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PREFACE

Preface Issue 2-2013

Hans-Christoph Grunau

"Curvature driven interface evolution" is one of the major current themes in the numerics resp. the numerical analysis resp. the analytical theory of partial differential equations. It is particularly attractive due to numerous physical applications on the one hand and due to close relations to differential geometry on the other hand. Harald Garcke gives a survey on the most important topics in this research area. Among these are models for crystal growth of snow flakes, melting ice in water or grain coarsening (aging) in two-phase mixtures: Stefan problem, Mullins-Sekerka-problem and variants. Moreover, the mean curvature and surface diffusion flow are covered as well as phase field models like the Allen-Cahn and Cahn-Hilliard equation. In the latter sharp classical interfaces are replaced by interfacial regions. All of these models are introduced, their physical applicability as well as their main properties are explained and they are illustrated by means of numerical simulations. The present survey will certainly prove to be a very helpful reference work for anybody working on resp. interested in these kinds of models.

The integral group ring $\mathbb{Z}G$ of a group G consists of elements of the form $\sum_{g\in G} z_g g$, where only finitely many of the coefficients $z_g \in \mathbb{Z}$ may be different from 0. These elements form a free $\mathbb{Z}$-module with G as a basis, while the multiplication in $\mathbb{Z}G$ is the $\mathbb{Z}$-linear extension of that in G. Wolfgang Kimmerle's survey article gives a brief account of classical results and focusses then on recent achievements and developments concerning the structure of the unit group of $\mathbb{Z}G$.

The Zentralblatt für Mathematik and the Jahresbericht der DMV begin a collaboration to review again classical books and the effect that they have had or could have had on the development of specific mathematical branches from today's point of view.

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In this issue Franz Lemmermeyer reviews the 2012-reprint of the 1893 edition of Richard Dedekind's book: "Was sind und was sollen die Zahlen?" It was Dedekind's purpose to provide a safe foundation for the natural numbers. However, his rather abstract approach was not appreciated by his contemporaries and Franz Lemmermeyer outlines why nowadays Peano's approach instead of Dedekind's is mostly used. Further book reviews concern recently released books dealing with combinatorial set theory and with regularity issues in obstacle problems.

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SURVEY ARTICLE

Curvature Driven Interface Evolution

Harald Garcke

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Abstract Curvature driven surface evolution plays an important role in geometry, applied mathematics and in the natural sciences. In this paper geometric evolution equations such as mean curvature flow and its fourth order analogue motion by surface diffusion are studied as examples of gradient flows of the area functional. Also in many free boundary problems the motion of an interface is given by an evolution law involving curvature quantities. We will introduce the Mullins-Sekerka flow and the Stefan problem with its anisotropic variants and discuss their properties.

In phase field models the area functional is replaced by a Ginzburg-Landau functional leading to a diffuse interface model. We derive the Allen-Cahn equation, the Cahn-Hilliard equation and the phase field system as gradient flows and relate them to sharp interface evolution laws.

Keywords Mean curvature flow · Gradient flow · Surface diffusion · Mullins-Sekerka problem · Stefan problem · Crystal growth · Phase field equation · Allen-Cahn equation · Cahn-Hilliard equation

Mathematics Subject Classification (2000) 53C44 · 53K93 · 35K91 · 35R35 · 35K55 · 49Q20 · 53A10 · 80A22 · 82B24

1 Introduction

The motion of hypersurfaces in Euclidean space driven by a law for the normal velocity which involves curvature quantities plays an important role in geometry, analysis and in many applications. The most prominent example is the *mean curvature flow*, where the normal velocity of a hypersurface is given as the mean curvature of the

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surface. This evolution law appears in geometry but variants of this flow have applications in image processing and they also describe the evolution of so-called grain boundaries in materials science.

We will explain later that mean curvature flow in some sense is the most efficient way to decrease the area of a surface, in particular mean curvature flow turns out to be a gradient flow of the area functional. For surfaces that bound a region mean curvature flow typically decreases the enclosed volume. If one wants to preserve the enclosed volume one could study a volume conserving *nonlocal mean curvature flow*. A flow with the same property which has more physical applications is the *surface diffusion flow*. Here, the normal velocity is given by minus the surface Laplacian of the mean curvature.

In some physical systems the surface energy of an interface is proportional to the total surface area of the interface. This is true for example for soap bubbles and related variational problems lead to the geometry and analysis of minimal surfaces and H-surfaces. We refer to the beautiful book by Hildebrandt and Tromba [103] and to [51, 126] for details.

We will also study the evolution of surfaces bounding a crystal. For crystals the surface area is not the appropriate energy. Instead the surface contribution to the energy will locally depend on the orientation of the surface in its surrounding space. The resulting surface energy will be anisotropic and related variational problems will lead to surfaces for which an anisotropic mean curvature will be either zero or constant. It is of course possible to consider gradient flows of such energies and this will lead us to the anisotropic mean curvature flow.

Often the evolution of a surface is influenced by quantities which are defined away from the surface. We will discuss the growth of a crystal and in this case the evolution of the crystal surface will be influenced for example by the temperature. In fact one has to solve a heat equation for the temperature away from the surface and in some models the temperature enters the mean curvature flow equation as an additional right-hand side.

Typically the topology of the surface will change during the evolution. If this happens a classical description of the surface involving parametrizations will develop singularities and hence will break down. We will discuss two approaches which will allow to pass through singularities. The first one involves Caccioppoli sets which are sets for which the characteristic function is of bounded variation. The second one is the phase field approach which describes the interface with the help of a smooth function which in an appropriate way approximates the characteristic function mentioned above. Another popular approach which is suitable to deal with topology changes is the level set method and we refer to [41, 75, 90, 137, 155, 157].

In this overview article I can of course only describe a few aspects of curvature driven interface evolution. I will focus on mean curvature flow, its fourth order analogue motion by surface diffusion and on crystal growth described by the Stefan problem with Gibbs–Thomson law. The latter is a paradigm free boundary problem. In a free boundary problem one seeks a solution of a partial differential equation on a domain which one has to find as part of the problem.

Curvature driven interfaces play an important role also in other areas. In geometry one is also interested in situations where the evolution of a surface is driven by

laws involving other curvature quantities such as the Gauss curvature, the scalar curvature or quantities involving the principal curvatures. Let me finally also mention applications in which curvature driven interface evolution plays a role. Examples are two-phase and free surface flow [144, 159], image analysis [7, 33, 34, 151], grain boundary motion [130], quantum dot formation [160], evolution of nanoporosity in dealloying [60], void evolution in electromigration [44], and flame propagation [156]. This list demonstrates that it is important both from a mathematical and from an applicational point of view to understand curvature driven interface evolution.

In order to illustrate the evolution laws discussed in this article we will frequently use numerical computations which were obtained with the help of parametric finite element methods which have been developed in the last years together with Barrett and Nürnberg [8–13].

2 Gradient Flows of the Area Functional

2.1 First Variation of the Area Functional

We consider a smooth, compact, oriented hypersurface Γ in $\mathbb{R}^d$ without boundary. The simplest surface energy of such a hypersurface Γ is proportional to the surface area of Γ. We hence consider the area functional

$$E(\Gamma) := \mathcal{H}^{d-1}(\Gamma) \tag{1}$$

where $\mathcal{H}^{d-1}$ is the $(d-1)$-dimensional surface measure. The goal now is to evolve Γ in such a way that the surface area decreases most rapidly. Roughly speaking this will be achieved by flowing Γ in the direction of the negative "gradient" of E. In order to define the gradient we first of all need to determine the first variation (the "derivative") of the area functional.

In order to compute a directional derivative of E we need to embed Γ in a one-parameter family of surfaces. This will be achieved with the help of a smooth vector field $\zeta : \mathbb{R}^d \to \mathbb{R}^d$. We define

$$\Gamma_t := \big\{ x + t\zeta(x) \mid x \in \Gamma \big\}, \quad t \in \mathbb{R}, \tag{2}$$

and a computation, see e.g. [49, 97, 117], gives

$$\frac{d}{dt} E(\Gamma_t)_{|t=0} = -\int_\Gamma H V \, d\mathcal{H}^{d-1}. \tag{3}$$

Here H is the mean curvature of Γ (which in this article will be, as often in the literature, the sum of the principal curvatures), $V = \zeta \cdot \nu$ is the normal velocity of the evolving surface $(\Gamma_t)_{t\in\mathbb{R}}$ at $t = 0$, $\zeta \cdot \nu$ is the Euclidean inner product of ζ and ν, and by $d\mathcal{H}^{d-1}$ we denote integration with respect to the $(d-1)$-dimensional surface measure. On Γ we have chosen a normal vector field ν and we here take the sign convention that the surface has positive mean curvature if it is curved in the direction of the normal. The formula (3) now shows that the surface area decreases if the surface moves in the direction of the mean curvature vector $H\nu$.

2.2 Gradient Flows

For a function $\Phi : \mathbb{R}^n \to \mathbb{R}$ with derivative $d\Phi_{x_0}$ at the point $x_0 \in \mathbb{R}^n$ we define the gradient $\operatorname{grad} \Phi(x_0) \in \mathbb{R}^n$ such that the following identity holds

$$d\Phi_{x_0}(v) = \big(\operatorname{grad} \Phi(x_0)\big) \cdot v \quad \text{for all } v \in \mathbb{R}^n.$$

Now $x : [0, T] \to \mathbb{R}^n$ is a solution of the gradient flow equation to Φ if

$$x'(t) = -\operatorname{grad} \Phi\big(x(t)\big) \tag{4}$$

holds for all $t \in [0, T]$. In particular, we have

$$\frac{d}{dt}\Phi\big(x(t)\big) = -\big\|\operatorname{grad} \Phi\big(x(t)\big)\big\|^2 \leq 0$$

where $\|\,.\,\|$ denotes the Euclidean norm in $\mathbb{R}^n$. In particular, we obtain that $\Phi(x(t))$ can only decrease in time.

For any $y : [0, T] \to \mathbb{R}^n$ with $\|y'(0)\| = \|\operatorname{grad} \Phi(x(0))\|$ and $y(0) = x(0)$ we have, using the Cauchy-Schwarz inequality

$$\begin{aligned}\frac{d}{dt}\Phi\big(y(0)\big) &= \big(\operatorname{grad} \Phi\big(y(0)\big)\big) \cdot y'(0) \\ &= \big(\operatorname{grad} \Phi\big(x(0)\big)\big) \cdot y'(0) \\ &\geq -\big\|\operatorname{grad} \Phi\big(x(0)\big)\big\|^2\end{aligned}$$

with an equality if and only if

$$y'(0) = -\operatorname{grad} \Phi\big(x(0)\big).$$

This shows that among all possible directions, the direction $-\operatorname{grad} \Phi(x(0))$ decreases Φ most efficiently.

The above considerations can be generalized to n-dimensional Riemannian manifolds M and functions $\Phi : M \to \mathbb{R}$. Denoting by $T_x M$ the tangent space at $x \in M$, by $\langle .,. \rangle$ the metric on $T_x M$ and by $d_x \Phi$ the differential of Φ, the gradient $\operatorname{grad}_M \Phi \in T_x M$ is defined such that

$$d_x \Phi(v) = \big\langle \operatorname{grad}_M \Phi(x), v \big\rangle \quad \text{for all } v \in T_x M$$

holds. Hence, as above the flow

$$x'(t) = -\operatorname{grad}_M \Phi\big(x(t)\big)$$

decreases Φ as fast as possible among all velocities with a given value for the norm of the velocity.

Choosing a time step $\tau > 0$, a natural approximation scheme for (4) would be to solve iteratively for $x^1, x^2, x^3 \ldots$ with a given initial value x^0 the (nonlinear) equation

$$\frac{x^n - x^{n-1}}{\tau} = -\operatorname{grad} \Phi\big(x^n\big), \quad n = 1, 2, 3, \ldots \tag{5}$$

which is an implicit Euler discretization for (4). The identity (5) is the Euler–Lagrange equation of the functional

$$\frac{1}{2\tau}\left\|x - x^{n-1}\right\|^2 + \Phi(x). \tag{6}$$

Now a natural approach to show existence of solutions to (5) is the study of the minimum problem to (6) which often can be solved by the direct method of the calculus of variations.

The approach can be naturally generalized to Hilbert spaces by replacing the Euclidean inner product by a general scalar product. It is also possible to generalize the scheme to metric spaces by replacing the norm of $x - x^{n-1}$ by the distance between x and x^{n-1}, i.e. one now considers

$$\frac{1}{2\tau}d^2\big(x, x^{n-1}\big) + \Phi(x).$$

We refer to Ambrosio, Gigli and Savaré [3] and Villani [170] for more details on the general gradient flow approach and to Luckhaus [115], Visintin [171], Almgren, Taylor, Wang [5], Luckhaus, Sturzenhecker [116], Mielke, Theil, Levitas [122], Otto [138] and Garcke, Schaubeck [85] for applications of the approach in specific situations.

2.3 Mean Curvature Flow as a Gradient Flow of the Area Functional

We formally endow the space $\mathcal{M}$ of all oriented hypersurfaces Γ in $\mathbb{R}^d$ with a tangent space which consists of all possible normal velocities, i.e. we set

$$T_\Gamma\mathcal{M} = \{V : \Gamma \to \mathbb{R}\}.$$

A function $V : \Gamma \to \mathbb{R}$ arises as a "tangent" vector, i.e. as a differential of a curve in $\mathcal{M}$, if we consider a vector field $\zeta : \mathbb{R}^d \to \mathbb{R}^d$ such that $\zeta \cdot \nu = V$ on Γ and define Γ_t as in (2). One natural choice of an inner product on $T_\Gamma\mathcal{M}$ is given by

$$\langle v_1, v_2\rangle_{L^2} = \int_\Gamma v_1 v_2 \, d\mathcal{H}^{d-1} \quad \text{for all } v_1, v_2 \in T_\Gamma\mathcal{M}.$$

Now the gradient $\operatorname{grad}_\mathcal{M} E$ of E needs to fulfill

$$\langle \operatorname{grad}_\mathcal{M} E, V\rangle_{L^2} = \frac{d}{dt}E(\Gamma_t)_{|t=0} = -\int_\Gamma HV \, d\mathcal{H}^{d-1}$$

for all $V : \Gamma \to \mathbb{R}$. Here $(\Gamma_t)_{t\in\mathbb{R}}$ is defined as above by choosing a $\zeta : \mathbb{R}^d \to \mathbb{R}^d$ such that $\zeta \cdot \nu = V$. We hence obtain

$$\operatorname{grad}_\mathcal{M} E = -H$$

and the gradient flow of the area functional E is the mean curvature flow

$$V = H.$$

More precisely, we say that a smooth one-parameter family $(\Gamma_t)_{t\geq 0}$ of hypersurfaces in $\mathbb{R}^d$ solves $V = H$ if for a local parametrization $X(t, p)$, $p \in U$, $U \subset \mathbb{R}^{d-1}$ open, it holds that

$$\partial_t X \cdot \nu = H.$$

In particular, we obtain

$$\frac{d}{dt}\mathcal{H}^{d-1}(\Gamma_t) = -\int_{\Gamma_t} H^2 \leq 0.$$

For more information on mean curvature flow we refer to the articles by Ecker [58] and White [178] and the books [22, 57, 90, 109, 117, 146].

Let me mention a few fundamental properties and results related to the mean curvature flow.

- An embedded curve in the plane evolving under curvature flow $V = H$ will become convex in finite time, see Grayson [98].
- A convex hypersurface in $\mathbb{R}^d$, i.e. a surface which is the boundary of a convex region, will shrink to a point in finite time. In doing so the surface will become more and more round, i.e. after rescaling to a surface enclosing a fixed volume the surface will converge to a sphere, see Gage, Hamilton [81] and Huisken [104].
- Mean curvature flow, written for example in local coordinates, leads to a parabolic equation of second order (the normal velocity leads to a time derivative and the mean curvature to two spatial derivatives). Second order parabolic partial differential equations fulfill *maximum and comparison principles* which play a fundamental role in the analysis of mean curvature flow. With the help of a comparison principle it can be shown that self-intersections during the flow are not possible and one can also show that if initially one surface is contained in another this property will be true for all later times, see e.g. the discussion in Ecker [58].
- Nonconvex surfaces in general can develop singularities, see Figs. 1, 2 for numerical computations with a torus as initial surface. Depending on the ratio of the torus's radii, the torus will either merge, see Fig. 1 or shrink to a circle, see Fig. 2. The possible singularities are well understood and in particular the famous monotonicity formula for mean curvature flow is important in order to classify the singularities, see [58, 106, 107, 117]. "The monotonicity formula intuitively says that under mean curvature flow the area of a hypersurface near any point is nonincreasing on any scale."[1]

Mean curvature flow does not preserve the volume enclosed by the surface. In many applications physical conservation laws lead to volume conservation and this motivates the discussion of volume preserving geometric gradient flows which are discussed in the next subsections.

2.4 Volume Preserving Geometric Flows

We now consider $\mathcal{M}_m$ to be the "manifold" of all smooth hypersurfaces $\Gamma \subset \mathbb{R}^d$ enclosing a bounded set $G \subset \mathbb{R}^d$ which has a prescribed volume $m \in \mathbb{R}^+$. It will

[1] Here we cite Ecker [54, p. 53].

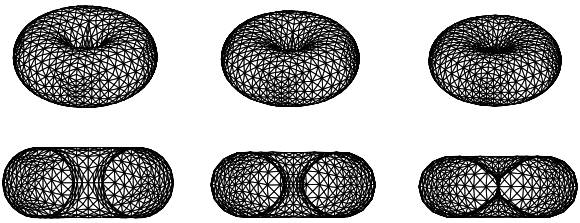


Fig. 1 Plots of a solution to mean curvature flow at times $t = 0, 0.05, 0.09$

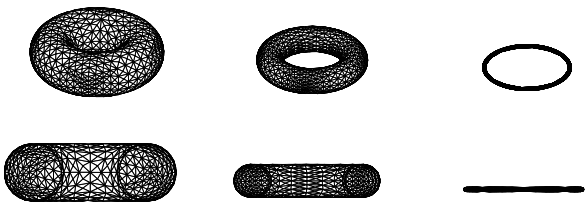


Fig. 2 Plots of a solution to mean curvature flow at times $t = 0, 0.1, 0.138$

turn out that the natural tangent space $T_\Gamma \mathcal{M}_m$ corresponds formally to all normal velocities $V : \Gamma \to \mathbb{R}$ with zero mean. If we choose perturbations $(\Gamma_t)_{t\in\mathbb{R}}$ of Γ as above we obtain for the enclosed volume $\mathrm{vol}(\Gamma_t)$ the following identity, see [49],

$$\frac{d}{dt}\,\mathrm{vol}(\Gamma_t) = \int_{\Gamma_t} V\, d\mathcal{H}^{d-1},$$

where we now choose ν as the outer unit normal to the set enclosed by the hypersurface Γ. This implies that the integral of the normal velocity needs to be zero to ensure that the volume is conserved. We can also endow $T_\Gamma \mathcal{M}_m$ with the L^2-inner product. The gradient $\mathrm{grad}_{\mathcal{M}_m} E$ has to fulfill $\int_\Gamma (\mathrm{grad}_{\mathcal{M}_m} E)\, d\mathcal{H}^{d-1} = 0$ and

$$\langle \mathrm{grad}_{\mathcal{M}_m} E, v\rangle = -\int_\Gamma H v\, d\mathcal{H}^{d-1} \tag{7}$$

for all v with $\int_\Gamma v\, d\mathcal{H}^{d-1} = 0$. The identity (7) does specify $\mathrm{grad}_{\mathcal{M}_m} E$ only up to a constant and since $\mathrm{grad}_{\mathcal{M}_m} E$ needs to have zero mean we obtain

$$\mathrm{grad}_{\mathcal{M}_m} E = -H + \fint_\Gamma H\, d\mathcal{H}^{d-1}$$

where $\fint_\Gamma H\, d\mathcal{H}^{d-1} = \int_\Gamma H\, d\mathcal{H}^{d-1}/(\int_\Gamma 1\, d\mathcal{H}^{d-1})$ is the average of H on Γ. The volume preserving mean curvature flow is hence given as

$$V = H - \fint_\Omega H\, d\mathcal{H}^{d-1}. \tag{8}$$

A more physical gradient flow is obtained when we choose an H^{-1}-inner product on $T_\Gamma \mathcal{M}_m$. For given $v_1, v_2 \in T_\Gamma \mathcal{M}_m$ we solve

$$-\Delta_\Gamma u_i = v_i \quad \text{on } \Gamma,$$

where Δ_Γ is the surface Laplacian on Γ. Since Γ has no boundary the Gauss theorem on manifolds gives $\int_\Gamma \Delta_\Gamma u_i \, d\mathcal{H}^{d-1} = 0$ which implies the solvability condition $\int_\Gamma v_i \, d\mathcal{H}^{d-1} = 0$ which is fulfilled due to $v_i \in T_\Gamma \mathcal{M}_m$. Setting

$$u_i := (-\Delta_\Gamma)^{-1} v_i$$

we define the H^{-1}-inner product on $T_\Gamma \mathcal{M}_m$ as

$$\begin{aligned}\langle v_1, v_2 \rangle_{H^{-1}} &:= \int_\Gamma \left(\nabla_\Gamma (-\Delta_\Gamma)^{-1} v_1\right) \cdot \left(\nabla_\Gamma (-\Delta_\Gamma)^{-1} v_2\right) d\mathcal{H}^{d-1} \\ &= \int_\Gamma v_1 (-\Delta_\Gamma)^{-1} v_2 \, d\mathcal{H}^{d-1}.\end{aligned}$$

In order to define the gradient $\operatorname{grad}_{H^{-1}} E$ of E with respect to the H^{-1}-inner product we observe that the following identities need to hold for all $v \in T_\Gamma \mathcal{M}_m$

$$\int_\Gamma v (-\Delta_\Gamma)^{-1} \operatorname{grad}_{H^{-1}} E \, d\mathcal{H}^{d-1} = \langle v, \operatorname{grad}_{H^{-1}} E \rangle_{H^{-1}} = -\int_\Gamma v H \, d\mathcal{H}^{d-1}.$$

We hence obtain

$$\operatorname{grad}_{H^{-1}} E = \Delta_\Gamma H$$

and the H^{-1}-gradient flow of E is given as

$$V = -\Delta_\Gamma H. \tag{9}$$

This evolution law is called motion by surface diffusion. In physics this evolution law is derived from mass conservation laws using appropriate constitutive assumptions, see [47, 131, 167]. It models phase transformation due to diffusion along the interface. This evolution law can be derived from the Cahn-Hilliard diffusion equation, see Sect. 5, when diffusion is restricted to an interfacial layer, see [32, 62], and this also motivates that the law (9) is called motion by surface diffusion.

