S})$. The following is then the *Bildgarbensatz*.

Theorem *If $F : X \to Y$ is a proper holomorphic map of complex spaces and $\mathcal{S}$ is a coherent sheaf of $\mathcal{O}_X$-modules, then for every $q \geq 0$ the direct image sheaf $R^q F_*(\mathcal{S})$ is coherent.*

There were germs of this result around at the time, e.g., Remmert had proved a result in special case of finite maps, Grauert and Remmert had proved it in the situation where $X = Y \times \mathbb{P}_n$ and F is the obvious projection and Grothendieck had proved the far simpler algebraic version. However, this result, along with the Oka Principle papers, brought complex analysis into a new era. Only five years before members of the Behnke seminar were trying to understand $\sqrt{xy}$!

Let us quote Grauert ([16, p. 446]) when discussing the main difficulties in the proof which involve a power series argument to handle the directions transversal to a fixed fiber X_0. *Roughly speaking, the proof of the direct image theorem uses power series expansion whose coefficients are cohomology classes on one fixed fiber. The coefficients are obtained recurrently and with estimates. Cohomology spaces carry natural Fréchet space structures. However, to get a convergent power series by recurrent formula with estimates, one needs a fixed norm for the iteration process instead of the infinite sequence of semi-norms. One key point of the proof of the direct image is to replace a cocycle with estimates for a weak norm by another cocycle with estimate for a stronger norm modulo a coboundary with estimates for an even weaker norm.*

Grauert began this article with a rather lengthy discussion of the importance of the cohomology of a certain direct image sheaf for the study of moduli spaces M. In that case $F : X \to M$ is a usual (surjective) proper holomorphic map of complex manifolds which is everywhere of maximal rank. The relevant sheaf is Θ_X, the sheaf of germs of holomorphic vector fields on X. For example, if X_y denotes the fiber over $y \in M$ and $r_q(y)$ is the dimension of $H^q(X_y, \Theta_y)$, then the semicontinuity of $r_q(y)$, which was proved by Kodaira and Spencer using the method of harmonic integrals, follows immediately from the direct image theorem. In fact, in much greater generality the direct image theorem implies the semicontinuity for any coherent sheaf provided the proper map $F : X \to Y$ is flat.

It took the complex analysis community a number of years to understand and somewhat simplify Grauert's proof (see, e.g., [5] and [25]). According to Grauert, the simplest proof can now be found in [18].

Akademischer Lehrer

Let me close this note with some personal comments. I was introduced to the "German school" of complex analysis in the late 1960's in the Stanford lectures of Aldo Andreotti, where I was the only student. In the first semester of these lectures Andreotti explained a number of Grauert's results (some of them discussed above) in a beautiful way. Although I had been a student for a while, this was the first time that I felt that I had seen "the truth". Of course Andreotti was a master lecturer, but the truth, I sensed, was embedded in Grauert's work. A few months later, when I realized I should prepare for my German language exam, luck struck again: My advisor, Halsey Royden, had explained in seminars some of his ideas on metrics on Teichmüller space. As a result I optimistically thought it might be good to go back to the basics and read Hermann Weyl's book *Die Idee der Riemannschen Fläche* and then jump to the modern developments and study Grauert's paper *Über Modifikationen und exzeptionelle analytische Mengen*. Up to this point I had very little experience reading mathematics and assumed this is the way it should be! Looking back, it is hard for me to believe that I had been so naively audacious! A year or so later during my postdoc time in Pisa, I told my friends that these two works were the only things I really understood. At the time I was embarrassed to admit this but soon realized my good fortune!

In the second semester of the above-mentioned course Andreotti "asked" me to lecture on various topics involving $\bar{\partial}$ at the boundary, Hans Levy's extension, etc. The audience consisted of Andreotti and Wilhelm Stoll. This began my lasting friendship with Stoll. He came from another "Schwerpunkt" of complex analysis, namely from the Tübingen group of Hellmuth Kneser. The complex analysis of Tübingen was, to a certain extent, related to that of the Münster school, e.g., they had competing theories of meromorphic maps. However, Kneser and Stoll went in other directions, proving continuation theorems under assumptions of bounded volume, and then building the foundations of value distribution theory in several complex variables.

My close relationship with Stoll continued during the almost 10 years I spent at Notre Dame where, particularly due to Stoll's connections, the faculty had close ties to German mathematicians. In my very first year there Stein visited for a semester. His energy, openness and obvious love of mathematics made a great impression on me. The next year Remmert came, and it was a great honor for me to drive him around town as he reviewed periods in which he had also been a guest at Notre Dame. In those days it was a bit non-standard to go from the US to Oberwolfach for a week, but when he invited me I jumped at the opportunity. In my Oberwolfach lecture Grauert, Remmert, Stein and Forster were in the first row. I figured if I could get through that I could get through anything!

Grauert was a kind, warm person of very few words. Sometimes he looked formal, but he was not. In the above-mentioned conference Douady lectured on his construction of the versal deformation of a complex space. Grauert, who had developed his own version of this theory, sat in the first row. Douady, who was dressed in a silk-like Hawaiian shirt which was not completely buttoned and was rather dirty because the night before he had slept in the forest, explained his puzzles and made silly jokes. Grauert remained quiet and respectful, only caring about the content. Later on at a memorial meeting in honor of Andreotti, who had passed away at a very early age, one could see that Grauert and Douady were very close.

Grauert didn't say much, but when he did he meant it! A comment of "good, continue on" to a young speaker after a talk really meant something. Similarly, his way of praising a student was often "Das können wir so machen". He was a key referee for one of our research concentrations sponsored by the DFG. As a 50-year-old I nervously appeared in front of Grauert for his comments: "almost everything is good, but the mathematics in this subproject is not important". Of course I dropped the subproject from the proposal. One might think of Grauert as being opinionated, but his opinions were based on serious thought; anything he said or wrote should be taken seriously!