Proposition 1 *Solutions* $(\Gamma_t)_{t \geq 0}$ *of* (9) *fulfill*

$$\begin{aligned}\frac{d}{dt} \operatorname{vol}(\Gamma_t) &= 0, \\ \frac{d}{dt} \operatorname{Area}(\Gamma_t) &\leq 0.\end{aligned}$$

Proof The second property follows from the fact that (9) is the gradient flow of the area functional with respect to the H^{-1}-inner product. The fact that the flow is volume preserving follows from

$$\frac{d}{dt} \operatorname{vol}(\Gamma_t) = \int_{\Gamma_t} V \, d\mathcal{H}^{d-1} = -\int_{\Gamma_t} \Delta_\Gamma H \, d\mathcal{H}^{d-1} = 0,$$

where the last identity is a consequence of Gauss' theorem. □

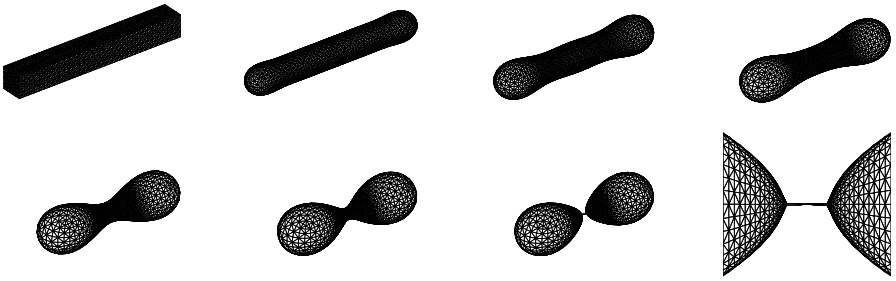


Fig. 3 Plots of a solution to the surface diffusion flow at times $t = 0, 0.01, 0.1, 0.2, 0.3, 0.36, 0.369$ (left to right, top to bottom). The final plot shows a blow up of the pinch-off at time $t = 0.369$

In the evolution law $V = -\Delta_\Gamma H$ the surface Laplacian—a second order operator—acts on the mean curvature and hence the flow leads to a fourth order parabolic partial differential equation. For fourth order parabolic equations maximum and comparison principles are in general not true. Also a monotonicity formula for the flow $V = -\Delta_\Gamma H$ is not known. These two facts are among the reasons why much less is known for the flow $V = -\Delta_\Gamma H$ in comparison to the mean curvature flow.

In the following I state a few results known for the surface diffusion flow.

- Short time existence and uniqueness of classical solutions is known. Spheres are asymptotically stable under the flow in the following sense. For initial data sufficiently close to a sphere a global solution exists and the solution will converge to a (possible different) sphere, see Elliott, Garcke [63] for curves in the plane and Escher, Mayer, Simonett [72] for higher dimensions.
- The flow $V = -\Delta_\Gamma H$ defines an analytic semigroup with some interesting properties. The set of equilibria is not isolated and in order to show stability of spheres either center manifold theory, see Escher, Mayer, Simonett [72], or a generalized principle of linearized stability has to be used, see Prüss, Simonett, Zacher [143].
- In the plane it can be shown that if a simple closed curve evolving under $V = -\Delta_\Gamma H$ exists for all time, it necessarily has to converge to a sphere, see Elliott and Garcke [63]. We also refer to a recent result by Wheeler [177] who proved that closed curves with initial data close to a round circle in the sense that the L^2-perturbation of the curvature remains small exist for all time and converges exponentially fast to a circle. A similar result also holds in higher dimensions, see [176].
- In contrast to mean curvature flow self intersections are possible, as was conjectured by Elliott, Garcke [63] and shown by Giga, Ito [94] and Mayer, Simonett [120], see also Blatt [19] for some recent results.
- The surface diffusion flow does not preserve convexity, see Fig. 3, which was shown by Giga, Ito [93], see also Blatt [19].
- A sharp criterion for finite time blow up of curves moving under the surface diffusion flow has been given by Chou [42].
- The sphere is stable under surface diffusion (see results above), while the cylinder is long wave unstable. Numerical simulations indicate that perturbations of a

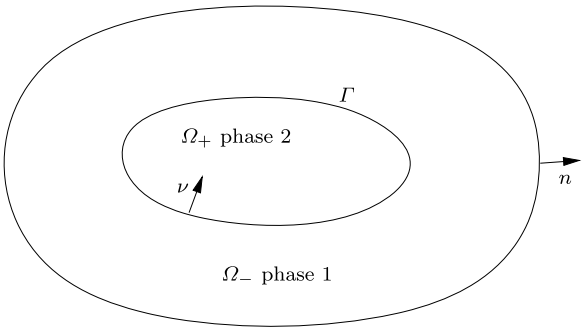


Fig. 4 An illustration of the geometry in the Mullins-Sekerka and the Stefan problem

cylinder can lead to finite-time pinch-off. A paper by Bernoff, Bertozzi and Witelski [18] studies the selfsimilar structure close to the pinch-off.
- Initial data which are given as a graph can loose this property during the evolution, see [64].
- Since $V = -\Delta_\Gamma H$ leads to a parabolic equation, the flow has a regularizing effect, i.e. in particular edges and corners will become smooth during the flow. As for the mean curvature flow also the surface diffusion flow in general will develop singularities. The regularizing effect, the fact that surface diffusion does not preserve convexity and the formation of singularities can be observed in the numerical simulation in Fig. 3.

2.5 The Mullins-Sekerka Free Boundary Problem as a Gradient Flow of the Area Functional

We now consider a compact hypersurface Γ in $\mathbb{R}^d$ which separates two open sets Ω_- and Ω_+ in an open domain $\Omega \subset \mathbb{R}^d$, see Fig. 4.

Let ν be the unit normal to Γ pointing into Ω_+ and n the outer unit normal to $\partial\Omega$. The Mullins-Sekerka free boundary problem describes the evolution of the spatial distribution of two phases (here the phases occupy the regions Ω_- and Ω_+) driven by the reduction of interfacial area and limited by diffusion. This evolution law can be derived from conservation laws taking the principles of thermodynamics into account, see [56, 100]. But here we choose an approach which derives the Mullins-Sekerka problem in the context of gradient flows, see e.g. [76, 77, 119, 133].

We again consider the set of surfaces $\mathcal{M}_m$ which enclose a volume $m \in \mathbb{R}^+$ and its tangent space $T_\Gamma\mathcal{M}_m$. In order to define the metric on $T_\Gamma\mathcal{M}_m$ we define functions $u_1, u_2 : \Omega \to \mathbb{R}$ for given $v_1, v_2 \in T_\Gamma\mathcal{M}_m$ which are solutions of

$$-\Delta u_i = 0 \quad \text{in } \Omega_- \cup \Omega_+, \tag{10}$$

$$-[\nabla u_i]^+_- \cdot \nu = v_i \quad \text{on } \Gamma, \tag{11}$$

$$[u_i]^+_- = 0 \quad \text{on } \Gamma, \tag{12}$$

$$\nabla u_i \cdot n = 0 \quad \text{on } \partial\Omega. \tag{13}$$

Here $[\,.\,]^+_-$ denotes the jump of a quantity across the interface Γ where we subtract the value in the $-$-phase from the value in the $+$-phase. The above system (10)–(13)

determines functions u_1, u_2 up to a constant which will be irrelevant for what follows. The metric $\langle .\,, .\rangle_{MS}$ on $T_\Gamma \mathcal{M}_m$ is now for all $v_1, v_2 \in T_\Gamma \mathcal{M}_m$ defined by

$$\langle v_1, v_2\rangle_{MS} := \int_{\Omega_- \cup \Omega_+} \nabla u_1 \cdot \nabla u_2 dx = \int_\Gamma v_1 u_2 \, d\mathcal{H}^{d-1},$$

where u_1, u_2 solve (10)–(13). We remark that the above system, which has to be solved in order to determine u_i, can be written in distributional form as

$$-\Delta u_i = v_i \delta_\Gamma, \tag{14}$$

where δ_Γ is a surface Dirac distribution defined by $\delta_\Gamma(\zeta) = \int_\Gamma \zeta$ for all test functions ζ. Hence formally $u_i = (-\Delta)^{-1}(v_i \delta_\Gamma)$ and hence $\langle .\,, .\rangle_{MS}$ can also be interpreted as an H^{-1}-inner product. We remark that a weak formulation of (14) is given as

$$\int_\Omega \nabla u_i \cdot \nabla \phi \, dx = \int_\Gamma v_i \phi \, d\mathcal{H}^{d-1}$$

which has to hold for all ϕ in the Sobolev space $H^1(\Omega) = \{f \in L^2(\Omega) \mid \nabla f \in L^2(\Omega, \mathbb{R}^d)\}$.

Now the negative gradient

$$w = -\mathrm{grad}_{MS} E \in T_\Gamma \mathcal{M}_m$$

needs to fulfill for all $v \in T_\Gamma \mathcal{M}_m$

$$\int_\Gamma v u \, d\mathcal{H}^{d-1} = \langle v, w\rangle_{MS} = \int_\Gamma v H \, d\mathcal{H}^{d-1}$$

where u is a solution to (10)–(13) with $v_i = w$.

In particular, we obtain that, up to an irrelevant constant, $u = H$ and hence the gradient flow $V = w = -\mathrm{grad}_{MS} E$ is for all $t > 0$ given as

$$-\Delta u = 0 \qquad \text{in } \Omega_-(t) \cup \Omega_+(t), \tag{15}$$

$$V = -[\nabla u]^+_- \cdot \nu \qquad \text{on } \Gamma_t, \tag{16}$$

$$u = H \qquad \text{on } \Gamma_t, \tag{17}$$

$$\nabla u \cdot n = 0 \qquad \text{on } \partial\Omega, \tag{18}$$

where $\Omega_-(t)$, $\Omega_+(t)$ are the sets occupied by the two phases at time t.

Proposition 2 *Solutions* $((\Gamma_t)_{t\geq 0}, u)$ *to* (15)–(18) *fulfill*

$$\frac{d}{dt} \mathrm{vol}\big(\Omega_-(t)\big) = 0, \tag{19}$$

$$\frac{d}{dt} \mathrm{Area}(\Gamma_t) = -\int_\Omega |\nabla u|^2 \leq 0 \tag{20}$$

where $\mathrm{vol}(\Omega_-(t))$ *is the volume of* $\Omega_-(t) \subset \Omega$.

Proof Although the area decrease follows from the gradient flow property we will show (20) directly. We have, using (15)–(18) and the Gauss theorem,

$$\begin{aligned}\frac{d}{dt}\operatorname{Area}(\Gamma_t) &= -\int_{\Gamma_t} HV\, d\mathcal{H}^{d-1} = \int_{\Gamma_t} u[\nabla u]_-^+ \cdot \nu\, d\mathcal{H}^{d-1} \\ &= -\int_{\Gamma_t} u\big(\nabla u^+\big)\cdot \nu^+ d\mathcal{H}^{d-1} - \int_{\Gamma_t} u\big(\nabla u^-\big)\cdot \nu^- d\mathcal{H}^{d-1} \\ &= -\int_{\Omega_+(t)} \operatorname{div}(\nabla u\, u)dx - \int_{\Omega_-(t)} \operatorname{div}(\nabla u\, u)dx = -\int_{\Omega} |\nabla u|^2 dx,\end{aligned}$$

where u^+ is u defined on $\Omega_+(t)$, u^- is u defined on $\Omega_-(t)$, $\nu^- = \nu$ is the outer unit normal to $\Omega_-(t)$ and $\nu^+ = -\nu$ is the outer unit normal to $\Omega_+(t)$. In addition, we have

$$\begin{aligned}\frac{d}{dt}\int_{\Omega_-(t)} 1\, dx &= \int_{\Gamma_t} V dx = -\int_{\Gamma_t} [\nabla u]_-^+ \cdot \nu\, d\mathcal{H}^{d-1} \\ &= \int_{\Omega_+(t)} \operatorname{div}(\nabla u) dx + \int_{\Omega_-(t)} \operatorname{div}(\nabla u) dx = 0.\end{aligned}$$

□

Mean curvature flow is a *second order* evolution equation and surface diffusion is a *fourth order* evolution equation. The operator which maps H to $[\nabla u]_-^+ \cdot \nu$ can be interpreted as a generalized *Dirichlet-to-Neumann operator* for the Laplace operator as it maps the Dirichlet data for the Laplace boundary value problem to Neumann data, see e.g. [68]. This operator is a nonlocal, pseudo-differential operator of first order and since the computation of the mean curvature already involves two spatial derivatives it turns out that the motion of the interface which separates the phases is determined by a nonlinear, nonlocal, pseudo-differential operator of *third order*, see [71].

The free boundary can also have different connected components and all the above is still valid as long as the components are disjoint. In particular, the overall volume of $\Omega_+(t)$ and $\Omega_-(t)$ is still preserved, see Fig. 5. Now one effect involving different connected components is that typically small particles, i.e. small connected components, shrink and the energetically more advantageous large particles grow.

2.6 Results on the Mullins-Sekerka Evolution

It is much more difficult to show existence of solutions to the Mullins-Sekerka problem in comparison to the geometric evolution equations discussed further above. This is due to the fact that the evolution of the hypersurface $(\Gamma_t)_{t\geq 0}$ determines the domains in which we have to solve Laplace equations and at the same time the solutions of the Laplace equations determine the normal velocity of the evolving hypersurface. In order to be able to formulate the problem in a suitable setting involving appropriate function spaces one has to transform the time dependent domains $\Omega_+(t)$ and $\Omega_-(t)$, to fixed reference domains. This is done with the help of a diffeomorphism which depend on the solution $(\Gamma_t)_{t\geq 0}$ and was first introduced by Hanzawa [102]. Now

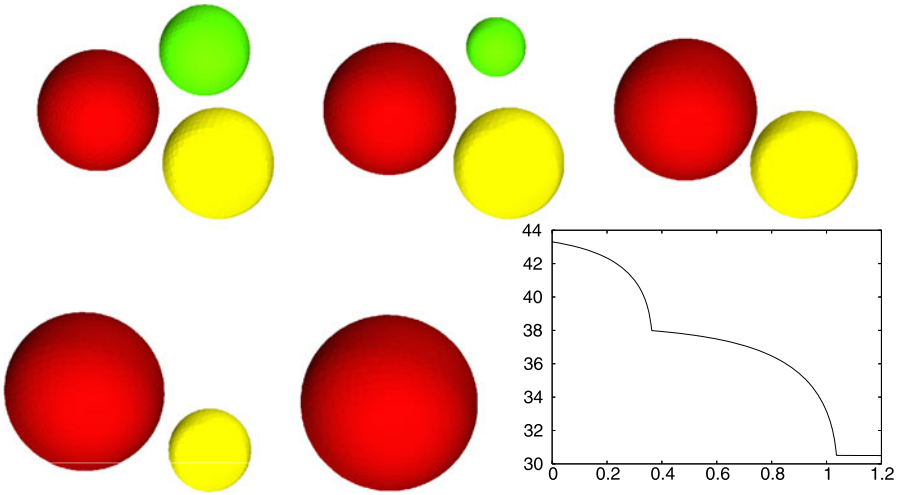
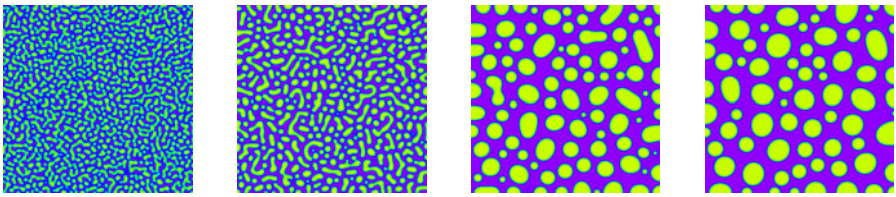


Fig. 5 The solution of a Mullins-Sekerka problem with three particles at times $t = 0, 0.3, 0.6, 0.9, 1.2$ (left to right, top to bottom). On the *bottom right*, a plot of the total surface area as a function of time



(a) $t = 0$ (b) $t = 0.04$ (c) $t = 1.4$ (d) $t = 2.9$

Fig. 6 Solutions to the Mullins-Sekerka problem reduce the total surface area by coarsening

the Laplace operator has to be transformed to the reference domain and altogether a highly nonlinear problem arises. If one formulates the transformed problem as an evolution equation for $(\Gamma_t)_{t\geq 0}$ one obtains a nonlinear, nonlocal, pseudo-differential operator of third order which carries a quasilinear structure. This approach was used independently by Escher, Simonett [70] and Chen, Hong, Yi [39] in order to show local existence of a unique local classical solution to (15)–(18) using a suitable contraction argument. Related earlier results in this direction are due to Duchon, Robert [53], Constantin, Pugh [43], Chen [37], Deckelnick, Elliott [48] who all showed local existence in two spatial dimensions of problems related to (10)–(13).

Of course similar geometric questions as for the geometric evolutions above arise and we mention for example Mayer [118] who showed that the Mullins-Sekerka evolution does not preserve convexity of the enclosed domain.

The Mullins-Sekerka model describes the aging of materials which consist of a binary mixture of two components which are e.g. two different metals. In many of these systems two different phases with different concentrations form and the evolution of the boundaries of the phase regions is given by (15)–(18). Typically many particles appear and the total surface area is high, see Fig. 6 (left).

Now large particles grow, while smaller ones shrink and eventually vanish. Hence the number of particles decreases and typical length scales such as particle size and inter-particle distance increase, see Fig. 6. This Ostwald ripening phenomenon has been studied intensively in the physics and metallurgical literature, see e.g. [172, 173] for reviews. For ensembles with a large number of particles, in which the volume fraction of one phase is small, it can be studied how the mean particle size increases. In fact, the mean particle size will grow like $t^{\frac{1}{3}}$. In addition, evolution laws for the particle size distribution can be derived. We refer to [45, 110, 114, 132–134, 172–174] for further details.

In general classical solutions to the Mullins-Sekerka problem do not exist for large times as topological changes and singularities can occur. For long-time existence results with general initial data one has to turn to weak formulations. Luckhaus and Sturzenhecker [116], see also [115], used a weak formulation of the identity $u = H$ in the setting of functions of bounded variations (BV-functions). The space of all functions of bounded variation is given as

$$BV(\Omega) = \left\{ f \in L^1(\Omega) \,\Big|\, \int_\Omega |\nabla f| < \infty \right\}$$

and $\int_\Omega |\nabla f|$ denotes the total variation of the distribution ∇f, i.e.

$$\int_\Omega |\nabla f| = \sup\left\{ \int_\Omega f \operatorname{div} g \, dx \,\Big|\, g \in C_0^1\big(\Omega, \mathbb{R}^d\big),\ \big|g(x)\big| \le 1 \text{ for all } x \in \Omega \right\}.$$

For $f \in BV(\Omega)$ one obtains that ∇f and $|\nabla f|$ are Radon measures on Ω with values in $\mathbb{R}^d$ and $\mathbb{R}$ respectively. A measurable set $E \subset \Omega$ with $\int_\Omega |\nabla \chi_E| < \infty$, where χ_E is the characteristic function of E, is called Caccioppoli set. In a generalized sense such a set E has bounded perimeter. We can now define a generalized unit normal to the boundary of E given by $\nu_E = \frac{\nabla \chi_E}{|\nabla \chi_E|}$ as the Radon-Nikodym derivative of $\nabla \chi_E$ with respect to $|\nabla \chi_E|$. We refer to Giusti [97] and Ambrosio, Fusco, Pallara [2] for more details on functions of bounded variation.

The BV-formulation of Luckhaus and Sturzenhecker now replaces the pointwise identity $u = H$ by

$$\int_0^T \int_\Omega \left(\operatorname{div} \xi - \frac{\nabla \chi}{|\nabla \chi|} \cdot \left(D\xi \frac{\nabla \chi}{|\nabla \chi|} \right) \right) d\big|\nabla \chi(t)\big| \, dt = \int_{\Omega_T} \operatorname{div}(u\xi) \chi \, d(x,t), \quad (21)$$

which has to hold for all $\xi \in C^1(\overline{\Omega}_T, \mathbb{R}^d)$, $\Omega_T := \Omega \times (0,T)$. Here $\chi : \Omega_T \to \{0,1\}$ is a phase function where phase 2 is given by the set $\{(x,t) \in \Omega \times (0,T) \mid \chi(x,t) = 1\}$ and phase 1 is given by the set $\{(x,t) \in \Omega \times (0,T) \mid \chi(x,t) = 0\}$ and one assumes that $\chi(.,t) \in BV(\Omega)$ for all $t \in (0,T)$.

If the interface is smooth and without boundary Eq. (21) leads to

$$\int_0^T \int_{\Gamma_t} \operatorname{div}_\Gamma \xi \, d\mathcal{H}^{d-1} dt = -\int_0^T \int_{\Gamma_t} u\xi \cdot \nu \, d\mathcal{H}^{d-1} dt$$

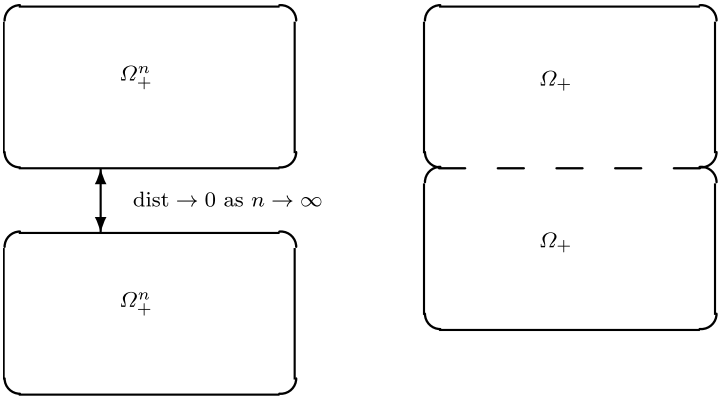


Fig. 7 An example where a loss of area appears in a limit $n \to \infty$

and using the Gauss theorem on manifolds we have

$$\int_0^T \int_{\Gamma_t} H\xi \cdot \nu \, d\mathcal{H}^{d-1} dt = \int_0^T \int_{\Gamma_t} u\xi \cdot \nu \, d\mathcal{H}^{d-1} dt$$

which shows that (21) is a weak formulation of $u = H$.

Luckhaus and Sturzenhecker [116] used a time discretization of the Mullins-Sekerka problem which is similar to the discussion in Sect. 2.2, see (5), (6), to obtain approximate solutions. However they were only able to show that limits of this approximation solve the weak formulation of the Mullins-Sekerka problem under an additional assumption. They had to exclude a loss of surface area for the interface in the limit when a time discretization parameter tends to zero. We refer to Fig. 7 for a situation in which there is a loss of area. The approach of Luckhaus and Sturzenhecker [116] was used for multi-phase situations in [86] and [24].

Later Röger [147] used methods from geometric measure theory and a result of Schätzle [153], who investigated the convergence of the equation $u_n = H_n$ in cases where u_n convergences as $n \to \infty$ in an approximate sense, to obtain a passage to the limit in time discrete approximations of the Mullins-Sekerka problem.