Hans Grauert was an "Akademischer Lehrer" in the sense of Humboldt. He didn't teach a subject because it was in the syllabus; he taught it because he had thought about it and knew it was important. He had a large number of doctoral students, more than 40, and he was proud of it. I know that in every case he had thought through their projects and made notes on the little pieces of paper that he carried around. One of our colleagues who was a student of Grauert remembers seeing the same piece of paper every time he came to Grauert's office hours.

Even though he was not the type of person to be heavily involved with the global organization of science, he did his duty, e.g., as managing editor of Mathematische Annalen and working in various capacities with the DFG. He certainly respected the traditions of Göttingen and was proud to have been President of the Göttingen Academy of Sciences. His strong points were, however, in the classroom where he focused on important phenomena in mathematics, in his one-on-one work with his students and of course in his remarkable research.

Those of us who have had the privilege of knowing Hans Grauert will not forget him. Fortunately, his deep ideas have survived him in his written works. Let us hope that his high standards of excellence in every aspect of our science will be carried on by future generations.

References

1. Andreotti, A., Grauert, H.: Algebraische Körper von automorphen Funktionen. Nachr. Akad. Wiss. Gött. Math.-Phys. Kl., 2B **1. 3**, 39–48 (1961)
2. Andreotti, A., Grauert, H.: Théorèmes de finitude pour la cohomologie des espaces complexes. Bull. Soc. Math. Fr. **90**, 193–259 (1962)
3. Cartan, H.: Espace fibrés analytique. In: Symposium International de Topologia Algebraica, Mexico, pp. 97–121 (1958)
4. Docquier, F., Grauert, H.: Levisches Problem und Rungescher Satz für Teilgebiete Steinscher Mannifaltigkeiten. Math. Ann. **140**, 94–123 (1960)
5. Forster, O., Knorr, K.: Ein Beweis des Grauertschen Bildgarbensatzes nach Ideen von B. Malgrange. Manuscr. Math. **5**, 19–44 (1971)
6. Forstnerič, F.: Stein Manifolds and Holomorphic Mappings. Ergebnisse der Mathematik und ihrer Grenzgebiete, 3. Folge, vol. 56. Springer, Berlin (2011)
7. Grauert, H.: Charakterisierung der holomorph vollständigen komplexen Räume. Math. Ann. **129**, 233–259 (1955)
8. Grauert, H.: Charakterisierung der Holomorphiegebiete durch die vollständige Kählersche Metrik. Math. Ann. **131**, 38–75 (1956)
9. Grauert, H.: Approximationssätze für holomorphe Funktionen mit Werten in komplexen Räumen. Math. Ann. **133**, 139–159 (1957)
10. Grauert, H.: Holomorphe Funktionen mit Werten in komplexen Lieschen Gruppen. Math. Ann. **133**, 450–472 (1957)
11. Grauert, H.: Analytische Faserungen über holomorph-vollständigen Räumen. Math. Ann. **135**, 263–273 (1958)
12. Grauert, H.: On Levi's problem and the imbedding of real-analytic manifolds. Ann. Math. **68**, 460–472 (1958)
13. Grauert, H.: Ein Theorem der analytischen Garbentheorie und die Modulräume komplexer Strukturen. Publ. Math. Paris IHES **5**, 233–292 (1960)
14. Grauert, H.: Über Modifikationen und exzeptionelle analytische Mengen. Math. Ann. **146**, 331–368 (1962)
15. Grauert, H.: Bemerkenswerte pseudokonvexe Mannifaltigkeiten. Math. Z. **51**, 377–391 (1963)
16. Grauert, H.: Selected Works with Commentaries, vols. I. and II. Springer, Berlin (1994)
17. Grauert, H., Remmert, R.: Komplexe Räume. Math. Ann. **136**, 245–318 (1958)
18. Grauert, H., Remmert, R.: Coherent Analytic Sheaves. Grundlehren der mathematischen Wissenschaften, vol. 265. Springer, Berlin (1984)

19. Gromov, M.: Oka's principle for holomorphic sections of elliptic bundles. J. Am. Math. Soc. **2**, 851–897 (1989)
20. Huckleberry, A.: Karl Stein (1913–2000). Jahresber. Dtsch. Math.-Ver. **110**(4), 195–206 (2008)
21. Huckleberry, A.: Hans Grauert: Mathematiker pur. Mitt. Dtsch. Math.-Ver. **16**(2), 75–77 (2008)
22. Huckleberry, A.: Hans Grauert: Mathematiker pur. Not. Am. Math. Soc. **56**(1), 38–41 (2009)
23. Huckleberry, A., Peternell, T. (eds.): A tribute to Hans Grauert. Notices of the AMS (to appear, 2013)
24. Hulek, K., Peternell, T.: Henri Cartan, ein französischer Freund. Jahresber. Dtsch. Math.-Ver. **111**(2), 85–94 (2009)
25. Narasimham, R.: Grauert's theorem on direct images of coherent sheaves. In: Séminaire de Mathématique Supérieures, No. 40 (Été 1969). Les Presses de l'Université de Montréal, Montreal (1969). 79 pp.
26. Narasimhan, R.: The Levi problem on complex spaces. Math. Ann. **142**, 355–365 (1960). Part II. in Math. Ann. **146**, 195–216 (1962)
27. Ohsawa, T.: On complete Kähler domains with C^1-boundary. Pub. RIMS Kyoto **16**, 929–940 (1980)
28. Remmert, R.: Complex Analysis in "Sturm und Drang". Mathematical Intelligencer, vol. 17, Springer, Berlin (1995)
29. Stein, K.: Analytische Funktionen mehrerer komplexer Veränderlichen zu vorgegebenen Periodizitätsmoduln und das zweite Cousinsche Problem. Math. Ann. **123**, 201–222 (1951)

Alan Huckleberry is a retired Professor of Mathematics from the Ruhr Universität Bochum, where he remains active in the graduate program and as Ortssprecher of SFB/Tr 12. Parallel to this he is *Wisdom Professor of Mathematics* at Jacobs University in Bremen. In 1963 he obtained his BA from Yale University in mathematics and biostatistics, continued with a Master's degree in probability theory in 1964 at Ball State University and received his Ph.D. from Stanford University in 1969 with a thesis in complex analysis. After a postdoctoral position in Pisa, he spent 10 years at Notre Dame University, becoming a Full Professor of Mathematics in 1978. During that period he visited Germany on numerous occasions, in particular supported by the Humboldt Foundation, and in 1980 became the successor to Friedrich Sommer in the Chair for Complex Analysis at the Ruhr Universität. He was awarded honorary doctor's degrees in Lille (1997) and in Nancy (2002). His current research activities involve areas of global complex geometry, actions and representations of Lie groups and applications in mesoscopic physics.
Photo: Bildarchiv des Mathematischen Forschurgsinstituts Oberwolfach.

Jahresber Dtsch Math-Ver (2013) 115:47–50
DOI 10.1365/s13291-013-0057-3

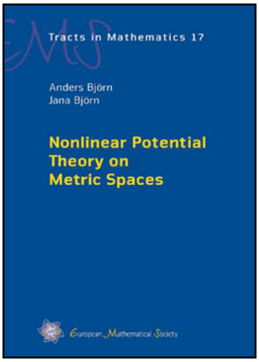
BOOK REVIEW

Anders Björn and Jana Björn: “Nonlinear Potential Theory on Metric Spaces”

European Mathematical Society Publishing House, 2011, 415 pp.

Juha Kinnunen

Published online: 6 February 2013



The reviewed monograph is related to analysis on metric measure spaces, in particular to Sobolev type spaces and potential theoretic aspects of certain nonlinear variational integrals. This has been a very active field of research during the last decades and by now it is a relatively well developed field with applications to many areas of mathematics. As we can see from the list of references of the monograph, hundreds of research papers have been published but so far there has not been a general reference in this field. The monograph by Anders and Jana Björn fills this gap in the literature. Some aspects of analysis on metric measure spaces have been previously covered by [1, 3, 5] and [6].

By the end of the last century it was realized that much of contemporary harmonic analysis, partial differential equations and the calculus of variations does not require much structure of the underlying space. The same concerns nonlinear potential theory. Classical potential theory studies properties of Laplace’s equation. It involves, for example, harmonic and superharmonic functions, maximum and comparison principles, potentials and capacities, the Dirichlet problem and the boundary behavior of the solutions. Similar potential theory can be developed also for certain nonlinear partial differential equations and related variational integrals, see [7] and [10] and the references therein. The theory was developed separately in the special cases, for example, of weighted Euclidean spaces with Muckenhoupt’s weights, Riemannian manifolds with a nonnegative Ricci curvature, graphs, Heisenberg groups and more general Carnot groups and Carnot–Carathéodory spaces. Later it was found out that

J. Kinnunen (✉)
Aalto University, Aalto, Finland
e-mail: juha.k.kinnunen@aalto.fi

there is a more general theory behind and this unified approach is the focus of the present monograph.

The standard assumptions for analysis on metric measure spaces are the volume doubling condition for the measure and the validity of a Poincaré type inequality. The former gives an upper bound for a dimension related to the measure and the latter gives a passage from the infinitesimal concept of the gradient to larger scales. A Borel measure is said to be doubling, if the measure of balls with the radius doubled can be controlled by the measure of the original balls independently of the scale and location. Much of harmonic analysis can be developed in the context of homogeneous spaces, which are quasi-metric spaces with a doubling measure, see [3]. However, this does not seem to be quite enough for the first order calculus needed for partial differential equations and for the calculus of variations. The Poincaré inequality states that the mean oscillation of the function is controlled by the mean value of the gradient in a scale and location invariant manner. Roughly speaking, if the gradient is small in average, then the function does not oscillate much.

In special cases, these conditions are either assumed a priori or proved by analyzing the structure of the space. Various examples of metric spaces with a doubling measure and the Poincaré inequality are given in the appendix of the monograph. In addition to the examples mentioned above, there are relatively exotic spaces that satisfy the standard assumptions. Indeed, there is a complete Cantor type space, with any Hausdorff dimension which is at least one, so that the corresponding Hausdorff measure is doubling and the space supports a Poincaré inequality. On the other hand, Semmes gave sufficient conditions for a metric space with a doubling measure to support a Poincaré inequality in [12]. Moreover, the doubling and Poincaré conditions are relatively robust assumptions in the sense that they are invariant under bi-Lipschitz changes of coordinates and they are preserved in the Gromov-Hausdoff limits of spaces. The doubling and Poincaré conditions are also assumed throughout the monograph under consideration and they are sufficient for regularity theory which, on the other hand, is the basis for nonlinear potential theory.

Historically, the direct methods of the calculus of variations show the existence of a solution in Sobolev spaces to a wide class of variational problems and partial differential equations. The main philosophy is the following: To obtain the existence of a weak solution, smoothness assumptions on the functions are relaxed and then regularity theory shows that, under favorable circumstances, the solution is smoother than assumed a priori. Indeed, the functions in Sobolev spaces are not necessarily even continuous, but however, they possess certain absolute continuity properties as in the fundamental theorem of calculus. Regularity theory is essentially based on two ingredients: Sobolev inequalities for arbitrary Sobolev functions and energy estimates for solutions of the partial differential equations or minimizers of the variational integrals. These rather general tools are available in a metric measure space under the standard assumptions.