3 Anisotropic Surface Energies

3.1 Anisotropic Variational Problems and the Wulff Shape

In many physical applications the surface energy density will depend on the local orientation of the surface in the surrounding space. Since we consider hypersurfaces, the local orientation can be expressed by a unit normal field $\nu : \Gamma \to \mathbb{R}^d$ ($|\nu| = 1$). We define the anisotropic surface energy as

$$E_\gamma(\Gamma) = \int_\Gamma \gamma(\nu) d\mathcal{H}^{d-1}$$

where

$$\gamma : \mathbb{R}^d \to [0, \infty)$$

is positively one-homogeneous, i.e. $\gamma(\lambda x) = \lambda \gamma(x)$ for all $\lambda > 0$, $x \in \mathbb{R}^d$. To define the energy E_γ it is enough to define γ on unit vectors but often it will be convenient to extend γ to all of $\mathbb{R}^d$ by requiring positive homogeneity of degree one. We remark that we obtain for $\gamma(x) = |x|$ the classical surface area. If γ is nonconstant on all unit normal vectors we have the phenomenon that some directions are energetically more favourable than others.

We can now consider a generalized isoperimetric problem for subsets $D \subset \mathbb{R}^d$ with smooth boundaries:

$$\text{minimize} \int_{\partial D} \gamma(\nu)\, d\mathcal{H}^{d-1} \quad \text{subject to } \operatorname{vol}(D) = const. \tag{22}$$

It is known that the Wulff set

$$\mathcal{W}_\gamma := \left\{ x \in \mathbb{R}^d \;\middle|\; x \cdot y \le \gamma(y) \text{ for all unit vectors } y \in \mathbb{R}^d \right\}$$

is the shape having the least anisotropic surface area among all sets enclosing the same volume. The problem (22) was formulated by Wulff [179] who also conjectured its solution. Dinghas [52] solved (22) among all convex polyhedra and later Taylor [164] and Fonseca, Müller [78, 79] gave existence and uniqueness proofs for very general interfacial energies.

It is helpful to visualize the interfacial energy density γ with the help of the Frank diagram

$$\mathcal{F}_\gamma = \left\{ x \mid \gamma(x) \le 1 \right\}$$

which is the one-ball of γ. In Fig. 8 we display the Frank diagram and the Wulff shape for a cubic and a hexagonal anisotropy. In these examples we observe the general fact that the Frank diagram and the Wulff shape are dual to each other, in the following sense. When γ is a convex, even function, it is a norm and has a dual norm. We obtain that $\mathcal{W}_\gamma$ is the unit ball of that dual norm, see [16, 168].

3.2 The First Variation of Anisotropic Energies

For a given hypersurface Γ we now construct, as in Sect. 2.1, a one parametric family $(\Gamma_t)_{t \ge 0}$ with the help of a smooth vector field $\zeta : \mathbb{R}^d \to \mathbb{R}^d$. In order to compute the first variation of $\int_\Gamma \gamma(\nu)\, d\mathcal{H}^{d-1}$ we need the following ingredients:

(i) $D\gamma(x) \cdot x = \gamma(x)$ for all $x \in \mathbb{R}^d \setminus \{0\}$,

(ii) $D_t \nu = -\nabla_\Gamma V$,

(iii) $\dfrac{d}{dt} \displaystyle\int_{\Gamma_t} f\, d\mathcal{H}^{d-1} = \int_{\Gamma_t} (D_t f - f H V) d\mathcal{H}^{d-1}$,

(iv) $\displaystyle\int_{\Gamma_t} \nabla_\Gamma \cdot F\, d\mathcal{H}^{d-1} = -\int_{\Gamma_t} H \nu \cdot F\, d\mathcal{H}^{d-1}$ for $F \in C^1\big(\Gamma_t, \mathbb{R}^d\big)$.

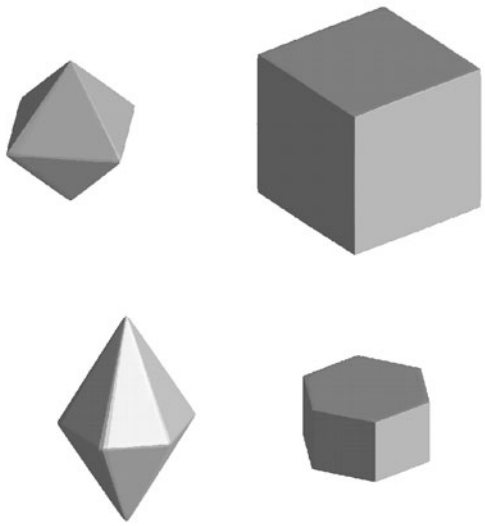


Fig. 8 Frank diagrams (*left*) and Wulff shapes (*right*) for different choices of the anisotropic energy E_γ. Above we see a cubic anisotropy and below a hexagonal anisotropy, see [13] for details

The first identity follows from the fact that γ is one-homogeneous. By $D_t f$ for a function f which is defined on an evolving surface we denote the normal time derivative of f, i.e. the time derivative following Γ_t with a velocity $V\nu$, see [99, formulae (15)–(21)], and [87]. More precisely we choose a path $z(\tau) \in \Gamma_\tau$ such that $z'(\tau) = (V\nu)(\tau, z(\tau))$ and define $D_t f(t, z(t)) = \frac{d}{d\tau} f(\tau, z(\tau))$, $\tau = 1$. The identity (ii) is shown e.g. in [99, formulae (15)–(24)], [58, 117]. For a proof of the transport theorem (iii) we refer to [49] or [87]. The equation (iv) in the Gauss theorem on manifolds for vector fields F which are not necessarily tangential and a proof can be found e.g. in [96, Sect. 16]. We remark that for non-tangential vector fields $\nabla_\Gamma \cdot F$ is given as

$$\nabla_\Gamma \cdot F = \sum_{i=1}^{d-1} (\partial_{\tau_i} F) \cdot \tau_i$$

where $\{\tau_1, \dots, \tau_{d-1}\}$ is an orthonormal basis of the tangent space.

We now compute

$$\begin{aligned}
\frac{d}{dt} \int_{\Gamma_t} \gamma(\nu) d\mathcal{H}^{d-1} &= \int_{\Gamma_t} \big(D_t \gamma(\nu) - \gamma(\nu) H V\big) d\mathcal{H}^{d-1} \\
&= \int_{\Gamma_t} \big(D\gamma(\nu) \cdot D_t \nu - \gamma(\nu) H V\big) d\mathcal{H}^{d-1} \\
&= \int_{\Gamma_t} \big(-D\gamma(\nu) \cdot \nabla_\Gamma V - \gamma(\nu) H V\big) d\mathcal{H}^{d-1} \\
&= \int_{\Gamma_t} \big(\nabla_\Gamma \cdot \big(D\gamma(\nu)\big) V + \big(D\gamma(\nu) \cdot \nu\big) V H - \gamma(\nu) H V\big) d\mathcal{H}^{d-1} \\
&= \int_{\Gamma_t} \nabla_\Gamma \cdot \big(D\gamma(\nu)\big) V \, d\mathcal{H}^{d-1}.
\end{aligned}$$

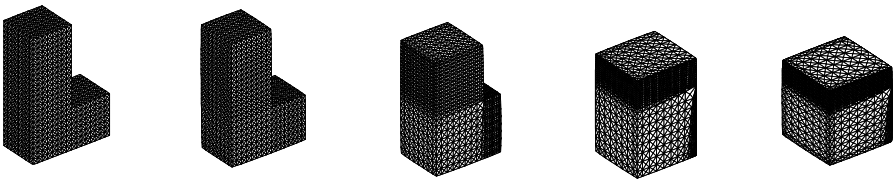


Fig. 9 Plots of a solution to anisotropic surface diffusion (23) with a cubic anisotropy at times $t = 0, 0.01, 0.05, 0.1, 0.25$

Here we used (i), (ii), (iii) and the Gauss theorem on manifolds, see (iv).

Hence the negative L^2-gradient of E_γ is given by

$$H_\gamma = -\nabla_\Gamma \cdot \big(D\gamma(\nu)\big)$$

which is known as the anisotropic mean curvature in the literature, see Taylor [166, 168]. Solutions of the classical isoperimetric problem have constant mean curvature. It turns out that solutions of the anisotropic version of the isoperimetric problem (22) lead to surfaces with constant anisotropic mean curvature.

3.3 Gradient Flows of the Anisotropic Surface Energy

We can replace the mean curvature H by the anisotropic mean curvature H_γ in all gradient flows studied in Sect. 2. We obtain in particular the anisotropic mean curvature flow

$$V = H_\gamma$$

and the anisotropic surface diffusion flow

$$V = -\Delta_\Gamma H_\gamma. \tag{23}$$

Both flows decrease the total anisotropic surface energy, i.e.

$$\frac{d}{dt}\int_{\Gamma_t} \gamma(\nu) d\mathcal{H}^{d-1} \leq 0$$

and the latter flow will preserve the enclosed volume. We will hence expect that the flow will converge to an appropriately scaled Wulff shape. We refer to Fig. 9 for a numerical computation with an anisotropy whose Wulff shape is a slightly regularized cube. We observe that for larger times the evolution will tend to the Wulff shape.

Often more general evolution laws of the form

$$\beta(\nu)V = H_\gamma$$

with a function $\beta : \mathbb{R}^d \to \mathbb{R}^+$ are of interest. Also these can be obtained as gradient flows by choosing an inner product of the form

$$\langle v_1, v_2\rangle_\beta = \int_\Gamma \beta(\nu) v_1 v_2 \, d\mathcal{H}^{d-1}.$$

We refer e.g. to Bellettini and Paolini [17] who studied the case $\beta = \frac{1}{\gamma}$ which naturally arises when interpreting the anisotropic curvature flows in the context of Finsler geometry.

4 The Stefan Problem

We have collected all ingredients to formulate the Stefan problem which describes solidification and melting phenomena in a general setting. The Stefan problem generalizes the Mullins-Sekerka problem studied in Sect. 3 as more physical effects are taken into account and in some sense it generalizes also the mean curvature flow as a forced mean curvature flow enters the overall system. In contrast to the Mullins-Sekerka problem where we solved Laplace's equation in the Stefan problem the heat equation has to be solved in the regions occupied by the phases and in the full Stefan problem also an additional time derivative can enter equation (17). We will not derive the Stefan problem from basic physical principles but refer to the books [100, 171] and [56] for a derivation.

The Stefan problem in a version taking anisotropic effects into account is now given as follows

$$\vartheta\, \partial_t u - \mathcal{K}_i\, \Delta u = 0 \qquad \text{in } \Omega_i(t), \text{ for } i \in \{-, +\}, \tag{24}$$

$$-[\mathcal{K}\nabla u]_-^+ \cdot \nu = \lambda V \qquad \text{on } \Gamma_t, \tag{25}$$

$$\beta(\nu) V = H_\gamma - a u \qquad \text{on } \Gamma_t \tag{26}$$

together with appropriate initial and boundary conditions and non-negative physical constants $\vartheta, \mathcal{K}_-, \mathcal{K}_+, \lambda$ and a.

4.1 The Classical Stefan Problem

Typically in the Stefan problem the unknown u describes the temperature in the system and the simplest modelling assumption for the temperature at the interface is that the temperature equals the melting temperature. Defining u to be the deviation from the melting temperature we have to choose $\beta = \gamma = 0$, $a \neq 0$ in (26) in order to obtain

$$u = 0 \tag{27}$$

as a boundary condition. The system (24), (25), (27) is called the classical Stefan problem and has been well-studied in the literature. We refer to the books by Elliott, Ockendon [65], Friedman [80], Meirmanov [121], Rubinstein [149] and Visintin [171] for more details, results and methods how to handle this problem analytically and numerically.

4.2 The Stefan Problem with Gibbs–Thomson Law

Setting $\beta = 0$, and for simplicity $a = 1$, (26) reduces to the Gibbs–Thomson law

$$u = H_\gamma \tag{28}$$

and the overall problem (24), (25), (28) is the Stefan problem with anisotropic Gibbs–Thomson law. An important contribution to this problem is due to Luckhaus [115]. He used an implicit time discretization, similar as the one discussed further above in the context of gradient flows, to show existence of a weak solution to (24), (25) together with

$$u = H \tag{29}$$

which is the isotropic version of the Gibbs–Thomson law. We now follow the lines of Rossi, Savaré [148] in order to describe the Stefan problem with Gibbs–Thomson law in the context of gradient flows. It is not so difficult to verify that the distributional formulation of (24), (25) is given by (for simplicity we set $\vartheta = \lambda = \mathcal{K}_- = \mathcal{K}_+ = 1$)

$$\partial_t (u + \chi) = \Delta u \tag{30}$$

where as before χ is the characteristic function of phase 2. Introducing the variable

$$e = u + \chi, \tag{31}$$

we can rewrite Eq. (30) as

$$\partial_t e = \Delta(e - \chi).$$

We now introduce

$$\Phi(e, \chi) = \int_\Omega \left(\frac{1}{2} |e - \chi|^2 + I_{\{0,1\}}(\chi) \right) dx + \int_\Omega |\nabla \chi|$$

where the function $I_{\{0,1\}}$ is zero at 0 and 1 and ∞ elsewhere. The overall problem (24), (25), (29) can be written as

$$\begin{aligned} (-\Delta)^{-1} \partial_t e &= -(e - \chi), \\ \Phi\big(e(t), \chi(t)\big) &\le \Phi\big(e(t), v\big) \quad \text{for all } v \in BV\big(\Omega, \{0, 1\}\big). \end{aligned}$$

Rossi and Savaré [148] remark that this formulation naturally leads to the reduced functional

$$\phi(e) := \inf_\chi \Phi(e, \chi)$$

and one can formally consider the Stefan problem with Gibbs–Thomson law as the H^{-1}-gradient flow for ϕ in $H^{-1}(\Omega)$ as follows

$$(-\Delta)^{-1} \partial_t e = -\frac{\delta \phi}{\delta e}$$

where $\frac{\delta \phi}{\delta e}$ is the first variation of ϕ.

The anisotropic case is more involved and an existence result following the strategy of Luckhaus has been given in Garcke, Schaubeck [85], see also Kraus [111] for

a slightly different approach. One crucial aspect is that the total variation has to be replaced by the anisotropic variation

$$\int_\Omega |\nabla f|_\gamma := \sup\left\{ -\int_\Omega f \operatorname{div} \varphi \, dx \mid \varphi \in C_0^1\big(\Omega, \mathbb{R}^d\big), \gamma^0\big(\varphi(x)\big) \le 1 \text{ a.e.} \right\},$$

where γ^0 is dual to γ, i.e.

$$\gamma^0(p) = \sup_{q \in \mathbb{R}^d \setminus \{0\}} \frac{p \cdot q}{\gamma(q)}.$$

In addition, an appropriate generalization of the weak formulation (21) has to be given, see [85].

4.3 Classical Solutions to the Stefan Problem

Local existence of classical solutions to the Stefan problem (24)–(26) can be shown by transforming the free boundary problem to a highly nonlinear problem on fixed domains. Similar as in the discussion at the beginning of Sect. 2.6 this approach uses the Hanzawa transformation. The problem on the fixed domain then has to be solved by a contraction argument using regularity theory for nonlinear parabolic equations. The isotropic case for the full Stefan problem (24)–(26) has been first considered by Radkevich [145] and Chen, Reitich [40].

Taking $\beta = 0$ in (26) leads to a certain quasi-static version of the Stefan problem and some new difficulties arise. Escher, Prüss and Simonett [73] were able to show local existence and uniqueness of analytic solutions in the isotropic case.

For the classical Stefan problem with $u = 0$ on Γ a comparison principle holds. For the Stefan problem with a curvature condition this is not possible any longer. Hence one has to come up with new ideas which replace methods which are based on comparison principles. Hadžić and Guo [101] were able to show stability of interfaces for the Stefan problem with curvature correction by developing a high-order energy method. We also mention a work by Prüss and Simonett [142] who study linear stability of spherical interfaces for (24)–(26) with $\beta > 0$.

Taking $\vartheta = 0$ leads to the Mullins-Sekerka problem. This problem in its anisotropic variant is relevant for many applications and also describes facetted growth of crystals. We only mention snow crystal growth as one example, see Figs. 10, 11.

If the Wulff shape or the Frank diagram has flat parts or corners new difficulties arise as the anisotropic mean curvature is not well-defined any longer and the nonlocal crystalline mean curvature will replace the anisotropic mean curvature. This concept was introduced by Taylor [165] and Angenent, Gurtin [6]. Well-posedness problems for the resulting motions are quite difficult and issues like "facet bending" or "facet breaking" appear, see Fig. 12 for an example of facet breaking. For more information on these problems we refer to the articles [14, 15, 88, 89, 92, 95, 127, 129]. Anisotropy, i.e. an orientation dependence through the normal, also appears in the kinetic coefficient β. The role played by the kinetic and the interfacial anisotropy in crystal growth is the subject of intense research. We refer to M.-H. Giga and Y. Giga [91] for details.

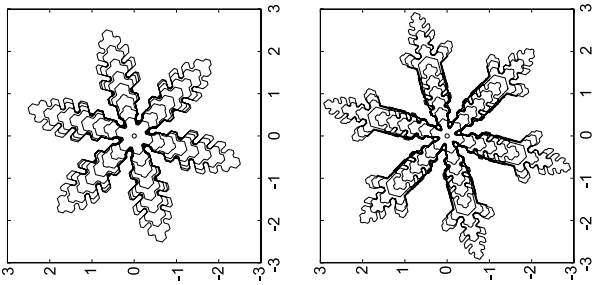


Fig. 10 Two dimensional numerical computations of snow crystal growth, see [11, 12]

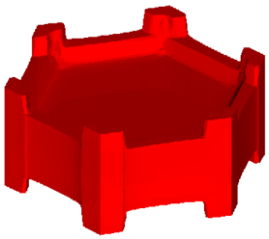


Fig. 11 Also snow crystals like in this numerical computation are observed in nature and they are called columns on plates, see [12]

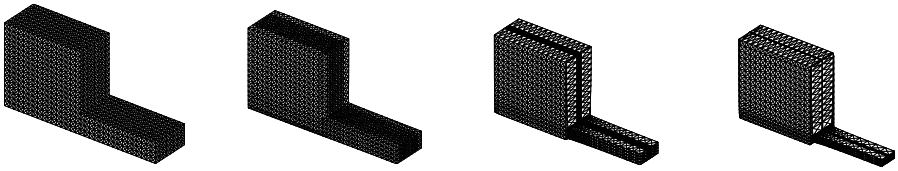


Fig. 12 Facet breaking in crystalline mean curvature flow

4.4 Numerical Approaches

There have been many numerical approaches to geometrical evolution equations and free boundary problems, we refer to [49, 155] for reviews. The numerical computations presented in this paper have been developed in cooperation with John Barrett and Robert Nürnberg from the Imperial College in London. The method is based on a parametric approach and in particular the interface is parametrized as a triangulated surface. In order to track the interface in time the triangles (or segments in 2d) move and in many numerical approaches the mesh degenerates during the evolution. In the papers [8–13] it was possible to come up with a variational formulation for geometrical evolution equations and free boundary problems which has the property that the mesh uses its tangential degrees of freedom in order to keep very good mesh properties. This together with a novel approach to discretize the anisotropic mean curvature H_γ made it possible to solve highly anisotropic geometric evolution equations, the Mullins-Sekerka problem and the Stefan problem with good mesh properties and a high precision. In this approach we heavily rely on an earlier work by Dziuk [55] who introduced a parametric finite element discretization of mean curvature and on work by Schmidt [154] who was the first to solve the three dimensional Stefan problem (24)–(26) with a parametric approach.

As a prototypical example for results obtained with the help of a parametric finite element method we show numerical computations of snow crystal growth, see Figs. 10, 11. They have been obtained with the model (24)–(26) with $\vartheta = 0$ and a hexagonal anisotropy, see Fig. 8 to the bottom. Figure 10 shows the evolution of snow crystals from an initial small seed leading to forms which most of us would consider to be a typical snow crystal. In fact many other forms are also possible and I refer to Fig. 11 for a form called columns on plates which also appears in nature. We refer to [11, 12] for more numerical computations and for a detailed introduction into the numerical methods. Somebody who is interested in the fascinating aspects of the physics and mathematics of snow crystal growth can find more details in [83, 112, 113].

5 Phase Field Equations as Gradient Flows

In the geometric evolution equations and in the free boundary problems discussed so far, the interface was described as a hypersurface. In the last thirty years phase field approaches have been another successful approach to describe the evolution of interfaces. In particular phase field methods allow for a change of topology. In a phase field description of interface evolution one uses instead of a characteristic function $\chi : \Omega \to \{0, 1\}$, which describes the two regions occupied by the phases, a smooth function which takes values close to given values, e.g. ± 1, and rapidly changes between these two values in a small interfacial region, see Figs. 14 and 15.

5.1 The Ginzburg–Landau Energy

The phase field approach is best motivated by considering the so-called Ginzburg–Landau energy

$$E_\varepsilon(\varphi) := \int_\Omega \left(\frac{\varepsilon}{2} |\nabla \varphi|^2 + \frac{1}{\varepsilon} \Psi(\varphi) \right) dx \tag{32}$$

where $\varepsilon > 0$ is a small parameter. For functions φ with a moderate energy $E_\varepsilon(\varphi)$ it will turn out that ε is proportional to the interfacial thickness between the region $\{\varphi \approx -1\}$ and $\{\varphi \approx 1\}$. The function $\Psi : \mathbb{R} \to \mathbb{R}_0^+$ is a double well potential having two global minima with value zero at ± 1, i.e. $\Psi(\pm 1) = 0$ and $\Psi(z) > 0$ for $z \notin \{-1, 1\}$, see Fig. 13 for an example. Typical choices are the quartic potential

$$\Psi(\varphi) = \frac{9}{32} \left(\varphi^2 - 1 \right)^2$$

and the double obstacle potential Ψ_{ob} which is defined as

$$\Psi_{ob}(\varphi) = \frac{1}{2} \left(1 - \varphi^2 \right) \quad \text{for all } \varphi \in [-1, 1]$$

and ∞ elsewhere, see [20], although different choices are possible, see e.g. Abels and Wilke [1].