The monograph of Anders and Jana Björn consists of two parts. In the first part, the authors develop a systematic theory of first order Sobolev spaces, called Newtonian spaces, and in the second part they consider potential theory related to minimizers of the nonlinear Dirichlet integrals on metric measure spaces. It is not clear how to define partial derivatives in metric spaces, but the notion of an upper gradient takes the role of the modulus of a gradient and this is sufficient in the definition of the Newtonian spaces and also in the definition of the Dirichlet integrals. In particular, the upper gradients, unlike the distributional gradients, do not rely on the linear structure of the Euclidean spaces. The interest in first order analysis in metric measure spaces was initiated by Hajłasz in [4] and the upper gradients were introduced by Heinonen and Koskela in [8]. Shanmugalingam conducted a first systematic study of the Newtonian spaces in [11].

The first part of the monograph includes upper gradients, doubling measures, Poincaré inequalities and capacities. They are all important tools in the calculus of variations on metric measure spaces. A particularly important result for regularity theory is that the volume doubling condition and the Poincaré inequality imply the Sobolev inequalities. Maximal function arguments are used to obtain a Gehring type lemma for reverse Hölder inequalities and the John-Nirenberg lemma for functions of bounded oscillation. The John-Nirenberg lemma is applied in Moser's proof of Harnack's inequality for minimizers and the Gehring lemma gives a higher integrability property for the upper gradients of minimizers.

The second part of the monograph deals with applications of Newtonian spaces to minimizers on metric spaces. Regularity theory showing that minimizers satisfy Caccioppoli type energy estimates and Harnack type inequalities is thoroughly presented, and in the sequel local Hölder continuity of the minimizers is also considered. These are fundamental results for the development of the finer aspects of nonlinear potential theory. In this respect the authors develop a systematic theory of comparison principles, superharmonic functions, and they also consider boundary regularity for the Dirichlet problem and removability results. Potential theoretic tools as the Perron method and the obstacle problem in the calculus of variations are used extensively in the arguments.

A brief discussion about alternative definitions of the first order Hajłasz and Cheeger type Sobolev spaces are given in the appendix. Under the standard assumptions different approaches to Sobolev spaces coincide and a coherent theory exists. In particular, it is shown that on weighted Euclidean spaces the theory coincides with the one developed by Heinonen, Kilpeläinen and Martio in [7]. In many respects the monograph by Anders and Jana Björn is a metric measure space version of [7], but it also contains many results that do not appear in [7] even in the Euclidean case. A careful and systematic analysis is required in the metric setting and the monograph under review is a self-contained and detailed exposition of the topic. The reader is only assumed to know rudiments of measure theory and functional analysis and after that the reader can immediately proceed to research questions of current interest.

Related topics that are not included in the monograph are the differentiation theory of Lipschitz functions developed by Cheeger [2]. This theory shows that, under the conditions of a doubling measure and a Poincaré inequality, Lipschitz functions are, in a suitable sense, differentiable almost everywhere. It also follows from Cheeger's theory, that the Newtonian spaces are reflexive in the appropriate range of the indices. Another topic that is not covered is a striking theorem of Keith and Zhong [9] about a self improving property of the Poincaré inequality. However, both issues are commented and the relevant references are given in the appendices.

References

1. Ambrosio, L., Tilli, P.: Topics on Analysis in Metric Spaces. Oxford Lecture Series in Mathematics and Its Applications, vol. 25. Oxford University Press, Oxford (2004)
2. Cheeger, J.: Differentiability of Lipschitz functions on metric measure spaces. Geom. Funct. Anal. **9**, 428–517 (1999)
3. Coifman, R.R., Weiss, G.: Analyse Harmonique Non-commutative sur Certains Espaces Homogènes. Lecture Notes in Mathematics, vol. 242. Springer, Berlin (1971)
4. Hajłasz, P.: Sobolev spaces on an arbitrary metric space. Potential Anal. **5**, 403–415 (1996)
5. Heinonen, J.: Lectures on Analysis on Metric Spaces. Springer, New York (2001)
6. Heinonen, J.: Nonsmooth calculus. Bull. Amer. Math. Soc. **44**, 163–232 (2007)
7. Heinonen, J., Kilpeläinen, T., Martio, O.: Nonlinear Potential Theory of Degenerate Elliptic Equations, 2nd edn. Dover, Mineola (2006)
8. Heinonen, J., Koskela, P.: Quasiconformal maps in metric spaces with controlled geometry. Acta Math. **181**, 1–61 (1998)
9. Keith, S., Zhong, X.: The Poincaré inequality is an open ended condition. Ann. of Math. **167**, 575–599 (2008)
10. Malý, J., Ziemer, W.P.: Fine Regularity of Solutions of Elliptic Partial Differential Equations. Math. Surveys and Monographs, vol. 51. Amer. Math. Soc., Providence (1997)
11. Shanmugalingam, N.: Newtonian spaces: an extension of Sobolev spaces to metric measure spaces. Rev. Mat. Iberoam. **16**, 243–279 (2000)
12. Semmes, S.: Finding curves on general spaces through quantitative topology, with applications to Sobolev and Poincaré inequalities. Selecta Math. **2**, 155–295 (1996)

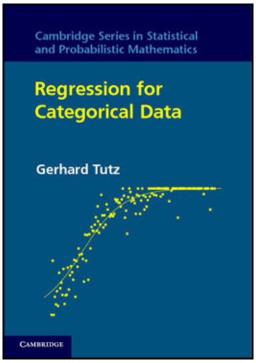
Jahresber Dtsch Math-Ver (2013) 115:51–55
DOI 10.1365/s13291-013-0058-2

BOOK REVIEW

Gerhard Tutz: „Regression for Categorical Data“
Cambridge University Press, 2012, 572 pp.

Thomas Kneib

Online publiziert: 6. Februar 2013



Lineare Regressionsmodelle stellen eines der wesentlichen Werkzeuge der Angewandten Statistik dar, wenn eine Zielgröße in Abhängigkeit von einer Reihe von erklärenden Variablen dargestellt werden soll. Während im klassischen linearen Modell die Zielgröße typischerweise als normalverteilt oder zumindest als stetig vorausgesetzt wird, ist diese Annahme in vielen praktischen Beispielen nicht mehr haltbar. Insbesondere im Falle kategorialer, also diskreter Zielgrößen müssen geeignete Erweiterungen betrachtet werden. In seinem Buch „Regression for Categorical Data“ behandelt Gerhard Tutz solche Erweiterungen, wobei kategoriale Daten in einem relativ weiten Sinn verstanden werden, so dass neben klassischen Ansätzen der kategorialen Regression für binäre und multinomiale Zielgrößen oder Zähldaten auch beispielsweise Modelle mit Zero-Inflation behandelt werden.

Ein typisches Beispiel eines kategorialen Regressionsmodells lässt sich für die Wahl eines Transportmittels für Fernreisen aus einer vorgegebenen Liste von Alternativen (Flugzeug, Zug, Bus, Auto) formulieren. Ziel der Analyse ist es hier, die Wahrscheinlichkeit für die Wahl eines bestimmten Transportmittels in Abhängigkeit beispielsweise von Preis, Reisezeit und anderen Einflussgrößen zu bestimmen. Da das lineare Modell eine stetige Zielgröße unterstellt, ist es zur Beschreibung der

T. Kneib (✉)
Lehrstuhl für Statistik, Georg-August-Universität Göttingen, Platz der Göttinger Sieben 5,
37073 Göttingen, Deutschland
e-mail: tkneib@uni-goettingen.de

kategorialen Zielgröße „gewähltes Transportmittel“ offenbar ungeeignet. Weitere im vorliegenden Buch besprochene Beispiele, die exemplarisch für andere Typen kategorialer Zielgrößen stehen können, sind etwa

- die Schätzung von Kreditausfallwahrscheinlichkeiten mit binärer Zielgröße (Kredit wurde zurückgezahlt oder nicht zurückgezahlt) in Abhängigkeit von Kredithöhe, Laufzeit des Kredits und früherem Zahlungsverhalten des Kunden (binäre Regressionsmodelle),
- die Modellierung der Anzahl insolventer Firmen in einem Monat in Abhängigkeit von der Kalenderzeit, um Konjunkturentwicklungen zu analysieren (Zähldaten-Regression), oder
- die Analyse des Behandlungserfolgs in einer Schmerztherapie nach Knie-Operationen, bei denen die Schmerzintensität auf einer geordneten Skala mit fünf Punkten (von „keine Schmerzen“ bis „starke Schmerzen“) beurteilt wird (Regression für ordinale Zielgrößen).

Neben der Beschreibung geeigneter Klassen von Regressionsmodellen für diese verschiedenen Typen von Zielgrößen werden die Prädiktorstruktur und die zum Schätzen verwendete Methodik im vorliegenden Buch dahingehend erweitert, dass moderne Ansätze der statistischen Regularisierung mit einem besonderen Fokus auf die Prädiktorselektion ebenfalls einbezogen werden können. Schließlich beinhaltet das Buch auch eigene Kapitel zu semiparametrischen Regressionsansätzen für kategoriale Daten, zu baum-basierten Verfahren und zur Prädiktion in kategorialen Regressionsmodellen.

Bei dem Buch des Kollegen Tutz handelt es sich um eine englische Übersetzung und erhebliche Erweiterung des 2000 im Oldenbourg Verlag erschienen Buchs „Analyse Kategorialer Daten“. Im Vergleich zu der deutschen Version hat sich der Umfang des Buchs deutlich erhöht (um nahezu 150 Seiten), so dass der Leser einen deutlich erweiterten und aktualisierten Inhalt vorfindet. Insbesondere die Methoden der Regularisierung sowie das Kapitel zur Prädiktion sind Ergänzungen, die das englischsprachige Buch auch für Besitzer des deutschen Bandes empfehlenswert machen.

Das Buch ist insgesamt auf einem mittleren mathematischen Niveau angesiedelt und setzt im Wesentlichen Basiswissen in Linearer Algebra und Wahrscheinlichkeitsrechnung voraus. Grundlagen des linearen Modells werden zu Beginn des Buchs kurz wiederholt, Hintergrundwissen zu linearen Modellen ist aber sicherlich dennoch empfehlenswert. Zielgruppe des Buchs sind Statistiker, Forscher unterschiedlicher Anwendungsbereiche mit solider statistischer Ausbildung sowie Studierende der Statistik, Mathematik oder anderer Fachgebiete mit quantitativem Schwerpunkt.

Ergänzend zum Buch werden auf der Homepage http://www.stat.uni-muenchen.de/~tutz/catdata eine Reihe von Datensätzen zur Verfügung gestellt. Darüber hinaus sind die Datensätze sowie R-Code für zahlreiche der durchgeführten Analysen in Form eines R-Pakets erhältlich. SAS-Code steht für einige ausgewählte Anwendungen zur Verfügung, wobei insbesondere die moderneren Erweiterungen und Verfahren nicht für SAS erhältlich sind, da entsprechende Implementationen in SAS bisher nicht vorhanden sind.

Kapitel 1 des Buchs bietet eine kurze Einleitung, in der eine Reihe von Beispielen vorgestellt und einige grundlegende Konzepte besprochen werden. Ergänzend wird ein kurzer Überblick zu linearen Modellen geboten, um die für das weitere Verständnis notwendigen Grundlagen bereitzustellen bzw. zu wiederholen.