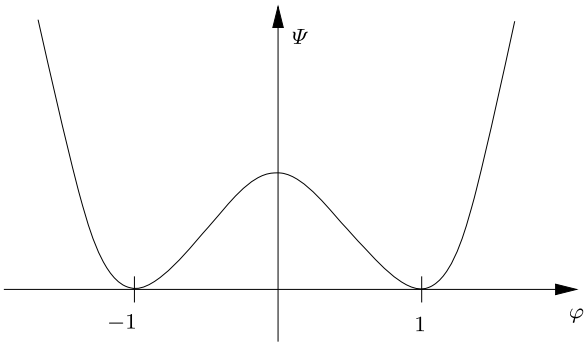


Fig. 13 The energy contribution $\Psi(\varphi)$ in (32) penalizes values of φ, which differ from ± 1

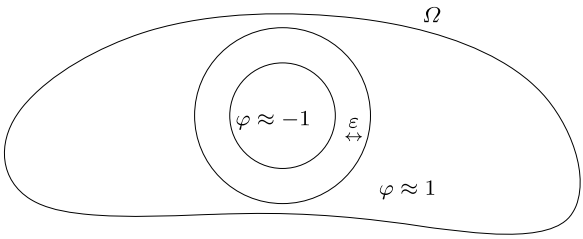


Fig. 14 A typical form of the phase field variable φ. Regions in which $\varphi \approx \pm 1$, are separated by a diffuse interfacial layer whose thickness is proportional to ε

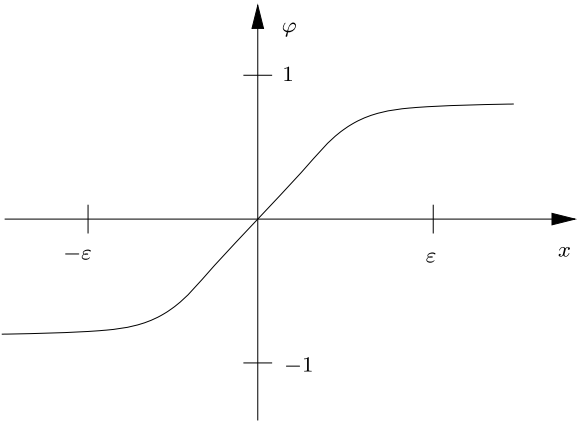


Fig. 15 The phase field variable typically has a profile with a phase transition on a diffuse interface of thickness ε

The term $\frac{1}{\varepsilon}\Psi(\varphi)$ in the energy E_ε penalizes values which differ from ± 1. In addition, the term $\frac{\varepsilon}{2}|\nabla\varphi|^2$ penalizes gradients of φ and hence too rapid changes of φ in space. It will turn out later that typical solutions of the phase field system have the form illustrated in Fig. 14, i.e. they are close to ± 1 in most parts of the domain and have an interfacial region with a thickness which is proportional to ε. In directions normal to the level sets of φ a typical solution of the phase field system has the form depicted in Fig. 15.

5.2 Relating Phase Field and Sharp Interface Energies

The Ginzburg-Landau energy (32) can be related to the surface energy in the limit $\varepsilon \to 0$. The appropriate notion to make this statement precise is the concept of Γ-limit.

Definition 1 Let (X, d) be a metric space and $(F_\varepsilon)_{\varepsilon>0}$ a family of functionals $F_\varepsilon : X \to [-\infty, \infty]$. We say that $(F_\varepsilon)_{\varepsilon>0}$ Γ-converges to a functional $F : X \to [-\infty, \infty]$ (which we will denote as $F_\varepsilon \xrightarrow{\Gamma} F$) if the following properties hold:

(i) (*lim inf inequality*) For every $u \in X$ and $u_\varepsilon \in X$, $\varepsilon > 0$, such that $u_\varepsilon \to u$ as $\varepsilon \to 0$ it holds

$$F(u) \leq \liminf_{\varepsilon \to 0} F_\varepsilon(u_\varepsilon).$$

(ii) (*lim sup inequality*) For every $u \in X$ there exist $u_\varepsilon \in X$, $\varepsilon > 0$, such that $u_\varepsilon \to u$ as $\varepsilon \to 0$ and

$$\limsup_{\varepsilon \to 0} F_\varepsilon(u_\varepsilon) \leq F(u).$$

We note that the concept of Γ-limits is more general and can be generalized to more general spaces, see [46]. The notion of Γ-limit is in particular appropriate for sequences of variational problems as under appropriate assumptions minima of F_ε will converge to minima of F, see [21].

It was shown in [124] and [125] that the Ginzburg-Landau energies E_ε defined in (32) Γ-converge to a multiple of the perimeter functional, see (1). It turns out that a suitable metric for this convergence is induced by the $L^1(\Omega)$-norm and hence we extend E_ε to $L^1(\Omega)$ by setting

$$E_\varepsilon(\varphi) := \begin{cases} \int_\Omega (\frac{\varepsilon}{2}|\nabla\varphi|^2 + \frac{1}{\varepsilon}\Psi(\varphi))\,dx & \text{if } \varphi \in H^1(\Omega), \\ \infty & \text{if } \varphi \in L^1(\Omega) \setminus H^1(\Omega). \end{cases}$$

Under appropriate assumptions on Ψ and Ω it can be shown that the functionals E_ε in fact Γ-converge to the functional

$$E(\varphi) := \begin{cases} c_\Psi \int_\Omega |\nabla \chi_{\{\varphi=1\}}| & \text{if } \varphi \in BV(\Omega, \{-1, 1\}), \\ \infty & \text{if } \varphi \in L^1(\Omega) \setminus BV(\Omega, \{-1, 1\}), \end{cases}$$

where $c_\Psi := \int_{-1}^{1} \sqrt{2\Psi(z)}dz$. This means we have

$$E_\varepsilon \xrightarrow{\Gamma} E \quad \text{as} \quad \varepsilon \to 0$$

with respect to the L^1-topology. This Γ-convergence result is stable under adding an integral constraint for E_ε in the functional which is important in many applications where this corresponds to a mass conservation property. We refer to [124, 125] and

[21] for more details. In later sections we will relate gradient flows of E_ε to the gradient flows of the area functional E discussed in Sect. 2.

5.3 Phase Field Models as Gradient Flows

We now consider different gradient flows involving the energy E_ε. Before discussing the gradient flows we note that the first variation $\frac{\delta E_\varepsilon}{\delta \varphi}$ of E_ε at $\varphi \in H^1(\Omega)$ in a direction $v \in H^1(\Omega)$ is given by

$$\frac{\delta E_\varepsilon}{\delta \varphi}(\varphi)(v) := \frac{d}{ds} E_\varepsilon(\varphi + sv)_{|s=0} = \int_\Omega \left(\varepsilon \nabla \varphi \cdot \nabla v + \frac{1}{\varepsilon} \Psi'(\varphi) v \right) dx.$$

5.3.1 The Allen-Cahn Equation

Choosing the L^2-inner product for functions defined on Ω we now obtain the equations for the L^2-gradient flow of E_ε as follows

$$\langle \partial_t \varphi, v \rangle_{L^2} = - \int_\Omega \left(\varepsilon \nabla \varphi \cdot \nabla v + \frac{1}{\varepsilon} \Psi'(\varphi) v \right) dx$$

which has to hold for all times and all suitable test functions v. For functions φ which are smooth enough the above is equivalent to

$$\begin{aligned} \partial_t \varphi &= \varepsilon \Delta \varphi - \frac{1}{\varepsilon} \Psi'(\varphi) \quad && \text{in } \Omega, \\ \frac{\partial \varphi}{\partial n} &= 0 && \text{on } \partial\Omega \end{aligned}$$

which follows after integration by parts with the help of the fundamental lemma of the calculus of variations.

5.3.2 The Cahn-Hilliard Equation

It is also possible, similar as in Sect. 2.4 where we considered gradient flows of the area functional, to consider an H^{-1}-gradient flow of the energy E_ε which preserves the integral of φ. We define

$$H^1_m(\Omega) = \left\{ u \in H^1(\Omega) \,\middle|\, \fint_\Omega u \, dx = m \right\}$$

with $m \in \mathbb{R}^+$ a given constant. For v_1, v_2 with $\int_\Omega v_i \, dx = 0$, $i = 1, 2$, we define $u_1, u_2 \in H^1_0(\Omega)$ as weak solutions of

$$\begin{aligned} -\Delta u_i &= v_i \quad && \text{in } \Omega, \\ \frac{\partial u_i}{\partial n} &= 0 && \text{on } \partial\Omega. \end{aligned}$$

Since the u_i are the solutions of a Neumann problem for the Laplace operator we set $u_i = (-\Delta_N)^{-1} v_i$. The H^{-1}-inner product is now given as

$$\begin{aligned}\langle v_1, v_2 \rangle_{H^{-1}} &:= \int_\Omega \left(\nabla(-\Delta_N)^{-1} v_1\right) \cdot \left(\nabla(-\Delta_N)^{-1} v_2\right) dx \\ &= \int_\Omega \nabla u_1 \cdot \nabla u_2 \, dx = \int_\Omega v_1 u_2 \, dx = \int_\Omega v_2 u_1 \, dx.\end{aligned}$$

For the H^{-1}-gradient flow we have

$$\langle \partial_t \varphi, v \rangle_{H^{-1}} = -\int_\Omega \left(\varepsilon \nabla\varphi \cdot \nabla v + \frac{1}{\varepsilon} \Psi'(\varphi) v \right) dx$$

for test functions v. Taking the definition of the H^{-1}-inner product into account we observe after integration by parts that a smooth solution of the gradient flow equation is a solution of the following boundary value problem:

$$\partial_t \varphi = \Delta \left(-\varepsilon \Delta \varphi + \frac{1}{\varepsilon} \Psi'(\varphi) \right) \quad \text{in } \Omega, \tag{33}$$

$$\frac{\partial \varphi}{\partial n} = 0, \qquad \frac{\partial \Delta \varphi}{\partial n} = 0 \qquad \text{on } \partial\Omega. \tag{34}$$

The facts that the Allen-Cahn equation and the Cahn-Hilliard equation are respectively the L^2- and the H^{-1}-gradient flow of the Ginzburg-Landau energy E_ε has first been discussed by Fife [76, 77]. Equation (33) is a parabolic partial differential equation of fourth order which is called the Cahn-Hilliard equation, see [61] and [136] for more details. Solutions of (33), (34) fulfill

$$\frac{d}{dt} \int_\Omega \varphi \, dx = 0, \qquad \frac{d}{dt} E_\varepsilon(\varphi) \le 0$$

which are the analogues of Proposition 1 which stated the related result for surface diffusion which is the H^{-1}-gradient flow of the area functional.

For later use we remark that a variant of the Cahn-Hilliard equation has a degenerate mobility $M(\varphi) := (1-\varphi^2)_+ := \max(0, 1-\varphi^2)$ and in this case we replace (33) by

$$\partial_t \varphi = \nabla \cdot \left(M(\varphi) \nabla \left(-\varepsilon \Delta \varphi + \frac{1}{\varepsilon} \Psi'(\varphi) \right) \right). \tag{35}$$

How to obtain this equation as a gradient flow of a suitably weighted H^{-1}-inner product is discussed by Taylor, Cahn [167].

5.3.3 The Phase Field System

It is also possible to formulate a phase field analogue of the full Stefan problem (24)–(26). We derive a simplified version of the phase field system, similar as in a paper by Penrose and Fife [140] with the help of the gradient flow perspective. To this end

we consider the unknowns interfacial energy e and phase field φ for which we define the functional

$$E_\varepsilon(e,\varphi) = \int_\Omega \left(s(e,\varphi) + \frac{\varepsilon}{2}|\nabla\varphi|^2 + \frac{1}{\varepsilon}\Psi(\varphi) \right) dx.$$

We now take the inner product $\langle e_1, e_2\rangle_{H^{-1}} + \langle \varphi_1, \varphi_2\rangle_{L^2}$ and obtain as gradient flow (not taking boundary conditions into account)

$$(-\Delta_N)^{-1}\partial_t e = -\frac{\delta E}{\delta e}, \tag{36}$$

$$\partial_t \varphi = -\frac{\delta E}{\delta \varphi}. \tag{37}$$

Defining $s(e,\varphi) = \frac{1}{2}(e-\varphi)^2$ and $u = e - \varphi$ we obtain

$$\frac{\partial s}{\partial e} = u, \qquad \frac{\partial s}{\partial \varphi} = -u$$

and hence we can rewrite (36), (37) as

$$\partial_t(u+\varphi) = \Delta u, \tag{38}$$

$$\partial_t \varphi = \varepsilon\Delta\varphi - \frac{1}{\varepsilon}\Psi'(\varphi) + u. \tag{39}$$

This is the phase field system and u typically is interpreted as temperature or chemical potential. We refer to [27, 35, 84, 140, 161, 175] for more information on the phase field model which in particular discuss thermodynamically consistent phase field models.

5.4 Mathematical Results on Phase Field Equations

In this subsection we give a brief discussion on known results for phase field type equations. As often in the literature the phrase phase field type equations includes also the Allen-Cahn equation, the Cahn-Hilliard equation and its variants. The phase field system in the stricter sense is given by (38), (39).

- The simplest versions of Allen-Cahn, Cahn-Hilliard and phase field equations are semi-linear parabolic equations. Global existence and uniqueness results rely on the fact that the energy E_ε is a Lyapunov functional. Existence can be shown for example with the help of a Galerkin approximation. We refer to Caginalp [28], Elliott, Zheng [66, 67] and Brokate, Sprekels [27] for details.
- It is more difficult to analyse the degenerate Cahn-Hilliard equation (35) as due to the fact that the mobility becomes zero some terms in a priori estimates degenerate and hence give no control of norms in function spaces. Elliott and Garcke [62] were able to show a global existence result using an additional entropy type estimate. Uniqueness is still not known and is one of the major open problems for fourth order degenerate parabolic equations.

– There has been a huge interest in the study of the long time behaviour of the Cahn-Hilliard equation and the phase field system and we cannot give a complete discussion of the results. It was shown that trajectories of the Cahn-Hilliard equation converge to equilibria as time goes to infinity, see Zheng [180] and Rybka, Hoffmann [150]. This result is non-trivial as equilibria in general are not isolated and the latter paper made use of the Łojasiewicz-Simon inequality. Many works deal with the study of the geometry of the attractors. In particular, results which bound the dimension of the attractor and results on the lower semi-continuity of the attractor when the phase field equation converges to the Cahn-Hilliard equation have been obtained. We refer to Temam [169], Efendiev, Gajewski, Zelik [59], Miranville, Zelik [123], Brochet, Chen, Hilhorst [23], Dupaix, Hilhorst, Kostin [54], but would like to mention that also many more recent contributions exist.

5.5 Sharp Interface Limits

It is possible to relate the gradient flows for the area functional and the gradient flows for the Ginzburg-Landau functional. In all situations discussed above the time dependent solutions $(\varphi_\varepsilon)_{\varepsilon>0}$ of a phase-field type equation converge to a function φ_0 taking only values ± 1 as ε tends to zero. The boundary between the sets $\{\varphi_0 = 1\}$ and $\{\varphi_0 = -1\}$ is a (maybe "generalized") surface for which an evolution law similar as the ones discussed in Sects. 2 and 3 will hold.

Roughly speaking two fundamental different approaches exist to study the limit $\varepsilon \to 0$ in the phase field equations. One approach assumes that a smooth local solution to the limiting surface evolution equation exists. With the help of this solution and a matched asymptotic expansion one then constructs local solutions to the phase field equations which converge to the original surface evolution equation when ε tends to 0. We refer to Caginalp, Fife [31], Caginalp [29] for formal results in this direction and De Mottoni, Schatzman [50] for a rigorous treatment.

Other approaches depend on a weak formulation of the limiting problem and lead to convergence results which are global in time. These methods are based e.g. on the theory of viscosity solutions [74], on varifold theory [108] use the BV-setting mentioned above [141] or use energy methods [163].

We now discuss what is known for the individual equations.

(1) *The Allen-Cahn equation*
After a suitable rescaling in time the Allen-Cahn equation is given as

$$\varepsilon \partial_t \varphi_\varepsilon - \varepsilon \Delta \varphi_\varepsilon + \frac{1}{\varepsilon} \Psi'(\varphi_\varepsilon) = 0.$$

In the limit $\varepsilon \to 0$ we obtain that the surface separating the sets $\{\varphi_0 = 1\}$ and $\{\varphi_0 = -1\}$, compare Figs. 14 and 15, will evolve by mean curvature flow

$$V = H.$$

There are many results on this limit using quite different methods. We refer e.g. to [25, 36, 50, 74, 108] for details.

(2) *The Cahn-Hilliard equation*
In order to discuss the sharp interface limit of the Cahn-Hilliard equation (33) we restate the equation as a system

$$\partial_t \varphi_\varepsilon = \Delta u_\varepsilon, \tag{40}$$

$$u_\varepsilon = -\varepsilon \Delta \varphi_\varepsilon + \frac{1}{\varepsilon} \Psi'(\varphi_\varepsilon). \tag{41}$$

In the limit the pairs $(\varphi_\varepsilon, u_\varepsilon)$ converge to a pair (φ_0, u_0) which is the solution of a Mullins-Sekerka problem, see (15)–(17),

$$0 = \Delta u_0 \qquad \text{in } \Omega_-(t) \cup \Omega_+(t), \tag{42}$$

$$2V = -[\nabla u_0]_-^+ \cdot \nu \quad \text{on } \Gamma_t, \tag{43}$$

$$2u_0 = c_\Psi H \qquad \text{on } \Gamma_t. \tag{44}$$

We obtain the factor 2 in (43) in comparison to (16) because φ_0 jumps from -1 to 1 whereas in (16) we considered the characteristic function of phase 2 which has the jump one across the interface. The different factor on the right hand side in (44) in comparison to (17) is explained by the fact that in E the surface area is weighted by the factor c_Ψ. The asymptotic limit of the Cahn-Hilliard equation has been studied in [4, 139, 162].

(3) *The nonlocal Allen-Cahn equation*
For the area functional we considered an L^2-gradient flow which preserves the enclosed volume by requiring that the normal velocities have mean zero. One can do something similar for the Ginzburg-Landau energy and obtains a gradient flow which, taking Neumann boundary conditions into account, preserves the integral of φ:

$$\varepsilon \partial_t \varphi_\varepsilon - \varepsilon \Delta \varphi_\varepsilon + \frac{1}{\varepsilon} \Psi'(\varphi_\varepsilon) = \frac{1}{\varepsilon} \fint_\Omega \Psi'(\varphi_\varepsilon)\, dx.$$

As asymptotic limit for $\varepsilon \to 0$ one obtains the nonlocal mean curvature flow, compare (8),

$$V = H - \fint_\Gamma H.$$

This asymptotic limit was studied by [26] in the radially symmetric case and in [38] for general geometries.

(4) *The Cahn-Hilliard equation with degenerate mobility*
In Sect. 2.4 we studied two different H^{-1}-gradient flows of the area functional. One was motion by surface diffusion, a local geometric evolution law, and the second was the Mullins-Sekerka free boundary problem. It turns out that we obtain both as asymptotic limits of Cahn-Hilliard equations. We already saw that the Mullins-Sekerka evolution is the sharp interface limit of the Cahn-Hilliard equation.

Taking a degenerate mobility in the Cahn-Hilliard equation and rescaling in time we have

$$\varepsilon \partial_t \varphi_\varepsilon = \nabla \cdot \left(\left(1 - \varphi_\varepsilon^2\right)_+ \nabla u_\varepsilon \right),$$
$$u_\varepsilon = -\varepsilon \Delta \varphi_\varepsilon + \frac{1}{\varepsilon} \Psi'(\varphi_\varepsilon)$$

and formal arguments by Cahn, Elliott and Novick-Cohen [32] indicate that solutions of this system converge to motion by surface diffusion, i.e. the limiting evolving surface fulfills

$$V = -d_\Psi \, \Delta_\Gamma H \quad \text{with a suitable constant } d_\Psi \in \mathbb{R}.$$

We remark here that it is still an open problem to justify this limit rigorously.

(5) *The phase field system*
It is also possible to relate the phase field equations to the Stefan problem (24)–(26) discussed in Sect. 4. We restrict ourselves to the isotropic case, i.e. in (24)–(26) we choose β as constant and $H_\gamma = \gamma H$ with a constant γ. In order to obtain the isotropic variant of (24)–(26) we formulate (38)–(39) with the help of physical constants as follows

$$\partial_t \left(\vartheta u + \frac{\lambda}{2} \varphi \right) = \mathcal{K} \Delta u, \tag{45}$$
$$c_\Psi \frac{a}{2} u = \varepsilon \beta \partial_t \varphi - \gamma \varepsilon \Delta \varphi + \frac{\gamma}{\varepsilon} \Psi'(\varphi). \tag{46}$$

It can be shown that this system converges to (24)–(26) (for $\mathcal{K}_- = \mathcal{K}_+$) in the sharp interface limit $\varepsilon \to 0$. This has been analyzed with the help of formally matched asymptotic expansions by Caginalp [28, 29], see also [56], and was later shown rigorously by Soner [158] and Caginalp and Chen [30]. We remark that (45), (46) also contains the Allen-Cahn equation and the Cahn-Hilliard equation as special cases by setting either $a = 0$ or ($\vartheta = 0$, $\beta = 0$). Also the case $\vartheta = 0$ is of importance and is called the viscous Cahn-Hilliard equation, see [135]. By choosing $\beta = 0$ one obtains the Stefan problem with Gibbs-Thomson law as the asymptotic limit, see [141, 152].

Acknowledgements Figures 1–3, 5, 8–12 are numerical computations by Robert Nürnberg (Imperial College, London) and they were performed in the context of the work [8–13]. Figure 6 has been provided by Ulrich Weikard. Helmut Abels, Klaus Deckelnick, Daniel Depner, Hans-Christoph Grunau, Claudia Hecht, Barbara Niethammer and Matthias Röger made helpful suggestions which improved the presentation. I would like to express my gratitude to all the above mentioned colleagues for their contributions and to Eva Rütz for typing my often very rough notes.