Die anschließenden Kapitel 2 bis 7 bilden einen größeren Block zu parametrischen Regressionsmodellen für verschiedene Typen von univariaten Zielgrößen. Als erstes grundlegendes Beispiel wird in Kapitel 2 („Binary Regression: The Logit Model“) ausführlich das Logit-Modell vorgestellt, mit dem sich binäre Zielgrößen wie die Kreditwürdigkeit eines Bankkunden beschreiben lassen. Insbesondere werden verschiedene Herleitungsmöglichkeiten des logistischen Regressionsmodells (latente Variablen, Bayes-Klassifikation durch quadratische Diskriminanzanalyse) sowie die Kodierung und Interpretation von verschiedenen Kovariablentypen behandelt. Kapitel 3 („Generalized Linear Models“) führt anschließend den allgemeinen Rahmen generalisierter linearer Modelle ein, die das Logit-Modell, das gewöhnliche lineare Modell und eine ganze Reihe weitere im Buch behandelte Modelle als Spezialfälle beinhalten. Damit lassen sich in einem vereinheitlichten Ansatz die Maximum Likelihood-Schätzung und weitere Methoden der statistischen Inferenz entwickeln, die dann unmittelbar für viele Spezialfälle zur Verfügung stehen. Als Erweiterung werden auch Verfahren der Quasi-Likelihood-Schätzung vorgestellt. Kapitel 4 („Modeling of Binary Data“) wendet sich dann erneut der Modellierung binärer Zielgrößen zu und bettet diese in die in Kapitel 3 entwickelte Methodik ein. Speziell wird die Maximum Likelihood-Schätzung ausführlich dargestellt ebenso wie Verfahren zur Beurteilung der Modellanpassung und der Modelldiagnose. Ebenfalls betrachtet werden verschiedene Möglichkeiten, die Kovariablen in einen geeigneten linearen Prädiktor zu übersetzen (insbesondere im Fall kategorialer Kovariablen) sowie den Erklärungsgehalt verschiedener Kovariablen zu beurteilen. Während sich Kapitel 4 weiterhin auf das Logit-Modell konzentriert, werden in Kapitel 5 („Alternative Binary Regression Models“) andere Formen der Regression für binäre Zielgrößen betrachtet. Dies beinhaltet sowohl die Diskussion alternativer Linkfunktionen als auch die simultane Schätzung der Link-Funktion mit den Kovariableneffekten und die Berücksichtigung von Überdispersion.

Kapitel 6 („Regularization and Variable Selection for Parametric Models“) wendet sich dann der Problematik der Variablenselektion und der Regularisierung im Falle hochdimensionaler Vektoren von erklärenden Variablen zu. Beispiele hierzu ergeben sich insbesondere im Genetikbereich, da hier mit modernen Technologien (High-Throughput-Experimente) sehr viele Einflussgrößen wie etwa Genexpressions-Niveaus oder Informationen zur Allelhäufigkeit von Einzelnukleotid-Polymorphismen erhoben werden können. Dagegen ist in diesen Beispielen die Anzahl der Beobachtungen typischerweise weiterhin relativ gering, so dass häufig die Zahl der Einflussgrößen die Zahl der Beobachtungen deutlich übersteigt und klassische Schätzansätze damit nicht mehr verwendet werden können. Entsprechend werden Penalisierungsansätze zur Regularisierung eingeführt, die es erlauben, die Komplexität eines Modells zu kontrollieren und somit auch hochdimensionale Modelle der Schätzung mit relativ geringen Fallzahlen zugänglich zu machen. Dieser Zweig der Statistik hat in den letzten Jahren eine rege Entwicklung erlebt und wird hier sehr schön in Ihrem aktuellen Stand und in breitem Überblick wiedergegeben. Neben Penalisierungsansätzen werden auch indirekte Regularisierungsansätze wie Boosting behandelt. Besonderen Raum nimmt die Regularisierung kategorialer Einflussgrößen ein, mit deren Hilfe beispielsweise automatisiert Kategorien einer Einflussgröße zusammengefasst werden können. Dabei muss insbesondere die unterschiedliche Skalierung der Einflussgrößen (ordinal versus nominal) berücksichtigt werden. Kapitel 7 („Regression Analysis of Count Data") beschließt dann den Block parametrischer Modelle für univariate Einflussgrößen mit einer Behandlung von Regressionsmodellen für Zählgrößen. Neben der ausführlichen Betrachtung der klassischen, loglinearen Poisson-Regression werden auch Erweiterungen wie Regressionsmodelle, basierend auf der negativen Binomialverteilung, oder Modelle mit Zero-Inflation behandelt.

Die beiden anschließenden Kapitel 8 („Multinomial Response Models") und 9 („Ordinal Response Models") wenden sich dann Modellen für mehrkategoriale Zielgrößen zu und unterscheiden dabei nach der Skalierung der Zielgröße. Kapitel 8 behandelt insbesondere das multinomiale Logit-Modell, aber auch eine Reihe von Erweiterungen beispielsweise für Paarvergleiche sowie die regularisierte Schätzung multinomialer Logit-Modelle. Kapitel 9 dagegen konzentriert sich auf spezielle Modelle für ordinale Zielgrößen wie das kumulative und das sequentielle Modell.

Während die Kapitel zu parametrischen Regressionsmodellen die Annahme eines linearen Prädiktors aufrecht erhalten, befassen sich die Kapitel 10 („Semi- and Non-Parametric Generalized Regression") und 11 („Tree-Based Methods") mit Verfahren, die eine allein datengesteuerte Bestimmung der funktionalen Form von Kovariableneinflüssen zulassen. Kapitel 10 konzentriert sich dabei auf Glättungsverfahren basierend auf Basisfunktionen und entwickelt hierzu eine entsprechende penalisierte Schätzmethodik. Dabei werden auch moderne Verfahren wie Boosting zur automatischen Wahl der Glattheit der zu schätzenden Funktionen herangezogen und Methoden für funktionale Daten behandelt. Kapitel 11 verwendet dagegen Regressions- und Klassifikationsbäume, die insbesondere dann Vorteile aufweisen, wenn komplexe Interaktionsformen in den Daten vorliegen.

Die verbleibenden Kapitel 12 bis 15 behandeln jeweils spezielle Fragestellungen der kategorialen Regression, die sich nicht unmittelbar einem der drei bisher skizzierten Themenblöcken zuordnen lassen. Kapitel 12 („The Analysis of Contingency Tables: Log-linear and Graphical Models") beschäftigt sich mit der Analyse von Kontingenztabellen insbesondere mit Hilfe log-linearer Modelle. Hierzu werden in aufsteigender Form ausgehend vom einfachsten Fall einer Kontingenztafel für zwei Merkmale komplexere Modelle für mehr Merkmale entwickelt. Kapitel 13 („Multivariate Response Models") wendet sich dann Modellen mit multivariater Zielgröße zu, wobei insbesondere Markov-Übergangsmodelle und marginale Modelle, basierend auf generalisierten Schätzgleichungen, behandelt werden. Kapitel 14 („Random Effects Models and Finite Mixtures") entwickelt dann einen alternativen Ansatz für multivariate Zielgrößen und gruppierte Daten, basierend auf gemischten Modellen mit zufälligen Effekten. Neben parametrischen Modellen mit normalverteilten zufälligen Effekten und den entsprechenden Schätzverfahren werden auch semiparametrische Erweiterungen und Modelle mit Mischverteilunsspezifikationen für die zufälligen Effekte betrachtet.

Das finale Kapitel 15 („Prediction and Classification") behandelt dann ausführlich die Probleme der Vorhersage der Zielgröße für neue Beobachtungen. Neben den grundlegenden Konzepten der Vorhersage und der Optimalität der Bayes-

Klassifikation werden eine ganze Reihe auch moderner Klassifikationsverfahren ausführlich vorgestellt und diskutiert. In einer Reihe von Anhängen werden benötigte Verteilungen, einige grundlegende mathematisch-statistische Konzepte, die Schätzung unter Nebenbedingungen, Informationskriterien, sowie Verfahren der numerischen Integration zusammengestellt. Jedes Kapitel beinhaltet Aufgaben zur vertiefenden Beschäftigung mit dem Inhalt.

Vergleicht man das Buch mit konkurrierenden Publikationen mit ähnlichem Fokus, so fällt insbesondere der Einbezug moderner Erweiterungen aus dem Bereich der Regularisierung auf. Insgesamt stellt das Buch eine Reihe sehr aktueller Entwicklungen vor und macht diese im Kontext der kategorialen Regression sehr gut zugänglich. Darüber hinaus zeichnet sich das Buch durch eine große Breite aus, so dass neben den Kernthemen der kategorialen Regression auch eine ganze Reihe angrenzender Themengebiete behandelt werden und somit auch dem mit den Grundprinzipien der kategorialen Regression vertrauten Leser zahlreiche neue Anregungen zur Verfügung gestellt werden. Das Buch erfüllt damit voll und ganz den vom Autor angestrebten Zweck, Statistikern und angewandten Forschern ebenso wie Studierenden das Gebiet der kategorialen Regression zu erschließen und ansprechend vorzustellen.

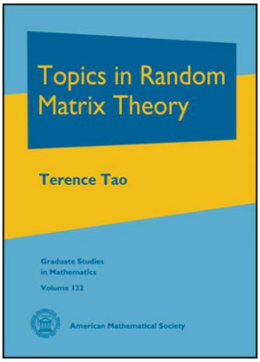
Jahresber Dtsch Math-Ver (2013) 115:57–59
DOI 10.1365/s13291-013-0059-1

BOOK REVIEW

Terence Tao: "Topics in Random Matrix Theory"
AMS, 2012, 282 pp.

Benjamin Schlein

Published online: 6 February 2013



The book under consideration is a basic introduction to the field of random matrix theory, which has been very active in the last years. It aims at graduate students interested in this subject or willing to start to work in this direction.

The general goal of random matrix theory is the understanding of the statistical properties of the eigenvalues and of the eigenvectors of $N \times N$ matrices, whose entries are random variables with a given probability law. Typically, one is interested in the behavior of the spectrum in the limit of large N.

Random matrices have first been introduced in the fifties by Wigner, to describe the excitation spectrum of heavy nuclei. Wigner's basic idea was the following. When dealing with very complex systems, it is impossible to write down the precise Hamilton operator. Instead, it makes sense to assume the matrix entries of the Hamiltonian to be random variables, and to establish properties of the eigenvalues which hold for typical realizations of the disorder (the eigenvalues of the Hamiltonian are the energy level of the system and determine the excitation spectrum observed in experiments). It turns out that Wigner's idea was very successful and, to this day, random matrices are widely used in nuclear physics to predict the spectrum of heavy nuclei, at least as a first approximation. Since their introduction, random matrices have been linked to several other branches of mathematics and physics. The spectrum of a large class of disordered and chaotic systems shares many similarities with the one of simple ensembles of random matrices. The success of Wigner's intuition and their ubiquitous appearance are signs

B. Schlein (✉)
Bonn, Germany
e-mail: benjamin.schlein@hausdorff-center.uni-bonn.de

for one of the most remarkable properties of random matrices; universality. In vague terms, universality refers to the fact that local spectral properties of systems with randomness depend on the underlying symmetry but are largely independent of further details, like the precise distribution of the randomness.

A simple ensemble of random matrices is the so called Gaussian Unitary Ensemble, or GUE, consisting of $N \times N$ hermitian matrices $H = (h_{ij})$ whose entries are, up to the symmetry constraints, independent centered Gaussian variables. Analogously, one can define the Gaussian Orthogonal Ensemble, or GOE, consisting of real symmetric matrices and the Gaussian Symplectic Ensemble, or GSE, consisting of quaternion hermitian matrices with Gaussian entries. The probability density for all these ensembles is proportional to $\exp(-\operatorname{tr} H^2)$. Thanks to their invariance (with respect to unitary, orthogonal and, respectively, symplectic conjugations), the joint probability density of Gaussian ensembles is explicit. The computation of statistics of eigenvalues in the limit of large N reduces, in this case, to the study of the asymptotics of families of orthogonal polynomials. Dyson established that the local eigenvalue correlations of GUE approaches the Wigner-Dyson sine-kernel distribution; similar results, with appropriately modified sine-kernels, hold for GOE and GSE.