References

1. Abels, H., Wilke, M.: Convergence to equilibrium for the Cahn-Hilliard equation with a logarithmic free energy. Nonlinear Anal. **67**(11), 3176–3193 (2007)

2. Ambrosio, L., Fusco, N., Pallara, D.: Functions of Bounded Variation and Free Discontinuity Problems. Oxford Mathematical Monographs. Clarendon Press, Oxford (2000)
3. Ambrosio, L., Gigli, N., Savaré, G.: Gradient Flows in Metric Spaces and in the Space of Probability Measures. Lectures in Mathematics ETH Zürich. Birkhäuser, Basel (2005), viii+333 pp.
4. Alikakos, N.D., Bates, P.W., Chen, X.: Convergence of the Cahn-Hilliard equation to the Hele-Shaw model. Arch. Ration. Mech. Anal. **128**(2), 165–205 (1994)
5. Almgren, F., Taylor, J.E., Wang, L.: Curvature-driven flows: a variational approach. SIAM J. Control Optim. **31**(2), 387–438 (1993)
6. Angenent, S.B., Gurtin, M.E.: Multiphase thermomechanics with interfacial structure. II. Evolution of an isothermal interface. Arch. Ration. Mech. Anal. **108**, 323–391 (1989)
7. Aubert, G., Kornprobst, P.: Mathematical Problems in Image Processing. Partial Differential Equations and the Calculus of Variations with a Foreword by Olivier Faugeras. Applied Mathematical Sciences, vol. 147. Springer, New York (2002), xxvi+286 pp.
8. Barrett, J.W., Garcke, H., Nürnberg, R.: A parametric finite element method for fourth order geometric evolution equations. J. Comput. Phys. **222**(1), 441–462 (2007)
9. Barrett, J.W., Garcke, H., Nürnberg, R.: On the parametric finite element approximation of evolving hypersurfaces in $\mathbb{R}^3$. J. Comput. Phys. **227**(9), 4281–4307 (2008)
10. Barrett, J.W., Garcke, H., Nürnberg, R.: Parametric approximation of surface clusters driven by isotropic and anisotropic surface energies. Interfaces Free Bound. **12**(2), 187–234 (2010)
11. Barrett, J.W., Garcke, H., Nürnberg, R.: On stable parametric finite element methods for the Stefan problem and the Mullins-Sekerka problem with applications to dendritic growth. J. Comput. Phys. **229**(18), 6270–6299 (2010)
12. Barrett, J.W., Garcke, H., Nürnberg, R.: Numerical computations of facetted pattern formation in snow crystal growth. Phys. Rev. E **86**(1), 011604 (2012)
13. Barrett, J.W., Garcke, H., Nürnberg, R.: A variational formulation of anisotropic geometric evolution equations in higher dimensions. Numer. Math. **109**(1), 1–44 (2008)
14. Bellettini, G.: An introduction to anisotropic and crystalline mean curvature flow. In: Proceedings of Minisemester on Evolution of Interfaces. Sapporo 210. Hokkaido University Technical Report Series in Math., vol. 145, pp. 102–159 (2010)
15. Bellettini, G., Novaga, M., Paolini, M.: Facet-breaking for three-dimensional crystals evolving by mean curvature. Interfaces Free Bound. **1**, 39–55 (1999)
16. Bellettini, G., Novaga, M., Paolini, M.: On a crystalline variational problem, part I: First variation and global L^∞ regularity. Arch. Ration. Mech. Anal. **157**, 165–191 (2001)
17. Bellettini, G., Paolini, M.: Anisotropic motion by mean curvature in the context of Finsler geometry. Hokkaido Math. J. **25**(3), 537–566 (1996)
18. Bernoff, A.J., Bertozzi, A.L., Witelski, T.P.: Axisymmetric surface diffusion: dynamics and stability of self-similar pinchoff. J. Stat. Phys. **93**(3–4), 725–776 (1998)
19. Blatt, S.: Loss of convexity and embeddedness for geometric evolution equations of higher order. J. Evol. Equ. **10**(1), 21–27 (2010)
20. Blowey, J.F., Elliott, C.M.: The Cahn-Hilliard gradient theory for phase separation with nonsmooth free energy. I. Mathematical analysis. Eur. J. Appl. Math. **2**(3), 233–280 (1991)
21. Braides, A.: Γ-Convergence for Beginners. Oxford Lecture Series in Mathematics and Its Applications, vol. 22. Oxford University Press, London (2005), xii+217 pp.
22. Brakke, K.A.: The Motion of a Surface by Its Mean Curvature. Mathematical Notes, vol. 20. Princeton University Press, Princeton (1978), i+252 pp.
23. Brochet, D., Chen, X., Hilhorst, D.: Finite dimensional exponential attractor for the phase field model. J. Anal. Appl. **49**, 197–212 (1993)
24. Bronsard, L., Garcke, H., Stoth, B.: A multi-phase Mullins-Sekerka system: matched asymptotic expansions and an implicit time discretisation for the geometric evolution problem. Proc. R. Soc. Edinb. A **128**, 481–506 (1998)
25. Bronsard, L., Kohn, R.V.: Motion by mean curvature as the singular limit of Ginzburg-Landau dynamics. J. Differ. Equ. **90**(2), 211–237 (1991)
26. Bronsard, L., Stoth, B.: Volume-preserving mean curvature flow as a limit of a nonlocal Ginzburg-Landau equation. SIAM J. Math. Anal. **28**(4), 769–807 (1997)
27. Brokate, M., Sprekels, J.: Hysteresis and Phase Transitions. Applied Mathematical Sciences, vol. 121. Springer, New York (1996), x+357 pp.
28. Caginalp, G.: An analysis of a phase field model of a free boundary. Arch. Ration. Mech. Anal. **92**, 205–245 (1986)

29. Caginalp, G.: Stefan and Hele-Shaw type models as asymptotic limits of the phase-field equations. Phys. Rev. A **39**(3), 5887–5896 (1989)
30. Caginalp, G., Chen, X.: Convergence of the phase field model to its sharp interface limits. Eur. J. Appl. Math. **9**, 417–445 (1998)
31. Caginalp, G., Fife, P.C.: Dynamics of layered interfaces arising from phase boundaries. SIAM J. Appl. Math. **48**(3), 506–518 (1988)
32. Cahn, J.W., Elliott, C.M., Novick-Cohen, A.: The Cahn-Hilliard equation with a concentration dependent mobility: motion by minus the Laplacian of the mean curvature. Eur. J. Appl. Math. **7**(3), 287–301 (1996)
33. Cao, F.: Geometric Curve Evolution and Image Processing. Lecture Notes in Mathematics, vol. 1805. Springer, Berlin (2003)
34. Chan, T.F., Shen, J.: Image Processing and Analysis. SIAM, Philadelphia (2005), xxi+184 pp.
35. Chen, L.-Q.: Phase-field models for microstructure evolution. Annu. Rev. Mater. Res. **32**, 113–140 (2002)
36. Chen, X.: Generation and propagation of interfaces for reaction-diffusion equations. J. Differ. Equ. **96**(1), 116–141 (1992)
37. Chen, X.: The Hele-Shaw problem and area-preserving curve shortening motion. Arch. Ration. Mech. Anal. **123**, 117–151 (1993)
38. Chen, X., Hilhorst, D., Logak, E.: Mass conserving Allen-Cahn equation and volume preserving mean curvature flow. Interfaces Free Bound. **12**(4), 527–549 (2010)
39. Chen, X., Hong, J., Yi, F.: Existence, uniqueness and regularity of classical solutions of the Mullins-Sekerka problem. Commun. Partial Differ. Equ. **21**, 1705–1727 (1996)
40. Chen, X., Reitich, F.: Local existence and uniqueness of the classical Stefan problem with surface tension and dynamical undercooling. J. Math. Anal. Appl. **162**, 350–362 (1992)
41. Chen, Y.G., Giga, Y., Goto, S.L.: Uniqueness and existence of viscosity solutions of generalized mean curvature flow equations. J. Differ. Geom. **33**, 749–786 (1991)
42. Chou, K.-S.: A blow-up criterion for the curve shortening flow by surface diffusion. Hokkaido Math. J. **32**(1), 1–19 (2003)
43. Constantin, P., Pugh, M.: Global solutions for small data to the Hele-Shaw problem. Nonlinearity **6**, 393–415 (1993)
44. Cummings, L.J., Richardson, G., Ben Amar, M.: Models of void electromigration. Eur. J. Appl. Math. **12**(2), 97–134 (2001)
45. Dai, S., Niethammer, B., Pego, R.L.: Crossover in coarsening rates for the monopole approximation of the Mullins-Sekerka model with kinetic drag. Proc. R. Soc. Edinb., Sect. A, Math. **140**(03), 553–571 (2010)
46. Dal Maso, G.: An Introduction to Γ-Convergence. Progress in Nonlinear Differential Equations and Their Applications, vol. 8. Birkhäuser, Boston (1993), xiv+339 pp.
47. Davi, F., Gurtin, M.E.: On the motion of a phase interface by surface diffusion. Z. Angew. Math. Phys. **41**(6), 782–811 (1990)
48. Deckelnick, K., Elliott, C.M.: Local and global existence results for anisotropic Hele-Shaw flows. Proc. R. Soc. Edinb. A **129**, 265–294 (1999)
49. Deckelnick, K., Dziuk, G., Elliott, C.M.: Computation of geometric partial differential equations and mean curvature flow. Acta Numer. **14**, 139–232 (2005)
50. De Mottoni, P., Schatzman, M.: Geometrical evolution of developed interfaces. Trans. Am. Math. Soc. **347**(5), 1533–1589 (1995)
51. Dierkes, U., Hildebrandt, S., Sauvigny, F.: Minimal Surfaces. Revised and Enlarged, 2nd edn. Grundlehren der Mathematischen Wissenschaften, vol. 339. Springer, Heidelberg (2010), xvi+688 pp.
52. Dinghas, A.: Über einen geometrischen Satz von Wulff für die Gleichgewichtsform von Kristallen. Z. Kristallogr. **105**, 304–314 (1944)
53. Duchon, J., Robert, R.: Evolution d'une interface par capillarité et diffusion de volume I. Existence locale en temps. Ann. Inst. Henri Poincaré, Anal. Non Linéaire **1**, 361–378 (1984)
54. Dupaix, C., Hilhorst, D., Kostin, I.N.: The viscous Cahn-Hilliard equation as a limit of the phase field model: lower semicontinuity of the attractor. J. Dyn. Differ. Equ. **11**(2), 333–353 (1999). (English summary)
55. Dziuk, G.: An algorithm for evolutionary surfaces. Numer. Math. **58**, 603–611 (1991)
56. Eck, C., Garcke, H., Knabner, P.: Mathematische Modellierung. Springer, Berlin (2011). xiv+513 pp., Revised second edn.
57. Ecker, K.: Regularity Theory for Mean Curvature Flow. Progress in Nonlinear Differential Equations and Their Applications, vol. 57. Birkhäuser, Boston (2004), xiv+165 pp.

58. Ecker, K.: Heat equations in geometry and topology. Jahresber. Dtsch. Math.-Ver. **110**(3), 117–141 (2008)
59. Efendiev, M.A., Gajewski, H., Zelik, S.: The finite dimensional attractor for a 4th order system of Cahn-Hilliard type with a supercritical nonlinearity. Adv. Differ. Equ. **7**(9), 1073–1100 (2002)
60. Eilks, C., Elliott, C.M.: Numerical simulation of dealloying by surface dissolution via the evolving surface finite element method. J. Comput. Phys. **227**(23), 9727–9741 (2008)
61. Elliott, C.M.: The Cahn-Hilliard model for the kinetics of phase separation. In: Mathematical Models for Phase Change Problems, Óbidos, 1988. Internat. Ser. Numer. Math., vol. 88, pp. 35–73. Birkhäuser, Basel (1989)
62. Elliott, C.M., Garcke, H.: On the Cahn-Hilliard equation with degenerate mobility. SIAM J. Math. Anal. **27**(2), 404–423 (1996)
63. Elliott, C.M., Garcke, H.: Existence results for diffusive surface motion laws. Adv. Math. Sci. Appl. **7**(1), 467–490 (1997)
64. Elliott, C.M., Maier-Paape, S.: Losing a graph with surface diffusion. Hokkaido Math. J. **30**, 297–305 (2001)
65. Elliott, C.M., Ockendon, J.R.: Weak and Variational Methods for Moving Boundary Problems. Pitman, Boston (1982)
66. Elliott, C.M., Zheng, S.: On the Cahn-Hilliard equation. Arch. Ration. Mech. Anal. **96**(4), 339–357 (1986)
67. Elliott, C.M., Zheng, S.: Global existence and stability of solutions to the phase field equations. In: Free Boundary Value Problems, Oberwolfach, 1989. Internat. Ser. Numer. Math., vol. 95, pp. 46–58. Birkhäuser, Basel (1990)
68. Escher, J.: The Dirichlet-Neumann operator on continuous functions. Ann. Sc. Norm. Super. Pisa, Cl. Sci. **4**(21), 235–266 (1994)
69. Escher, J.: Funktionalanalytische Methoden bei freien Randwertaufgaben. Jahresber. Dtsch. Math.-Ver. **109**(4), 195–219 (2007)
70. Escher, J., Simonett, G.: Classical solutions for Hele-Shaw models with surface tension. Adv. Differ. Equ. **2**(4), 619–642 (1997)
71. Escher, J., Simonett, G.: A center manifold analysis for the Mullins-Sekerka model. J. Differ. Equ. **143**(2), 267–292 (1998)
72. Escher, J., Mayer, U., Simonett, G.: The surface diffusion flow for immersed hypersurfaces. SIAM J. Math. Anal. **29**(6), 1419–1433 (1998)
73. Escher, J., Prüss, J., Simonett, G.: Analytic solutions for a Stefan problem with Gibbs-Thomson correction. J. Reine Angew. Math. **563**, 1–52 (2003)
74. Evans, L.C., Soner, H.M., Souganidis, P.E.: Phase transitions and generalized motion by mean curvature. Commun. Pure Appl. Math. **45**(9), 1097–1123 (1992)
75. Evans, C., Spruck, J.: Motion by mean curvature I. J. Differ. Geom. **33**, 635–681 (1991)
76. Fife, P.C.: Models for phase separation and their mathematics. In: Mimura, M., Nishida, T. (eds.) Nonlinear Partial Differential Equations and Applications. KTK, Tokyo (1993)
77. Fife, P.C.: Barrett Lecture Notes (1991). University of Tennessee
78. Fonseca, I.: The Wulff theorem revisited. Proc. R. Soc. Lond. Ser. A, Math. Phys. Sci. **432**(1884), 125–145 (1991)
79. Fonseca, I., Müller, S.: A uniqueness proof for the Wulff theorem. Proc. R. Soc. Edinb. A **119**(1–2), 125–136 (1991)
80. Friedman, A.: Variational Principles and Free Boundary Problems. Wiley/Interscience, New York (1982)
81. Gage, M., Hamilton, R.S.: The heat equation shrinking convex plane curves. J. Differ. Geom. **23**, 69–95 (1986)
82. Garcke, H.: Mechanical effects in the Cahn-Hilliard model: a review on mathematical results. In: Miranville, A. (ed.) Mathematical Methods and Models in Phase Transitions, pp. 43–77. Nova Science, New York (2005)
83. Garcke, H.: Kepler, Kristalle und Computer. Mathematik und numerische Simulationen helfen Kristallwachstum zu verstehen. MDMV **20**, 219–228 (2012)
84. Garcke, H., Nestler, B., Stinner, B.: A diffuse interface model for alloys with multiple components and phases. SIAM J. Appl. Math. **64**(3), 775–799 (2004)
85. Garcke, H., Schaubeck, S.: Existence of weak solutions for the Stefan problem with anisotropic Gibbs-Thomson law. Adv. Math. Sci. Appl. **21**(1), 255–283 (2011)
86. Garcke, H., Sturzenhecker, T.: The degenerate multi-phase Stefan problem with Gibbs-Thomson law. Adv. Math. Sci. Appl. **8**(2), 929–941 (1998)

87. Garcke, H., Wieland, S.: Surfactant spreading on thin viscous films: nonnegative solutions of a coupled degenerate system. SIAM J. Math. Anal. **37**(6), 2025–2048 (2006)
88. Giga, Y.: Anisotropic curvature effects in interface dynamics. Sūgaku Expo. **52**, 113–117 (2000). English translation, Sugaku Expositions **16**, 135–152 (2003)
89. Giga, Y.: Singular diffusivity—facets, shocks and more. In: Hill, J.M., Moore, R. (eds.) Applied Math. Entering the 21st Century, pp. 121–138. ICIAM, Sydney (2003). SIAM, Philadelphia 2004
90. Giga, Y.: Surface Evolution Equations. A Level Set Approach. Monographs in Mathematics, vol. 99. Birkhäuser, Basel (2006), xii+264 pp.
91. Giga, M.-H., Giga, Y.: On the role of kinetic and interfacial anisotropy in the crystal growth theory. Preprint (2013)
92. Giga, M.-H., Giga, Y.: Very singular diffusion equations: second and fourth order problems. Jpn. J. Ind. Appl. Math. **27**, 323–345 (2010)
93. Giga, Y., Ito, K.: Loss of convexity of simple closed curves moved by surface diffusion. In: Topics in Nonlinear Analysis. Progr. Nonlinear Differential Equations Appl., vol. 35, pp. 305–320. Birkhäuser, Basel (1999)
94. Giga, Y., Ito, K.: On pinching of curves moved by surface diffusion. Commun. Appl. Anal. **2**(3), 393–406 (1998)
95. Giga, Y., Rybka, P.: Quasi-static evolution of 3-D crystals grown from supersaturated vapor. J. Differ. Equ. **15**, 1–15 (2003)
96. Gilbarg, D., Trudinger, N.S.: Elliptic Partial Differential Equations of Second Order. Grundlehren der mathematischen Wissenschaften, vol. 224. Springer, Berlin (1998), xiii+517 pp.
97. Giusti, E.: Minimal Surfaces and Functions of Bounded Variation. Monographs in Mathematics, vol. 80. Birkhäuser, Basel (1984), xii+240 pp.
98. Grayson, M.A.: The heat equation shrinks embedded plane curves to round points. J. Differ. Geom. **26**, 285–314 (1987)
99. Gurtin, M.E.: Configurational Forces as Basic Concepts of Continuum Physics. Applied Mathematical Sciences, vol. 137. Springer, New York (2000), vii+249 pp.
100. Gurtin, M.G.: Thermodynamics of Evolving Phase Boundaries in the Plane. Clarendon, Oxford (1993)
101. Hadžić, M., Guo, Y.: Stability in the Stefan problem with surface tension (I). Commun. Partial Differ. Equ. **35**(2), 201–244 (2010)
102. Hanzawa, E.: Classical solutions of the Stefan problem. Tohoku Math. J. **33**(3), 297–335 (1981)
103. Hildebrandt, S., Tromba, A.: The Parsimonious Universe. Shape and Form in the Natural World. Copernicus, New York (1996), xiv+330
104. Huisken, G.: Flow by mean curvature of convex surfaces into spheres. J. Differ. Geom. **20**(1), 237–266 (1984)
105. Huisken, G.: The volume preserving mean curvature flow. J. Reine Angew. Math. **382**, 35–48 (1987)
106. Huisken, G.: Asymptotic behavior for singularities of the mean curvature flow. J. Differ. Geom. **31**, 285–299 (1990)
107. Huisken, G.: Local and global behaviour of hypersurfaces moving by mean curvature. Proc. Symp. Pure Math. **54**, 175–191 (1993)
108. Ilmanen, T.: Convergence of the Allen-Cahn equation to Brakke's motion by mean curvature. J. Differ. Geom. **38**(2), 417–461 (1993)
109. Ilmanen, T.: Elliptic regularization and partial regularity for motion by mean curvature. Mem. Amer. Math. Soc. **108**(520) (1994), x+90 pp.
110. Kohn, R.V., Otto, F.: Upper bounds on coarsening rates. Commun. Math. Phys. **229**(3), 375–395 (2002)
111. Kraus, C.: The degenerate and non-degenerate Stefan problem with inhomogeneous and anisotropic Gibbs-Thomson law. Eur. J. Appl. Math. **22**(5), 393–422 (2011)
112. Libbrecht, K.G.: The Snowflake. Winter's Secret Beauty (2003). Voyageur Press
113. Libbrecht, K.G.: Morphogenesis on Ice: the physics of snow crystals. Engineering & Science **1** (2001)
114. Lifshitz, I.M., Slyozov, V.V.: The kinetics of precipitation from supersaturated solid solutions. J. Phys. Chem. Solids **19**, 35–50 (1961)
115. Luckhaus, S.: Solutions for the two-phase Stefan problem with the Gibbs–Thomson law for the melting temperature. Eur. J. Appl. Math. **1**(2), 101–111 (1990)
116. Luckhaus, S., Sturzenhecker, T.: Implicit time discretization for the mean curvature flow equation. Calc. Var. Partial Differ. Equ. **3**(2), 253–271 (1995)

117. Mantegazza, C.: Lecture Notes on Mean Curvature Flow. Progress in Mathematics, vol. 290. Springer, Basel (2011), xii+166 pp.
118. Mayer, U.F.: Two-sided Mullins-Sekerka flow does not preserve convexity. Electr. J. Differ. Equ. **1**, 171–179 (1998)
119. Mayer, U.F.: A numerical scheme for moving boundary problems that are gradient flows for the area functional. Eur. J. Appl. Math. **11**, 61–80 (2000)
120. Mayer, U.F., Simonett, G.: Self-intersections for the surface diffusion and the volume-preserving mean curvature flow. Differ. Integral Equ. **13**(7–9), 1189–1199 (2000)
121. Meirmanov, A.M.: The Stefan Problem. De Gruyter, Berlin (1992)
122. Mielke, A., Theil, F., Levitas, V.I.: A variational formulation of rate-independent phase transformations using an extremum principle. Arch. Ration. Mech. Anal. **162**(2), 137–177 (2002)
123. Miranville, A., Zelik, S.: Robust exponential attractors for Cahn-Hilliard type equations with singular potentials. Math. Methods Appl. Sci. **27**(5), 545–582 (2004)
124. Modica, L.: The gradient theory of phase transitions and the minimal interface criterion. Arch. Ration. Mech. Anal. **98**, 123–142 (1987)
125. Modica, L., Mortola, S.: Un esempio di Γ-convergenza. Boll. Unione Mat. Ital, B **14**(5), 285–299 (1977)
126. Morgan, F.: Geometric Measure Theory. A Beginner's Guide, 4th edn. Elsevier, Amsterdam (2009)
127. Mucha, P.: Regular solutions to a monodimensional model with discontinuous elliptic operator. Interfaces Free Bound. **14**, 145–152 (2012)
128. Mucha, P.: On weak solutions to the Stefan problem with Gibbs-Thomson correction. Differ. Integral Equ. **20**(7), 769–792 (2007)
129. Mucha, P., Rybka, P.: A note on a model system with sudden directional diffusion. J. Stat. Phys. **146**, 975–988 (2012)
130. Mullins, W.W.: Two-dimensional motion of idealized grain boundaries. J. Appl. Phys. **27**, 900–904 (1956)
131. Mullins, W.W.: Theory of thermal grooving. J. Appl. Phys. **28**, 333–339 (1957)
132. Niethammer, B.: Derivation of the LSW-theory for Ostwald ripening by homogenization methods. Arch. Ration. Mech. Anal. **147**(2), 119–178 (1999)
133. Niethammer, B., Otto, F.: Ostwald ripening: the screening length revisited. Calc. Var. PDE **13**(1), 33–68 (2001)
134. Niethammer, B., Pego, R.L.: Non-self-similar behavior in the LSW theory of Ostwald ripening. J. Stat. Phys. **95**(5–6), 867–902 (1999)
135. Novick-Cohen, A.: On the Viscous Cahn-Hilliard Equation, Edinburgh, 1985–1986. Material Instabilities in Continuum Mechanics, pp. 329–342. Oxford Sci. Publ., New York (1988)
136. Novick-Cohen, A.: The Cahn-Hilliard equation: mathematical and modeling perspectives. Adv. Math. Sci. Appl. **8**(2), 965–985 (1998)
137. Osher, S., Fedkiw, R.: Level Set Methods and Dynamic Implicit Surfaces. Applied Mathematical Sciences, vol. 153. Springer, New York (2003), xiv+273 pp.
138. Otto, F.: The geometry of dissipative evolution equations: the porous medium equation. Commun. Partial Differ. Equ. **26**(1–2), 101–174 (2001)
139. Pego, R.L.: Front migration in the nonlinear Cahn-Hilliard equation. Proc. R. Soc. Lond. Ser. A, Math. Phys. Sci. **422**(1863), 261–278 (1989)
140. Penrose, O., Fife, P.C.: Thermodynamically consistent models of phase-field type for the kinetics of phase transitions. Physica D **43**(1), 44–62 (1990)
141. Plotnikov, P.I., Starovoitov, V.N.: Stefan problem with surface tension as a limit of the phase field model. Differ. Equ. **29**(3), 395–404 (1993)
142. Prüss, J., Simonett, G.: Stability of equilibria for the Stefan problem with surface tension. SIAM J. Math. Anal. **40**(2), 675–698 (2008)
143. Prüss, J., Simonett, G., Zacher, R.: On normal stability for nonlinear parabolic equations. Discrete Contin. Dyn. Syst. **2009**(suppl.), 612–621 (2009). 7th AIMS Conference on Dynamical Systems, Differential Equations and Applications
144. Prüss, J., Simonett, G.: On the two-phase Navier-Stokes equations with surface tension. Interfaces Free Bound. **12**(3), 311–345 (2010)
145. Radkevich, E.V.: The Gibbs-Thompson correction and conditions for the existence of a classical solution of the modified Stefan problem. Dokl. Akad. Nauk SSSR **316**(6), 1311–1315 (1991). Translation in Soviet Math. Dokl. **43**(1), 274–278 (1991)
146. Ritoré, M., Sinestrari, C.: Mean Curvature Flow and Isoperimetric Inequalities. Birkhäuser, Basel (2010)

147. Röger, M.: Existence of weak solutions for the Mullins-Sekerka flow. SIAM J. Math. Anal. **37**(1), 291–301 (2005)
148. Rossi, R., Savaré, G.: Gradient flows of non convex functionals in Hilbert spaces and applications. ESAIM Control Optim. Calc. Var. **12**, 564–614 (2006)
149. Rubinstein, L.I.: The Stefan Problem. AMS Translation, vol. 27. Am. Math. Soc., Providence (1971)
150. Rybka, P., Hoffmann, K.-H.: Convergence of solutions to Cahn-Hilliard equation. Commun. Partial Differ. Equ. **24**(5–6), 1055–1077 (1999)
151. Sapiro, G.: Geometric Partial Differential Equations and Image Analysis. Cambridge University Press, Cambridge (2001)
152. Schätzle, R.: The quasistationary phase field equations with Neumann boundary conditions. J. Differ. Equ. **162**(2), 473–503 (2000)
153. Schätzle, R.: Hypersurfaces with mean curvature given by an ambient Sobolev function. J. Differ. Geom. **58**(3), 371–420 (2001)
154. Schmidt, A.: Computation of three dimensional dendrites with finite elements. J. Comput. Phys. **195**, 293–312 (1996)
155. Sethian, J.A.: Level Set Methods and Fast Marching Methods. Evolving Interfaces in Computational Geometry Fluid Mechanics, Computer Vision, and Materials Science, 2nd edn. Cambridge Monographs on Applied and Computational Mathematics, vol. 3. Cambridge University Press, Cambridge (1999), xx+378
156. Sethian, J.A.: Curvature and the evolution of fronts. Commun. Math. Phys. **101**(4), 487–499 (1985)
157. Soner, H.M.: Motion of a set by the curvature of its boundary. J. Differ. Equ. **101**, 313–372 (1993)
158. Soner, H.M.: Convergence of the phase field equations to the Mullins-Sekerka problem with kinetic undercooling. Arch. Ration. Mech. Anal. **131**, 139–197 (1995)
159. Solonnikov, V.A.: Unsteady flow of a finite mass of a fluid bounded by a free surface. Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) **152** (1986), Kraev. Zadachi Mat. Fiz. i Smezhnye Vopr. Teor. Funktsii 18, 137–157, 183–184; translation in J. Soviet Math. **40**(5), 672–686 (1988)
160. Spencer, B.J., Voorhees, P.W., Davis, S.H.: Morphological instability in epitaxially strained dislocation-free solid films. Phys. Rev. Lett. **67**, 3696–3699 (1991)
161. Steinbach, I.: Phase-field models in materials science. Model. Simul. Mater. Sci. Eng. **17**, 073001 (2009)
162. Stoth, B.: Convergence of the Cahn-Hilliard equation to the Mullins-Sekerka problem in spherical symmetry. J. Differ. Equ. **125**(1), 154–183 (1996)
163. Stoth, B.: A sharp interface limit of the phase field equations: one-dimensional and axisymmetric. Eur. J. Appl. Math. **7**(6), 603–633 (1996)
164. Taylor, J.E.: Crystalline variational problems. Bull. Am. Math. Soc. **84**(4), 568–588 (1978)
165. Taylor, J.E.: Constructions and conjectures in crystalline nondifferential geometry. In: Lawson, B., Tanenblat, K. (eds.) Differential Geometry. Proceedings of the Conference on Differential Geometry, Rio de Janeiro, 1991. Pitman Monographs Surveys Pure Appl. Math., vol. 52, pp. 321–336 (1991)
166. Taylor, J.E.: Mean curvature and weighted mean curvature. Acta Metall. Mater. **40**(7), 1475–1485 (1992)
167. Taylor, J.E., Cahn, J.W.: Linking anisotropic sharp and diffuse surface motion laws via gradient flows. J. Stat. Phys. **77**(1–2), 183–197 (1994)
168. Taylor, J.E., Cahn, J.W., Handwerker, C.A.: Geometric models of crystal growth. Acta Metall. Mater. **40**(7), 1443–1474 (1992)
169. Temam, R.: Infinite-Dimensional Dynamical Systems in Mechanics and Physics. Applied Mathematical Sciences, vol. 68. Springer, New York (1988), xxi+648 pp.
170. Villani, C.: Optimal Transport. Old and New. Grundlehren der Mathematischen Wissenschaften, vol. 338. Springer, Berlin (2009), xxii+973 pp.
171. Visintin, A.: Models of Phase Transitions. Progress in Nonlinear Differential Equations and Their Applications, vol. 28. Birkhäuser, Boston (1996)
172. Voorhees, P.W.: The theory of Ostwald ripening. J. Stat. Phys. **38**, 231–252 (1985)
173. Voorhees, P.W.: Ostwald ripening of two-phase mixtures. Annu. Rev. Mater. Sci. **22**, 197–215 (1992)
174. Wagner, C.: Theorie der Alterung von Niederschlägen durch Umlösen. Z. Elektrochem. **65**, 581–594 (1961)
175. Wang, S.-L., Sekerka, R.F., Wheeler, A.A., Murray, B.T., Coriell, S.R., Braun, R.J., McFadden, G.B.: Thermodynamically-consistent phase-field models for solidification. Physica D: Nonlinear Phenomena **69**(1–2), 189–200 (1993)
176. Wheeler, G.: Surface diffusion flow near spheres. Calc. Var. Partial Differ. Equ. **44**(1–2), 131–151 (2012)

177. Wheeler, G.: On the Curve Diffusion Flow of Closed Plane Curves. Annali di Matematica Pura ed Applicata (2012)
178. White, B.: Evolution of curves and surfaces by mean curvature. In: Proceedings of the International Congress of Mathematicians, vol. 1, Beijing, 2002, pp. 525–538 (2002)
179. Wulff, G.: Zur Frage der Geschwindigkeit des Wachstums und der Auflösung der Kristallflächen. Z. Kristallogr. **34**, 449–530 (1901)
180. Zheng, S.: Asymptotic behavior of solutions to the Cahn-Hilliard equation. Appl. Anal. **23**, 165–184 (1986)

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SURVEY ARTICLE

Unit Groups of Integral Group Rings: Old and New

Wolfgang Kimmerle

Abstract The article provides a survey on the unit group of integral group rings. It leads from classical theorems to recent advances on this topic. The focus is on large subgroups, i.e. subgroups of finite index in U, as well as on units of finite order. A short look at the methods used in the proofs is given. The interplay between group theory, representation theory and algebraic number theory should become transparent.

Keywords Integral group rings · Unit groups · Zassenhaus conjecture · Gruenberg–Kegel graph

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1 Introduction

Let G be a group. The integral group ring of G is denoted by $\mathbb{Z}G$. Elements of $\mathbb{Z}G$ are written as sums of the form

$$\sum_{g\in G} z_g g \quad \text{with } z_g \in \mathbb{Z},$$

where almost all coefficients z_g are zero, i.e. $\mathbb{Z}G$ is as $\mathbb{Z}$-module the free $\mathbb{Z}$-module with G as basis. The multiplication of the integral group ring $\mathbb{Z}G$ is the $\mathbb{Z}$-linear extension of the one of the group G. The unit group is denoted by $U(\mathbb{Z}G)$. The ring homomorphism

$$\varepsilon : \mathbb{Z}G \longrightarrow \mathbb{Z}, \qquad \sum_{g\in G} z_g g \mapsto \sum_{g\in G} z_g$$

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is called the augmentation of $\mathbb{Z}G$. A unit of $\mathbb{Z}G$ has augmentation ± 1 and the units with augmentation 1 form a normal subgroup $V(\mathbb{Z}G)$ called the group of normalized units of $\mathbb{Z}G$. Clearly $U(\mathbb{Z}G) \cong V(\mathbb{Z}G) \times C_2$ and in order to study $U(\mathbb{Z}G)$ it suffices to study $V(\mathbb{Z}G)$.

The object of this expository article[1] is to illustrate some of the recent achievements and developments concerning the structure of the unit group of integral group rings. We concentrate upon the generation of large subgroups of $V(\mathbb{Z}G)$, e.g. free subgroups, as well as on the question whether the torsion units of $V(\mathbb{Z}G)$ are more or less determined by G. Here we report on recent results in the context of the Zassenhaus conjecture and the related prime graph question. We also try to give a short insight into the methods used for attacking the problems. The hope is that a nonexpert gets an impression of the nice interplay between arithmetic and number theory on one side and group theory and representation theory on the other side. We also touch computational methods which contribute not only in an experimental way to partial solutions and answers of challenging conjectures.

For the general background on units in integral group rings we refer to [34] and [37], for further trends concerning arithmetic of group rings see [19].

2 Generic Types of Units

Given $\mathbb{Z}G$ it is a first question how to find units different to elements of $\pm G$. It may happen that there are no other units and if G is torsion free (i.e. each nontrivial element of G has infinite order) this is in the context of the following famous still open problems due to I. Kaplansky.

The Unit Conjecture (UC) [38, 3.16]. Let F be a field and let G be a torsion free group. Then all units of FG are of the form kg, $k \in F^*$, $g \in G$. In particular $\mathbb{Z}G$ has only trivial units.

The Zero Divisor Conjecture (ZDC) [38, 11.1]. Let F be a field and let G be a torsion free group. Then FG has no zero divisors.

(UC) implies (ZDC). This follows from a theorem of Connell [5] that FG is a prime ring if G is torsion free, cf. [32, 13.1.2]. For suppose that a and b are nonzero elements of FG with $a \cdot b = 0$. Because FG is prime it follows that $aFGb \neq 0$, i.e. there exists an element $x \in FG$ such that $y := axb \neq 0$. However $y^2 = 0$ and thus $1 - y$ is a unit of FG. If $1 - y = kg$ with $k \in F^*$ and $g \in G$ then y is a nilpotent element of $F\langle g \rangle$. But for the infinite cyclic group (UC) is valid. Thus $1 - y$ is a nontrivial unit of FG contradicting the hypothesis that the (UC) holds for G.

With respect to (ZDC) the best known result obtained so far seems to be that it holds provided G is soluble by finite [24]. (UC) holds provided G is a right ordered group [32, p. 588]. A group G with a subnormal series

$$1 = G_k \lhd \cdots \lhd G_1 \lhd G_0 = G$$

such that G_i/G_{i+1} is torsion free Abelian is right ordered [1].

[1] The article is an extended version of a survey given by the author at the annual DMV-Tagung 2012 in Saarbrücken.

The finite groups G with $U(\mathbb{Z}G) = \pm G$ were classified by G. Higman in his thesis 1940 [16, Theorem 8]. Let R be the ring of integers in an algebraic number field K. Then G. Higman pointed out that the unit group of RG has finite index in the unit group of any maximal order of KG containing it [16, Theorem 4]. As remarked in Higman's thesis this is a particular case of a result of O. Schilling. So if G is Abelian then the classical Dirichlet unit theorem may be applied. This explains widely that in the case when G is finite $U(\mathbb{Z}G) = \pm G$ if, and only if, G is an Abelian group of exponent 2, 3, 4 or 6 or a Hamiltonian 2-group.[2] If G is finite Abelian G. Higman established a perfect analogue to Dirichlet's unit theorem:

$$U(\mathbb{Z}G) = \pm G \times F,$$

where F is a free Abelian group of finite rank.

But even with the knowledge that there are non-trivial units it is not that easy to write them down explicitly. Note that the fundamental units, i.e. generators of the free part occurring in Dirichlet's unit theorem, are already unknown for general cyclotomic fields. The following examples reflect more or less the known generic "ground" types for units of integral group rings.

- The elements $\pm g$, $g \in G$ are called the *trivial units* of $\mathbb{Z}G$.
- Let $h, g \in G$, assume that g has finite order $o(g) = n$. Put $\widehat{g} = 1 + g + \cdots + g^{n-1}$. Then

  $$\mu(g, h) := 1 + (g - 1)h\widehat{g}$$

  is a unit of $\mathbb{Z}G$. If $\mu(g, h) \neq 1$ then $o(\mu(g, h)) = \infty$. Units of this type are called *bicyclic units*.
- Again let $g \in G$ with $o(g) = n$ and let $k, m \in \mathbb{N}$ with $k^m \equiv 1 \bmod n$. Then

  $$b_{k,m}(g) = \left(1 + g + \cdots + g^{k-1}\right)^m + \frac{1 - k^m}{n}\widehat{g}$$

  is a unit. Such units are called *Bass units*.

  Note that if $k \not\equiv \pm 1 \bmod n$ then $b_{k,m}(g)$ has infinite order.

As will be explained in Sect. 3 bicyclic units and Bass units play an essential role as generators of large subgroups of $U(\mathbb{Z}G)$, i.e. subgroups of finite index. Using the elementary definition of the multiplication of integral group rings in order to determine units usually leads immediately to problems in diophantine number theory and gives no insight into the structure of $U(\mathbb{Z}G)$.

A better way is to consider $\mathbb{Z}G$ as $\mathbb{Z}$-order in the rational group algebra $\mathbb{Q}G$. With this description it is easier to recognize units. As a typical example we consider the integral group ring of the symmetric group S_3.

$$\begin{array}{ccc} \mathbb{Q}S_3 = \mathbb{Q} \times \mathbb{Q} \times \begin{pmatrix} \mathbb{Q} & \mathbb{Q} \\ \mathbb{Q} & \mathbb{Q} \end{pmatrix} \\ \cup \\ \Lambda = \mathbb{Z} \times \mathbb{Z} \times \begin{pmatrix} \mathbb{Z} & 3\mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} \end{pmatrix} \end{array}$$

[2] A non-Abelian finite group G is called Hamiltonian if each subgroup is normal in G.

and

$$\mathbb{Z}S_3 = \left\{ \left(a, b, \begin{pmatrix} c & d \\ e & f \end{pmatrix} \right) \in A;\, a \equiv c \bmod 3,\, a \equiv b \bmod 2,\, b \equiv f \bmod 3 \right\},$$

where the permutations $s = (1,2)$ and $t = (1,2,3)$ are given as

$$(1,2) = \left(1, -1, \begin{pmatrix} -2 & -3 \\ 1 & 2 \end{pmatrix}\right), \qquad (1,2,3) = \left(1, 1, \begin{pmatrix} -2 & -3 \\ 1 & 1 \end{pmatrix}\right).$$

Now clearly $v = \left(1, -1, \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)\right)$ is an involution of $V(\mathbb{Z}S_3)$ and an easy calculation with matrices shows that

$$v \sim (1,2) \quad \text{within } \mathbb{Q}S_3 \text{ but } v \nsim (1,2) \text{ within } \mathbb{Z}_2 S_3.$$

In terms of the group basis $S_3 = \{1, t, t^2, s, st, ts\}$ the unit v reads as

$$v = t - t^2 - s + ts + st.$$

Note also, if v is contained in a subgroup U of $V(\mathbb{Z}S_3)$ isomorphic to S_3 there must be a (2×2)-matrix $A \in \left(\begin{smallmatrix} \mathbb{Z} & 3\mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} \end{smallmatrix}\right)$ such that

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} A \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = A^2.$$

The resulting system of diophantine equations has no solution.

3 Large Subgroups

In this section we assume that G is a finite group. F_n denotes the free group of rank n.

Theorem 3.1 (B. Hartley–P.F. Pickel [10, Theorem 2]) *If $U(\mathbb{Z}G)$ does not contain a subgroup isomorphic to F_2 then G is Abelian or Hamiltonian.*

So the unit group of Abelian or Hamiltonian group rings obviously plays again a specific role. For that one of other groups it is a natural task to find generators for the free subgroups. If $u = \sum_{g \in G} z_g g$ then $u^* = \sum_{g \in G} z_g g^{-1}$.

Theorem 3.2 (Z. Marciniak–S.K. Sehgal [28]) *If u is a non-trivial bicyclic unit, then $\langle u, u^* \rangle = F_2$.*

Now it is an obvious question whether there is a similar statement with respect to Bass units. A first result into this direction is the following.

Theorem 3.3 (J. Goncalves–A. del Rio [7, Main Theorem]) *Suppose that G is finite soluble. Let g be a non-central element of G of prime order and let $u = u_{k,m}(g)$ be a Bass unit of infinite order. Then there is a unit v which is either a Bass unit or a bicyclic unit with*

$$\langle u^n, v^n \rangle = F_2, \quad \textit{for suitable } n \in \mathbb{N}.$$

The search for a minimal possible counterexample to Theorem 3.3 for insoluble groups leads to simple groups with special elements. If G is a finite group and $a \in G$ then

$$D_G(a) = \left\{ g \in G; a^g \in \left\{ a, a^{-1} \right\} \right\}$$

is called the *dihedralizer* of a in G. Suppose that $a \in G$ has finite order. The Bass unit $u = u_{k,m}(a)$ has finite order over the centre of $U(\mathbb{Z}G)$ if, and only if, u is of finite order or $D_G(a) = G$ [7, Lemma 2.1]. This leads to the following definition. $a \in G$ is called *dihedral p-critical* if it has prime order $p \geq 5$ and the following three conditions are satisfied

- $D_G(a) \neq G$.
- $D_U(a) = U$ for all proper subgroups U of G with $a \in U$.
- $D_{G/N}(\bar{a}) = G/N$ for all normal subgroups N of G with $N \neq 1$.

There are only few simple groups with such elements.

Theorem 3.4 (J. Goncalves, R. Guralnick, A. del Rio [8, Theorem 1]) *If G is a simple group with a dihedral p-critical element a then $G \cong PSL(2, q)$, $q = l^f$. Moreover $p = l^f = 5$ or $l \neq p > 5$ and $2f$ is the multiplicative order of l modulo p.*

The proof uses the classification of the finite simple groups, abbreviated by CFSG. In [30] it could be shown that CFSG may be avoided if $|C_G(a)|$ is even. But if the centralizer order of the dihedral p-critical element is odd there seems to be no proof without CFSG.

Bicyclic units and Bass units play also a prominent role in the general question to find explicit generators for $U(\mathbb{Z}G)$.

Theorem 3.5 (J. Ritter–S.K. Sehgal [33]) *Suppose that G is nilpotent of odd order. Then the bicyclic units and the Bass units generate a subgroup of finite index of $U(\mathbb{Z}G)$.*

In general this theorem is not valid for nilpotent groups of even order [37, Sect. 5.3]. In order to complete these studies for finite nilpotent groups large subgroups of units of cyclotomic Hamiltonian quaternions should be constructed, cf. [37, 5.12]. Units in orders of division algebras seems to be a worthwhile challenging research area.

To write units explicitly as a product of Bass units is an old question. The original methods of H. Bass and J. Milnor involved K-theory. Recently a constructive approach has been established in [18].

4 Torsion Units

Assume in this section that G is finite. Looking again at $\mathbb{Z}G$ inside its maximal order and on units of finite order leads naturally to invertible $n \times n$-matrices over $\mathbb{Z}$ of finite order. The crystallographic restriction theorem determines precisely these orders for

a given dimension n. E.g. for $n = 3$ the well known result is $1, 2, 3, 4, 6$. It becomes clear that the order of a torsion unit of $\mathbb{Z}G$ will be restricted depending on the degree of its representations and therefore depending on the order of G.

Subgroups of finite order of $U(\mathbb{Z}G)$ are closely related to symmetry groups of lattices. By a well known theorem of Burnside the finite subgroups of $GL(n, \mathbb{Q})$ occur after suitable conjugation as finite subgroups of $GL(n, \mathbb{Z})$. Thus the theory on finite rational matrix groups, cf. [31], is naturally connected with the study of torsion subgroups of $\mathbb{Z}G$.

Let u be a unit of finite order of $V(\mathbb{Z}G)$ and assume further that G is finite. J.A. Cohn and D. Livingstone showed that $o(u)$ divides the exponent of G [4]. H. Zassenhaus stimulated further work on torsion units by posing the conjecture [43]

(**ZC**) u is within $\mathbb{Q}G$ conjugate to a trivial unit, i.e. $u \sim g \in G$.

It is not possible to replace $\mathbb{Q}G$ by $\mathbb{Z}G$. This shows the involution v of $\mathbb{Z}S_3$ defined in Sect. 1.

The first striking positive results on (ZC) concerning p-groups were achieved by K.W. Roggenkamp and L.L. Scott [36] and then with different and stronger methods by A. Weiss [40, 41].

Theorem 4.1 (A. Weiss [41]) *(ZC) holds provided G is nilpotent.*

Many investigations have then been undertaken in the case when G is metacyclic. The final positive answer to (ZC) for this class of groups was given by M. Hertweck 2003 in [14]. In the mean time the following appears to be the most far reaching result concerning the class of metabelian groups.

Theorem 4.2 (M. Caicedo, A. del Rio, L. Margolis [3]) *Let G be a finite group. (ZC) holds if G has a normal cyclic subgroup C such that G/C is Abelian.*

The Zassenhaus conjecture (ZC) is still open and is certainly one of the most prominent open questions in the theory of integral group rings. Especially for symmetric groups S_n it is far away from being solved. For these groups it is known if $n \leq 5$.