A natural extension of GUE/GOE/GSE are so called invariant ensembles, consisting of $N \times N$ hermitian matrices H with density proportional to $\exp(-\operatorname{tr} V(H))$, for a regular function V which grows sufficiently fast at infinity. These ensembles are still invariant, but their entries are not independent (unless $V(s) = s^2$). Universality for invariant ensembles was established in [1, 7]; the local eigenvalue correlations are independent of the choice of V and are always described by the same distribution observed in the Gaussian case. Another natural extension of the Gaussian ensembles are (hermitian, real symmetric or quaternion hermitian) Wigner matrices, whose entries are, up to the symmetry constraints, independent and identically distributed centered random variables. Because of the absence of an explicit expression for the joint distribution of the eigenvalues, the analysis of the spectral properties of Wigner matrices requires completely different techniques.

A first fundamental result about Wigner matrices was established by Wigner in [11]. After appropriate rescaling of the entries, the density of the eigenvalues, also known as the density of states, converges in probability towards the famous semicircle law, independently of the specific distribution of the entries. Wigner's original result concerns the density of states on intervals containing typically a very large number of eigenvalues, of the order N. In the last years, this result has been substantially improved, to show that the density of states converges towards the semicircle law already on much smaller intervals, containing typically much fewer eigenvalues (up to order one). This local semicircle law (proven, for example, in [5]) gives much finer information about the spectrum of Wigner matrices, compared with Wigner's original result. In particular, it has been applied to show universality; the local eigenvalue correlations of Wigner matrices converge, in the limit of large N, to the same distributions observed for the corresponding Gaussian ensembles (we are referring here to eigenvalue correlations in the bulk; edge universality has been previously established in [8]). For hermitian Wigner matrices this result was first obtained, with different approaches and under different conditions, in [4, 9]. For ensembles with arbitrary symmetry (hermitian, real symmetric or quaternion hermitian), it was established in [6]. For more recent and stronger result, see [2, 10].

One of the main challenges facing researchers in random matrices in the next years consists in extending universality from mean-field type models like Wigner matrices towards systems with non-trivial structure. In this sense, band matrices provide a very interesting and intriguing model. Band matrices are $N \times N$ matrices, whose entries are independent (up to the symmetry constraints) random variables, with a variance which rapidly decays to zero, outside a strip of size W around the diagonal. Depending on W, one expects two different regimes. For $W \ll \sqrt{N}$, the eigenvalues are expected to be independent of each others, following a Poisson distribution. For $W \gg \sqrt{N}$, on the other hand, one expects the same local eigenvalue correlations observed for Wigner matrices. Some important partial results in this direction has been recently obtained in [3], but new ideas are probably required to solve this fascinating question.

The book "Topics in Random Matrix Theory" by Terence Tao is based on a graduate course that the author gave at UCLA in 2010. It contains three chapters. The first chapter contains a general introduction to probability theory and additional preparatory material related to eigenvalues of sums of hermitian matrices. Chapter two starts with a review of the phenomenon of concentration of measure, which turns out to be an important tool for the analysis of random matrices. It continues with a discussion of basic topics in random matrix theory, including the operator norm of random matrices, the semicircle law of Wigner matrices, the circular law for matrices without symmetry and an introduction to free probability. Chapter three, on the other hand, is dedicated to a selection of related subjects; Dyson Brownian motion, the Golden-Thompson inequality and the derivation of Wigner-Dyson's sine-kernel for the (bulk) correlations of GUE (and of the Airy kernel for the edge correlations).

References

1. Deift, P., Kriecherbauer, T., McLaughlin, K.T.-R., Venakides, S., Zhou, X.: Uniform asymptotics for polynomials orthogonal with respect to varying exponential weights and applications to universality questions in random matrix theory. Commun. Pure Appl. Math. **52**, 1335–1425 (1999)
2. Erdős, L., Knowles, A., Yau, H.-T., Yin, J.: Spectral Statistics of Erdős–Rényi Graphs. II. Eigenvalue Spacing and the Extreme Eigenvalues. Preprint arXiv:1103.3869
3. Erdős, L., Knowles, A., Yau, H.-T., Yin, J.: Delocalization and Diffusion Profile for Random Band Matrices. Preprint arXiv:1205.5669
4. Erdős, L., Péché, S., Ramírez, J., Schlein, B., Yau, H.-T.: Bulk universality for Wigner matrices. Commun. Pure Appl. Math. **63**, 895–925 (2010)
5. Erdős, L., Schlein, B., Yau, H.-T.: Wegner estimate and level repulsion for Wigner random matrices. Int. Math. Res. Not. **3**, 436–479 (2010)
6. Erdős, L., Schlein, B., Yau, H.-T.: Universality of random matrices and local relaxation flow. Invent. Math. **185**(1), 75–119 (2011)
7. Pastur, L., Shcherbina, M.: Bulk universality and related properties of Hermitian matrix models. J. Stat. Phys. **130**(2), 205–250 (2008)
8. Soshnikov, A.: Universality at the edge of the spectrum in Wigner random matrices. Commun. Math. Phys. **207**(3), 697–733 (1999)
9. Tao, T., Vu, V.: Random Matrices: universality of the local eigenvalue statistics. Acta Math. **206**, 127–204 (2011)
10. Tao, T., Vu, V.: The Wigner-Dyson-Mehta bulk universality conjecture for Wigner matrices. Preprint arXiv:1101.5707
11. Wigner, E.: Characteristic vectors of bordered matrices with infinite dimensions. Ann. Math. **62**, 548–564 (1955)