Note that if u is contained in a *group basis*, i.e. a subgroup of $V(\mathbb{Z}G)$ of the same order as G, then it is well known that u is rationally conjugate to a trivial unit. This follows from the class sum correspondence, cf. [34, IV], [20]. If h is an element of a group basis H of $\mathbb{Z}G$ then the sum over its conjugates in H is called its *class sum*. There is a bijection between H and G such that the class sums of corresponding elements coincide. Thus corresponding elements have the same character values on each ordinary irreducible representation and are therefore conjugate within $\mathbb{C}G$. A Noether-Deuring argument shows that they are then also conjugate within $\mathbb{Q}G$. But in general torsion units are not contained in group bases, e.g. the unit $v \in \mathbb{Z}S_3$ given in Sect. 1 is such a unit.

The Zassenhaus conjecture may be used to construct large normal torsion free subgroups of $U(\mathbb{Z}G)$. Starting with a faithful rational representation of G of degree m we extend this representation to a ring homomorphism

$$\tau : \mathbb{Q}G \longrightarrow \mathbb{Q}^{m \times m}.$$

After suitable conjugation we may assume that

$$\tau(\mathbb{Z}G) \subset \mathbb{Z}^{m\times m}.$$

Assume that ZC holds for $\mathbb{Z}G$. Then every image of a torsion unit of $U(\mathbb{Z}G)$ is conjugate within $GL(m,\mathbb{Q})$ to $\pm\tau(g)$, for some $g \in G$. Because the restriction of τ to G is injective it follows that $\mathrm{Ker}\tau \cap U(\mathbb{Z}G)$ is a torsion free normal subgroup of $U(\mathbb{Z}G)$. Let q be a prime. Since $\tau(U(\mathbb{Z}G)) \subset GL(m,\mathbb{Z})$, we may apply the group homomorphism $\rho : GL(m,\mathbb{Z}) \longrightarrow GL(m,\mathbb{Z}/q\mathbb{Z})$ defined by reducing the entries of the matrices modulo q.

A classical result of Minkowski says that ρ is injective on finite subgroups provided q is odd.[3] The kernel of the composition $\rho \circ \tau$ yields now a large torsion free normal subgroup of $U(\mathbb{Z}G)$. In other words the whole information on the torsion part of $U(\mathbb{Z}G)$ is somehow hidden in the torsion part of $GL(m,\mathbb{Z})$ as well as in $GL(m,\mathbb{Z}/q\mathbb{Z})$.

In the case of the symmetric group S_3 the description of $\mathbb{Z}S_3$ given in Sect. 2 may easily be used to show that the torsion free normal subgroup constructed as above with the 2-dimensional rational representation and $q = 3$ is a complement to $\pm S_3$ in $U(\mathbb{Z}S_3)$. For a long time it was a major question whether G has a normal complement in $V(\mathbb{Z}G)$. 1983 K.W. Roggenkamp and L.L. Scott showed that this is in general—even for metacyclic groups—not the case [35].

5 Methods

One main tool to study torsion units in integral group rings are *partial augmentations*. Throughout this section G is still assumed to be finite. Let $g \in G$ and $u = \sum_{x\in G} u_x x \in U(\mathbb{Z}G)$. Then

$$\varepsilon_g(u) = \sum_{h\in g^G} u_h$$

is called the partial augmentation of u with respect to the conjugacy class g^G. The following criterion demonstrates the relevance of partial augmentations in the context of the Zassenhaus conjecture.

Theorem 5.1 [26, 29] *A unit $u \in V(\mathbb{Z}G)$ is rationally conjugate to a trivial unit $g \in G$ if, and only if, $\varepsilon_x(v) \geq 0$ for every $v \in \langle u\rangle$ and every $x \in G$.*

Representation theoretical arguments and a special case of Chouinard's criterion on projectivity of modules [11, 3.2] yield the following

Theorem 5.2 (M. Hertweck [12, Theorem 2.4]) *Let $u \in V(\mathbb{Z}G)$ of finite order. Then*

$$\varepsilon_g(u) \neq 0 \Longrightarrow o(g) \ \textit{divides}\ o(u).$$

[3] If $q = 2$, then only elements conjugate to diagonal matrices of order 2 may be in the kernel of ρ restricted to a finite subgroup.

This theorem improves earlier results in [29, 2.7] considerably. Together with Theorem 5.1 and the Berman–Higman result that $\varepsilon_1(v) = 0$ for a non-trivial $v \in V(\mathbb{Z}G)$ [37] it has the remarkable consequence that the Zassenhaus conjecture holds for units of prime order p provided G has precisely one conjugacy class of elements of order p.

In [26] I. Luthar and I.B.S. Passi noticed the close connection between partial augmentations and the eigenvalues of u on a Wedderburn block of $\mathbb{C}G$. M. Hertweck observed that the same arguments may be applied in the modular situation [12, §3, §4].

More precisely let K be a field whose characteristic does not divide the order n of the unit u and which contains a primitive n-th root of unity. Assume that the partial augmentations of u and all its powers are known. Then if D is a K-representation of G, the multiplicities of the eigenvalues of $D(u)$ are determined by Fourier inversion. Let χ be the character of D and let ξ a possible eigenvalue of $D(u)$. Denote by $\mu(u, \xi, \chi)$ the multiplicity of ξ as an eigenvalue of $D(u)$. Then

$$\mu(u, \xi, \chi) = \frac{1}{n} \sum_{d|n} \mathrm{Tr}_{\mathbb{Q}(\zeta^d)/\mathbb{Q}}\big(\chi\big(u^d\big)\xi^{-d}\big) \in \mathbb{Z}_{\geq 0}.$$

The eigenvalues of $D(u)$ and the character values are considered within $\mathbb{Q}(\zeta)$ where ζ is a primitive n-th root of unity. $\mathrm{Tr}_{\mathbb{Q}(\zeta^d)/\mathbb{Q}}(x) = \sum_{\sigma \in Gal(\mathbb{Q}(\zeta^d)/\mathbb{Q})} \sigma(x)$ denotes the sum over the Galois conjugates.

Clearly the degrees of the irreducible K-representations bound the possibilities for $\mu(u, \xi, \chi)$. Note that $\chi(u^d)$ is given by the partial augmentations of u^d. Thus there are only finitely many possible partial augmentations of u. These may be easily computed via computer algebraic methods nowadays called the HeLP-algorithm [25]. In some situations this algorithm shows that u and its powers have only the trivial partial augmentations, i.e. for u^m, $m \in \mathbb{N}$ there is precisely one conjugacy class C_m of group elements with $\varepsilon_{C_m}(u^m) = 1$ and the partial augmentations of all other classes vanish. By Theorem 5.1 it follows then that u is conjugate within $\mathbb{Q}G$ to a group element. In any case the algorithm gives bounds for the possible partial augmentations of u.

Another main approach is due to A. Weiss. It is a consequence of the following result on p-permutation lattices.

Theorem 5.3 (A. Weiss [40], [37, (50.1)]) *Let G be a finite group and p be a prime. Let R be the integral closure of the p-adic integers $\mathbb{Z}_p$ in a finite extension of $\mathbb{Q}_p$. Let M be a RG-lattice and $N \lhd G$. Suppose that the RN-lattice M_N is free and that the fixed-point module M^N is a permutation lattice for G/N over R.*

Then M is a permutation lattice for G over R.

Let N be a normal p-subgroup of G. Assume that $u \in V(RG)$ is of finite order and maps to 1 under the reduction $\rho : RG \longrightarrow RG/N$. Then it follows with partial augmentations that u has to be a p-element. Denote by C the group generated by u. Let $P \in Syl_p(G)$ and consider $M = RG$ as $P \times C$-bimodule, where P acts by left multiplication and C by right multiplication. With $N = N \times 1$ it is clear that M_N is a free RN-lattice and $M^N \cong RG/N$ with P and C acting from left and right rsp. via ρ. So C acts trivially on M^N and Theorem 5.3 yields that $M_{P \times C}$ is a permutation

lattice for $P \times C$ and so also $M_{N\times C}$ for $N \times C$. Because C acts trivially on M^N a look at a permutation basis detects an isomorphism between C and a subgroup of N. Character theory finally shows that u is rationally conjugate to an element of N. For a detailed description see [11, Sect. 4].

Note that the arguments above may be applied also in the situation when C is replaced by a finite subgroup U of $V(\mathbb{Z}G)$ [11, Proposition 4.2].

It is a natural question whether the torsion units are even p-adically conjugate. In special situations, see [11, 5.4], this can be achieved. In general however rational conjugacy is best possible.

6 The Gruenberg-Kegel Graph

The prime graph $\Pi(G)$ of a group G is the graph whose vertices are the primes dividing the order of a torsion element of G. Two different vertices p and q are connected by an edge if, and only if, G has an element g with $o(g) = pq$. If G is finite the result of Cohn-Livingstone mentioned above shows that the vertices of the prime graph of G and that one of $V(\mathbb{Z}G)$ coincide. This remains true for arbitrary groups [38, Theorem 3.14]. In particular $V(\mathbb{Z}G)$ is torsion free if, and only if, G is a torsion free group.

The interest on the prime graph of a finite group comes from work of K.W. Gruenberg and K.W. Roggenkamp on the decomposition of the augmentation ideal (i.e. the kernel of the augmentation) of $\mathbb{Z}G$ in the seventies [9]. Using the classification of the finite simple groups J.S. Williams finally established that for G finite augmentation ideals of $\mathbb{Z}G$ decompose if, and only if, the prime graph of G is disconnected. The proof contains a purely group-theoretical result of K.W. Gruenberg and O. Kegel on the structure of finite groups with disconnected prime graph [42, Theorem A]. J.S. Williams and finally A.S. Kondratiev described the prime graphs of all simple groups with more than one connected component [23, 42]. It follows that a finite group has at most six different prime graph components. The same result holds with respect to $V(\mathbb{Z}G)$ provided G is finite whileas infinite groups in general may have arbitrarily many prime graph components. The Janko group J_4 is the only finite simple group with six components. Nowadays the prime graph of a group is often called its Gruenberg-Kegel graph.

In the case of the normalized unit group $V(\mathbb{Z}G)$ we call $\Pi(V(\mathbb{Z}G))$ the Gruenberg-Kegel graph of $\mathbb{Z}G$ denoted by $\Gamma(\mathbb{Z}G)$. It is an obvious question whether $\Gamma(\mathbb{Z}G) = \Pi(G)$. Clearly, if (ZC) is valid, this question has an affirmative answer. So the determination of $\Gamma(\mathbb{Z}G)$ should be regarded as a first step towards the Zassenhaus Conjecture (ZC). Note that (ZC) might hold for all finite groups even if one has to say that the evidence by the known results is small [17]. Thus it makes sense to investigate first weaker versions. The prime graph question is that one which has been intensively studied during the last years. One reason is that with the HeLP-method this question may be attacked using the ordinary character table and the Brauer tables for the group in question. In particular sporadic simple groups have been considered that way, mainly by V. Bovdi and A. Konovalov. For about half of the 26 sporadic simple groups the prime graph question could be positively settled, cf. [2]. That these

computations do not play the role of single examples shows the following reduction to almost simple groups. This reduction has been obtained for soluble extensions in [21, Proposition 4.3] and with respect to arbitrary extensions in [22].

Proposition 6.1 *Let N be a proper minimal normal subgroup of the finite group G. Assume that N is not non-Abelian simple and that $\Gamma(\mathbb{Z}G/N) = \Pi(G/N)$ then*

$$\Gamma(\mathbb{Z}G) = \Pi(G).$$

Because for Abelian (simple) groups the Zassenhaus conjecture (ZC) is valid it follows immediately that the Gruenberg-Kegel graph of $\mathbb{Z}G$ coincides with the prime graph of G provided G is soluble [21]. Thus it remains for a given non-Abelian simple group S to check all almost simple groups coming from S, i.e. all groups H sandwiched between $S = \mathrm{Inn}S$ and $\mathrm{Aut}S$. So far mainly simple groups, especially sporadic simple groups, have been investigated with the HeLP-method. Proposition 6.1 shows that these investigations should be extended also to their automorphism groups and the almost simple groups in between. In many situations the character tables for these groups, in particular with respect to sporadic simple groups, are available [39].

Proposition 6.2 [22] *Let G be a finite group whose order is divisible by at most three primes. Then*

$$\Pi\big(U(\mathbb{Z}G)\big) = \Pi(G \times C_2).$$

If G does not map onto $\mathrm{PGL}(2,9)$ *or* M_{10} *then*

$$\Gamma(\mathbb{Z}G) = \Pi(G).$$

By the remarks above we need to check only the almost simple groups coming from simple groups whose order is divisible by precisely three primes. By CFSG there are precisely 8 such simple groups, see also [15] or [6],

$$\mathrm{PSL}(2,5) \cong \mathrm{A}_5, \mathrm{PSL}(2,7), \mathrm{PSL}(2,8), \mathrm{PSL}(2,9) \cong \mathrm{A}_6,$$
$$\mathrm{PSL}(2,17), \mathrm{PSL}(3,3), \mathrm{PSP}(3,4) \cong \mathrm{U}(4,2), \mathrm{U}(3,3).$$

Using the HeLP-method [26, 27], [12, §6], [22] and Hertweck's results for the alternating group A_6 [13] the prime graph question can be settled for all almost simple groups whose order is divisible by three primes except the cases $\mathrm{PGL}(2,9)$ and M_{10}. For these two groups the question whether $V(\mathbb{Z}G)$ has elements of order six remains open.[4] Because the prime 2 is connected in $\pm G$ to all other primes the result on $U(\mathbb{Z}G)$ follows nevertheless.

[4] It seems to be the case that A. Bächle and L. Margolis found recently a way to settle the open cases using additionally p-adic integral representations, cf. A. Bächle, L. Margolis, Rational conjugacy of torsion units in integral group rings of non-solvable groups, arXiv:1305.7419.

References

1. Bovdi, A.A.: Group rings of torsion free groups. Sib. Mat. Zh. **1**, 555–558 (1960)
2. Bovdi, V., Jespers, E., Konovalov, A.: Torsion units in integral group rings of Janko simple groups. Math. Comput. **80**(273), 593–615 (2011)
3. Caicedo, M., del Rio, A., Margolis, L.: Zassenhaus conjecture for cyclic-by-Abelian groups. J. London Math. Soc. doi:10.1112/jlms/jdt002
4. Cohn, J.A., Livingstone, D.: On the structure of group algebras I. Can. J. Math. **17**, 583–593 (1965)
5. Connell, I.G.: On the group ring. Can. J. Math. **15**, 650–685 (1963)
6. Feit, W.: The current situation in the theory of finite simple groups. In: Actes Congres Int. Math., vol. 1, pp. 55–93 (1970)
7. Goncalves, J., del Rio, A.: Bass cyclic units as factors in a free group in integral group ring units. Int. J. Algebra Comput. **21**, 531–545 (2011)
8. Goncalves, J., Guralnick, R.M., del Rio, A.: Bass units in integral group rings of simple groups (2013), arXiv:1302.1751v1 [mathGR]
9. Gruenberg, K.W., Roggenkamp, K.W.: Decomposition of the augmentation ideal and of the relation modules of a finite group. Proc. Lond. Math. Soc. **31**(3), 149–166 (1975)
10. Hartley, B., Pickel, P.F.: Free subgroups in the unit groups of integral group rings. Can. J. Math. **32**(6), 1342–1352 (1980)
11. Hertweck, M.: On the torsion units of some integral group rings. Algebra Colloq. **13**(2), 329–348 (2006)
12. Hertweck, M.: Partial augmentations and Brauer character values of torsion units in group rings. Preprint (2007), arXiv:math.RA/0612429
13. Hertweck, M.: Zassenhaus Conjecture for A_6. Proc. Indian Acad. Sci. Math. Sci. **118**(2), 189–195 (2008)
14. Hertweck, M.: Torsion units in integral group rings of certain metabelian groups. Proc. Edinb. Math. Soc. **51**(2), 363–385 (2008)
15. Herzog, M.: On finite simple groups divisible by three primes only. J. Algebra **10**, 383–388 (1968)
16. Higman, G.: Units in group rings, D. Phil. thesis, Oxford Univ. (1940)
17. Höfert, C., Kimmerle, W.: On torsion units of integral group rings of small order. In: Giambruno, A., Milies, C.P., Seal, S.K. (eds.) Groups, Rings and Group Rings. Lect. Notes in Pure Math., vol. 248, pp. 243–252. Chapman & Hall, London (2006)
18. Jespers, E., del Rio, A., Van Gelder, I.: Writing units as a product of Bass units. Math. Comput. in press
19. Jespers, E., Kimmerle, W., Marciniak, Z., Nebe, G.:. Mini-workshop: Arithmetik von Gruppenringen, Report no. 55/2007 MFI Oberwolfach
20. Kimmerle, W.: More on the class sum correspondence. In: Roggenkamp, K.W., Taylor, M. (eds.) Group Rings and Class Groups. DMV Sem., vol. 18, pp. 27–37. Birkhäuser, Basel (1992)
21. Kimmerle, W.: On the prime graph of the unit group of integral group rings of finite groups. Contemp. Math. **420**, 215–228 (2006)
22. Kimmerle, W., Konovalov, A.: On the Prime Graph of the Unit Group of Integral Group Rings of Finite Groups II. Stuttgarter Mathematische Berichte (2012)
23. Kondrat'ev, A.S.: On prime graph components of finite simple groups (Russian). Mat. Sb. **180**(6), 787–797 (1989). Translated in Math. USSR Sb. **67**, 864 (1990)
24. Krophuller, P.H., Linnell, P., Moody, J.A.: Applications of a new K-theoretic theorem to soluble group rings. Proc. Am. Math. Soc. **104**, 675–684 (1988)
25. LAGUNA—Lie AlGebras and Units of group Algebras, version 3.0, developed by V. Bovdi, A.B. Konovalov, A.R. Rossmanith and Cs. Schneider (2003) (http://www.ukrgap.exponenta.ru/laguna.htm)
26. Luthar, I.S., Passi, I.B.S.: Zassenhaus conjecture for A_5. J. Natl. Acad. Math. India **99**, 1–5 (1989)
27. Luthar, I.S., Trama, P.: Zassenhaus conjecture for S_5. Commun. Algebra **19**(8), 2353–2362 (1991)
28. Marciniak, Z., Sehgal, S.K.: Constructing free subgroups of integral group rings. Proc. Am. Math. Soc. **125**, 1005–1009 (1997)
29. Marciniak, Z.S., Ritter, J., Sehgal, S.K., Weiss, A.: Torsion units in integral group rings of some metabelian groups, II. J. Number Theory **25**, 340–352 (1987)
30. Margolis, L.: Freie Untergruppen in $U(\mathbb{Z}G)$ und dieder-kritische Elemente, Diplomarbeit Stuttgart (2011)
31. Nebe, G., Plesken, W.: Finite Rational Matrix Groups. Memoirs AMS, vol. 556 (1991)

32. Passman, D.S.: The Algebraic Structure of Group Rings. Wiley-Interscience, New York (1977)
33. Ritter, J., Sehgal, S.K.: Construction of units in integral group rings of finite nilpotent groups. Trans. Am. Math. Soc. **324**, 602–621 (1991)
34. Roggenkamp, K.W.: Group rings: units and the isomorphism problem, with contributions by W. Kimmerle and A. Zimmermann. In: Roggenkamp, K.W., Taylor, M. (eds.) Group Rings and Class Groups. DMV Sem., vol. 18, pp. 1–152. Birkhäuser, Basel (1992)
35. Roggenkamp, K.W., Scott, L.L.: Non-splitting examples for normalized units. J. Pure Appl. Algebra **27**, 299–314 (1983)
36. Roggenkamp, K.W., Scott, L.L.: Isomorphisms of p-adic group rings. Ann. Math. **126**, 593–647 (1987)
37. Sehgal, S.K.: Units in Integral Group Rings (with an Appendix of A. Weiss), Pitman Monographs and Surveys. Pure and Applied Mathematics, vol. 69. Longman Scientific & Technical, Harlow (1993)
38. Sehgal, S.K.: Group rings. In: Hazewinkel, M. (ed.) Handbook of Algebra, vol. 3, pp. 455–541. Elsevier, Amsterdam (2003)
39. The GAP Group: GAP—Groups, Algorithms, and Programming, version 4.6.4 (2013). http://www.gap-system.org
40. Weiss, A.: p-adic rigidity of p-torsion. Ann. Math. **127**, 317–332 (1987)
41. Weiss, A.: Torsion units in integral group rings. J. Reine Angew. Math. **415**, 175–187 (1991)
42. Williams, J.S.: Prime graph components of finite groups. J. Algebra **21**, 487–513 (1981)
43. Zassenhaus, H.: On the torsion units of finte group rings. In: Studies in Math. (in Honour of A. Almeida Costa), pp. 119–126. Instituto de Alta Cultura, Lisboa (1974)

Wolfgang Kimmerle hat in Stuttgart Mathematik studiert und im Jahr 1978 promoviert. Von Januar–Mai 1980 war er Assistant Professor an der University of Virginia, danach Hochschulassistent an der Universität Stuttgart. Habilitation 1990 mit einer Habilitationsschrift über ganzzahlige Darstellungstheorie endlicher Gruppen. Er ist seit 1996 außerplanmäßiger Professor und war von Februar bis Juni 2004 Professor und Head of Department of Mathematics an der GUC Cairo. Seit 2004 ist er Akademischer Direktor an der Universität Stuttgart und war von April 2010 bis März 2013 Studiendekan in Mathematik. Forschungsgebiete: Gruppen- und Darstellungstheorie, Gruppenringe, Burnsideringe, Computeralgebra, kombinatorische Topologie.

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CORRIGENDUM

Corrigendum zu „Wilhelm Klingenberg, 1924–2010“

Jost-Hinrich Eschenburg

Online publiziert: 8. Mai 2013

In meinem Nachruf auf Wilhelm Klingenberg, Jahresbericht der DMV 114 (2012), 163–170, habe ich am Anfang eine möglicherweise missverständliche Formulierung gewählt, die ich gerne zurechtrücken möchte: „Wilhelm Klingenberg hat für die Differentialgeometrie in Deutschland nach dem Zweiten Weltkrieg eine ähnliche Bedeutung wie Marcel Berger für Frankreich oder Manfredo Do Carmo für Brasilien: Fast alle, die heute in Deutschland mit diesem Fach zu tun haben, sind in irgendeiner Weise mit Klingenberg verbunden, als direkte oder indirekte Schüler von ihm selbst oder von Kollegen, die mit ihm eng verbunden waren.“

Diese Formulierung hat für Irritationen gesorgt, die nicht beabsichtigt waren und die ich bedauere. Meine Formulierung trägt vielleicht der Tatsache nicht genügend Rechnung, dass es sowohl in der damaligen Bundesrepublik als auch in der DDR wichtige Entwicklungen in der Differentialgeometrie und globalen Analysis gab, die von Wilhelm Klingenberg wenig oder gar nicht beeinflusst waren. Statt „fast alle“ hätte ich daher lieber „viele“ schreiben sollen.

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BOOK REVIEW

Richard Dedekind: "Was sind und was sollen die Zahlen?"

Reprint of the 1893 edition (German). Cambridge University Press, New York 2012, 58 pp

Franz Lemmermeyer

Published online: 23 April 2013

What is a number? The Pythagoreans had a clear answer: a number is a proper multiple of the unit. This was the basis of their number theory as laid down in Book VII of Euclid's Elements. Since the early philosophers regarded the unit as indivisible, the Greeks did not develop a theory of fractions, although their astronomers (and, later, Diophantus) freely used them.

It was also clear to the Greeks that lengths of line segments and the areas of polygons can be measured quantitatively. They stopped short of attaching "numbers" to geometric objects, and instead considered ratios of "magnitudes of the same kind". Examples of such ratios were the ratio of the diagonal and the side of a square, or that of the circumference and the diameter of a circle. It is not at all trivial to compare such ratios: the theory of ratios was developed by Eudoxus and is presented in Book V of Euclid's Elements.

Comparing certain propositions in Book V (on ratios of magnitudes) and Book VII (on numbers and their proportions) it becomes clear that the Greeks realized that ratios do behave a lot like proportions of numbers. It took many centuries of working with these objects until the ratio of the diagonal and the side of the square (or π, the ratio of the circumference and the diameter of a circle) was accepted as some kind of "number".

This is a slightly revised version of a review which appeared first in Zentralblatt der Mathematik (Zbl. 1257.01035).

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This vague concept of "number" sufficed for developing calculus. But then Abel, Dirichlet, Cauchy, Weierstraß and others finally started putting calculus on a firm foundation by replacing arguments based on geometric intuition (a continuous function changing signs must have a root) by proofs. It was soon realized that such fundamental theorems depended crucially on the "completeness of numbers" (a notion which even Dedekind still called "continuity" (Stetigkeit)). Instead of accepting the completeness of the reals as a geometric "axiom", many mathematicians developed this result by extending the natural numbers step by step to the rational and then the real numbers. Perhaps the most familiar method of constructing the reals is the one used by Cantor, who considered Cauchy sequences of rational numbers.

Dedekind's tool for constructing the real numbers out of the rationals is the "Dedekind cut", which he explained in his book [2]. The basic idea is that every real number α defines a decomposition of the rational numbers into two disjoint subsets, those smaller than α and those larger than α. Conversely, the real numbers may be constructed by looking at such subsets of the rationals. The core idea of Dedekind's theory is the same as that by Eudoxus in Book V of Euclid's Elements, as Dedekind himself remarked in the preface of the book at hand: "It is exactly this age-old conviction which is certainly the source of my theory [...]". In this respect, see [7].

With the construction of the real numbers from the rationals in place, Dedekind looked at the foundations of arithmetic, namely the natural numbers. This is what Dedekind's memoir [1] is all about. Dedekind introduced numbers using sets and functions in what was an extremely abstract approach at the time; Hilbert, upon his arrival in Berlin on a journey through various German university towns in 1888, recalled that "in Berlin, I heard in all mathematical circles young and old talk—mostly in a hostile sense—about Dedekind's work "Was sind und was sollen die Zahlen?", which had just appeared." This quotation from Hilbert's defense of the "tertium non datur" in his talk in Hamburg in 1930 (see [8] and [9]) is one out of many that show the attitude of Dedekind's contemporaries towards his abstract mathematics in general, and, in particular, towards the exposition found in "Was sind und was sollen die Zahlen?". In this respect see also Dedekind's correspondence with Keferstein published by M.A. Sinaceur [10].

The negative reaction to Dedekind's memoir is also reflected in Dedekind's own remarks in the preface of the second edition of his booklet, which is reprinted here. In fact, to finitists like Kronecker it must have been an act of blasphemy that Dedekind defined the natural numbers with the help of infinite sets, whose existence he "proved" in a way that modern readers will find rather strange (see Satz 66). Perhaps Dedekind had the audacity to perform such an act because he regarded numbers as "free creations of the human mind" and not, as Kronecker did, as creations of God.

Today, Peano's approach to natural numbers is much better known than Dedekind's, and Dedekind's clarification of the notion of a number is interesting mainly to historians. I do not know how many of them read German (there exist English translations, in particular in "The Nature and Meaning of Numbers" in [3] and [5]), and there are even translations into Russian [4] and Spanish [6]; but there must be even fewer who can read the present reissue, which is typeset in fraktur. Perhaps Dedekind's ground-breaking book would have deserved more than a simple reprint.

References

1. Dedekind, R.: Was sind und was sollen die Zahlen? Vieweg, Braunschweig (1888). JFM 20.0049.05
2. Dedekind, R.: Stetigkeit und irrationale Zahlen. Vieweg, Braunschweig (1892). JFM 24.0248.02
3. Dedekind, R.: Essays on the Theory of Numbers. I. Continuity and Irrational Numbers. II. The Nature and Meaning of Numbers. The Open Court, Chicago (1901). London: Kegan Paul and Co. 116 S. Nature 64, 374. (1901; JFM 32.0185.01); Chicago: The Open Court Publishing Co. 115 S (1909; JFM 40.0220.02)
4. Dedekind, R.: Was sind und was sollen die Zahlen? Kasan Ges. (2) **15**(2), 25–105 (1905). JFM 36.0087.04 (Russian)
5. Dedekind, R.: What Are Numbers and What Should They Be? RIM Monographs in Mathematics. Research Institute for Mathematics, Orono (1995). Zbl 0826.01037
6. Dedekind, R.: Qué son y para qué sirven los números? Y otros escritos sobre los fundamentos de la matemática. Alianza Editorial, Madrid (1998). Zbl 1041.01012
7. Gardies, J.L.: Eudoxe et Dedekind. Rev. Hist. Sci. **37**, 111–125 (1984). Zbl 0571.01021
8. Hilbert, D.: Die Grundlegung der elementaren Zahlentheorie. Math. Ann. **104**, 485–494 (1931). JFM 57.0054.04
9. Sieg, W., Schlimm, D.: Dedekind’s analysis of number. Synthese **147**(1), 121–170 (2005). Zbl 1084.01008
10. Sinaceur, M.A.: L’infini et les nombres. Commentaires de R. Dedekind à “Zahlen”. La correspondance avec Keferstein. Rev. Hist. Sci. **27**, 251–278 (1974). Zbl 0293.01030

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BOOK REVIEW

Lorenz J. Halbeisen: "Combinatorial Set Theory"

Springer-Verlag, 2012, 453 pp

Ralf Schindler

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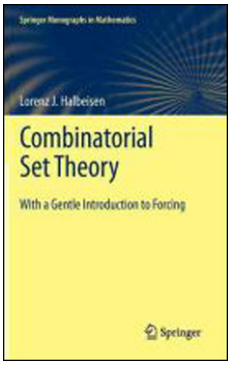

Combinatorics is that area of mathematics where we abstract that much from any structural properties of the objects under study that the Pigeonhole Principle is of prominent use. This principle says that if 5 pigeons sit in 3 holes, then at least one hole is occupied by more than 1 pigeon, or more generally: there can be no injection $f: A \to B$ if the cardinality of the set A is strictly bigger than the cardinality of the set B.

The Pigeonhole Principle yields Ramsey's Theorem [5]. It's infinite version tells us that for all n and $k \in \mathbb{N}$, if every subset of $\mathbb{N}$ with n elements is colored by one of the colors $1, 2, \ldots, k$, then there is an infinite $X \subset \mathbb{N}$ and some color $c \in \{1, 2, \ldots, k\}$ such that every subset of X with n elements is colored by c. Ramsey Theory studies (finite and infinite) variants and generalizations of Ramsey's Theorem and their applications.

An old result which exploits Ramsey theoretic methods, even though it actually predates F. Ramsey's formulation of his theorem, is a theorem of I. Schur [6] according to which for every $n > 1$ and for every sufficiently large prime p, the equation

$$x^n + y^n \equiv z^n \pmod{p}$$

has a solution x, y, z which is nontrivial, i.e., $p \nmid xyz$. A proof of this may be found in the classic [2] on Ramsey Theory.

By a theorem of Paris and Harrington [4] there is a *finite* variant of Ramsey's Theorem which is true but cannot be proven in Peano Arithmetic. Analogously, also many statements of *infinite* Ramsey Theory are independent in the following fashion.

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For an arbitrary set B, $\mathcal{P}(B)$ denotes the power set of B, i.e., the set of all subsets of B; let us write $\mathcal{P}_\infty(B) \subset \mathcal{P}(B)$ for the set of all infinite subsets of B. A set $A \subset \mathcal{P}_\infty(\mathbb{N})$ is called *Ramsey* iff there is some infinite $X \subset \mathbb{N}$ such that

$$\mathcal{P}_\infty(X) \subset A \quad \text{or} \quad \mathcal{P}_\infty(X) \cap A = \emptyset.$$

It is not difficult to use AC, the Axiom of Choice, to show that there is some $A \subset \mathcal{P}_\infty(\mathbb{N})$ which is not Ramsey. With the help of concepts from Descriptive Set Theory, we may however make sense of the following question, which then naturally arises: In the presence of AC, how "definable" can a non-Ramsey $A \subset \mathcal{P}_\infty(\mathbb{N})$ be? In order to answer this question, we have to specify the axiomatic background in which we work. This is because, as it turns out, the answer is independent from ZFC, the standard Zermelo–Fraenkel axiomatization of set theory with AC. The same applies if we drop assuming AC. A theorem of A.R.D. Mathias shows that in the absence of AC, every $A \subset \mathcal{P}_\infty(\mathbb{N})$ can be Ramsey [3].

The book under review provides a thorough and nicely written account of combinatorial set theory and infinite Ramsey theory together with a treatment of the underlying set theoretical axioms as well as of sophisticated methods which are involved in proving independence results.

Part I of the book introduces variants of Ramsey's Theorem and ZFC (with and without atoms) and discusses AC, the Banach–Tarski paradox, and ultrafilters on $\mathbb{N}$. It also deals with the topic of cardinal arithmetic in the absence of AC, where not every cardinality needs to be an $\aleph$. An amusing open problem is: Without AC, if there is a surjection from $A \times A$ onto $\mathcal{P}(A)$, must it be the case that A has at most 4 elements?

Part I also introduces several cardinal characteristics related to combinatorial questions, such as $\mathfrak{p}$, $\mathfrak{b}$, $\mathfrak{d}$, $\mathfrak{s}$, $\mathfrak{r}$, $\mathfrak{a}$, and $\mathfrak{i}$. E.g., the concept of $A \subset \mathcal{P}_\infty(\mathbb{N})$ being Ramsey is related to the shattering number $\mathfrak{h}$. To give an example, let us present the almost disjoint number $\mathfrak{a}$ which is a bit easier to define than $\mathfrak{h}$. If $A, B \in \mathcal{P}_\infty(\mathbb{N})$, then A and B are called almost disjoint if $A \cap B$ is finite. A *mad* (maximal almost disjoint) family is a collection $\mathcal{A} \subset \mathcal{P}_\infty(\mathbb{N})$ such that any two distinct elements of $\mathcal{A}$ are almost disjoint and for every $A \in \mathcal{P}_\infty(\mathbb{N})$ there is some $B \in \mathcal{A}$ such that $A \cap B$ is infinite. By a diagonal argument, no mad family can be countable. Also, there is always a mad family of cardinality $2^{\aleph_0}$, the size of the continuum. The cardinal characteristic $\mathfrak{a}$ is defined to be the smallest cardinality of a mad family, so that $\aleph_0 < \mathfrak{a} \leq 2^{\aleph_0}$. An exciting area of set theory studies the possible values of cardinal characteristics and their relations to each other, cf. e. g. [1].

Part II of the book introduces the key method of this area: the technique of *forcing*, which originally was developed by P. Cohen to prove the independence of the Continuum Hypothesis from ZFC. Especially relevant for the separation of various cardinal characteristics from each other is *iterated* forcing. To give just one example (Proposition 18.5 in the book), it may be shown by adding a lot of Cohen reals to a set theoretic parent universe that $\mathfrak{a}$ can consistently be strictly smaller than $2^{\aleph_0}$. This is done by showing that in this special situation, a mad family from the parent universe is still mad in the forcing extension after adding Cohen reals.

In order to produce deeper results a more detailed analysis of different types of reals which may be added by forcing is called for. This is the topic of Part III of

the book, which revisits Cohen reals but also introduces Laver, Silver, Miller, and Mathias reals. The forcings which add such reals may be iterated by *proper* forcing, a concept isolated by S. Shelah. Part III of the book brings us to the frontier of present-day research. For instance, it is shown how to build models of $\mathfrak{a} < \mathfrak{d} = \mathfrak{r}$ (Proposition 21.10), $\mathfrak{s} = \mathfrak{b} < \mathfrak{d}$ (Proposition 21.13), $\mathfrak{d} < \mathfrak{r}$ (Proposition 22.4), $\mathfrak{r} < \mathfrak{d}$ (Proposition 23.7), and $\mathfrak{p} < \mathfrak{h}$ (Proposition 2.12), respectively. All these statements may be translated into natural combinatorial statements about infinite sets of natural numbers.

Each chapter of the book comes with historical information, suggestions for further reading, and it lists open problems.

Lorenz Halbeisen wrote a marvelous book. I can recommend this book to all graduate students, PostDocs, and researchers who are interested in set theoretical combinatorics, set theory in the absence of AC, (iterated) forcing, and cardinal invariants. However, also mathematicians from other areas who are interested in the foundational aspects of their subject will enjoy this book.

References

1. Brendle, J.: Cardinal invariants of the continuum—a survey. Surikaisekikenkyusho Kokyuroku **1202**, 7–32 (2001)
2. Graham, R.L., Rothschild, B.L., Spencer, J.H.: Ramsey Theory. Wiley, New York (1980)
3. Mathias, A.R.D.: Happy families. Ann. Math. Log. **12**, 59–111 (1977)
4. Paris, J.B., Harrington, L.: A mathematical incompleteness in Peano arithmetic. In: Barwise, J. (ed.) Handbook of Mathematical Logic. North-Holland, Amsterdam, pp. 1133–1142 (1977)
5. Ramsey, F.: On a problem of formal logic. Proc. Lond. Math. Soc. Ser. II **30**, 264–286 (1930)
6. Schur, I.: Über die Kongruenz $x^m + y^m \equiv z^m \pmod{p}$. Jahresber. Dtsch. Math.-Ver. **25**, 114–117 (1916)

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BOOK REVIEW

Arshak Petrosyan, Henrik Shahgolian, Nina Uraltseva: "Regularity of Free Boundaries in Obstacle-Type Problems"

AMS, 2012, 221 pp

Georg S. Weiss

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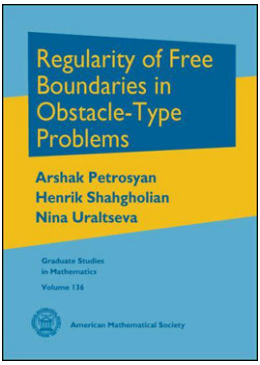

Die mathematische Analyse von Problemen mit freier Oberfläche hat eine lange Tradition, möglicherweise bis zu Newton, der sich bereits mit der mathematischen Analyse von Wasserwellen beschäftigt hat. Danach haben eine Reihe prominenter Naturwissenschaftler (darunter Laplace, Lagrange, Cauchy, Poisson und Stokes) Beiträge dazu geleistet. In den 1970er und 1980er Jahren fand mit neuen Methoden ein Boost statt, in dessen Zentrum das klassische *Hindernisproblem* stand. Im ursprünglichen Modell des Hindernisproblems wird eine elastische Membran über ein gegebenes Hindernis gespannt, und es ergibt sich eine Kontaktzone, deren Rand als „freier Rand" bezeichnet wird. Unter bestimmten Voraussetzungen an das Hindernis kann das mathematische Problem in einer nichtlinearen partiellen Differentialgleichung, nämlich

$$\Delta u = c\chi_{\{u>0\}} \quad \text{in } D,$$

zusammengefasst werden, wobei sich die im Allgemeinen variablen Koeffizienten c aus dem Hindernis ergeben und χ_A die charakteristische Funktion der Menge A bezeichnen soll. In den 1970er und 1980er Jahren ist eine Reihe von Büchern [1, 3–9] zu dem Thema erschienen, aber obwohl die Forschung seitdem große Fortschritte gemacht hat, mangelt es, abgesehen von dem eher auf Viskositätslösungen ausgerichteten Buch [2], an neueren Fach- und Lehrbüchern zu dem Thema.

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Das vorliegende Buch von Arshak Petrosyan, Henrik Shahgholian und der berühmten Lehrbuchautorin Nina Uraltseva hat sich zur Aufgabe gesetzt, neuere Methoden und Trends in der Forschung zur Regularitätstheorie freier Randwertprobleme zu dokumentieren.

Der Fokus liegt auf Problemen des Typs

$$\Delta u = f(x, u, \nabla u) \quad \text{in } D, \tag{1}$$

wo $f(x, z, p)$ einen Sprung in der z-Variablen und/oder der p-Variablen hat und der *freie Rand* die Sprungmenge der Funktion $x \mapsto f(x, u(x), \nabla u(x))$ sein soll.

Das Buch enthält eine Fülle von Übungsaufgaben sowie Hinweise zur Benutzung in Lehrveranstaltungen, und es ist vor dem Erscheinen bereits als Vorlesungsgrundlage verwendet und getestet worden, so dass es auch hervorragend als Lehrbuch geeignet ist. Zum Verständnis genügen Grundkenntnisse in elliptischen partiellen Differentialgleichungen wie z.B. Maximumprinzip, Harnack-Ungleichung sowie Regularität und Randregularität.

Das erste Kapitel enthält *Beispielprobleme* zu (1): das klassische Hindernisproblem, Probleme aus der Potentialtheorie, ein Problem aus der Supraleitfähigkeit, eine elastische Membran in einem Medium mit zwei Phasen, ein Problem aus der Kontrolltheorie, eine Kompositmembran, *optimal stopping* aus der stochastischen Analysis sowie niederdimensionale Hindernisse.

In Kapitel 2 wird *optimale Regularität der Lösung u* behandelt. Intuitiv sollte Beschränktheit der Funktion f Beschränktheit der zweiten Ableitung von u implizieren und dies sollte auch die optimale Schranke für die Regularität sein. Für eine Reihe von Problemen wird dies in Kapitel 2 mit unterschiedlichen Methoden bewiesen, es wird aber auch ein Gegenbeispiel zur Beschränktheit der zweiten Ableitungen beschrieben.

Kapitel 3 behandelt Eigenschaften wie *Nichtdegeneriertheit* der Lösung nahe des freien Randes (d.h. Wachstumsabschätzungen der Lösung von unten), *Dichten* der Koinzidenzmenge und ihres Komplementes sowie Dimension des freien Randes und Abschätzungen seines *Hausdorff-Maßes*. Außerdem werden wichtige Methoden wie z.B. Blow-ups und Monotonieformeln vorgestellt.

In Kapitel 4 wird *Regularität des freien Randes* (unter bestimmten Voraussetzungen bis zur Analytizität) in der Nähe sogenannter *flacher Randpunkte* (d.h. der freie Rand liegt zwischen 2 parallelen Hyperebenen mit beliebig kleinem Abstand, wenn man nahe genug an den festen Randpunkt herangeht) hergeleitet.

Um gleichmäßige Regularitätsabschätzungen zu erhalten und die Regularität nahe Singularitäten oder dem Gebietsrand ∂D zu untersuchen, ist das Wissen über *globale Lösungen*, d.h. Lösungen im ganzen Raum, sehr nützlich und wird in Kapitel 5 behandelt.

Als Anwendungen werden in Kapitel 6 zunächst gleichmäßige Lipschitz-Abschätzungen für den freien Rand hergeleitet. Des Weiteren wird Regularität in sogenannten *branch points*, d.h. von $\{u > 0\}$, $\{u < 0\}$ und dem offenen Kern von $\{u = 0\}$ berührte Punkten, diskutiert. Hier wird auch ein Gegenbeispiel beschrieben, in dem die Regularität des freien Randes nicht viel besser als C^1 ist.

Kapitel 7 beschreibt Resultate zu *singulären freien Randpunkten*. So ist die Menge der singulären freien Randpunkte unter Umständen in einer C^1-Mannigfaltigkeit enthalten.

In Kapitel 8 wird das Verhalten des freien Randes in der Nähe des „festen“ Randes ∂D untersucht.

Das letzte Kapitel behandelt *niederdimensionale Hindernisse*. Auf diesem Gebiet sind erst kürzlich entscheidende Fortschritte gemacht worden, so dass dieser Teil besonders spannend ist.

Abschließend lässt sich sagen, dass das Buch als *survey* Fachleute anspricht, aber mit seiner Fülle an Übungsaufgaben und seiner didaktischen Organisation ebensogut als Einführung für Nicht-Fachleute mit Grundkenntnissen in elliptischen partiellen Differentialgleichungen geeignet ist. Auf dem Gebiet des Hindernisproblems wird es mit Sicherheit ein Standardwerk werden.

Literatur

1. Baiocchi, C., Capelo, A.: Applications to free boundary problems. In: Variational and Quasivariational Inequalities. A Wiley-Interscience Publication. Wiley, New York (1984). Translated from the Italian by Lakshmi Jayakar
2. Caffarelli, L., Salsa, S.: A Geometric Approach to Free Boundary Problems. Graduate Studies in Mathematics, Bd. 68. American Mathematical Society, Providence (2005)
3. Chipot, M.: Variational Inequalities and Flow in Porous Media. Applied Mathematical Sciences, Bd. 52. Springer, New York (1984)
4. Crank, J.: Free and Moving Boundary Problems. The Clarendon Press Oxford University Press, New York (1987)
5. Díaz, J.I.: Elliptic equations. In: Nonlinear Partial Differential Equations and Free Boundaries. Vol. I. Research Notes in Mathematics, Bd. 106. Pitman, Boston (1985) (Advanced Publishing Program)
6. Duvaut, G., Lions, J.-L.: Inequalities in Mechanics and Physics. Grundlehren der Mathematischen Wissenschaften, Bd. 219. Springer, Berlin (1976). Translated from the French by C.W. John
7. Friedman, A.: Variational Principles and Free-Boundary Problems. Pure and Applied Mathematics. Wiley, New York (1982)
8. Kinderlehrer, D., Stampacchia, G.: An Introduction to Variational Inequalities and Their Applications. Pure and Applied Mathematics, Bd. 88. Academic Press, New York (1980)
9. Rodrigues, J.-F.: Obstacle Problems in Mathematical Physics. North-Holland Mathematics Studies, Bd. 134. North-Holland, Amsterdam (1987). Notas de Matemática [Mathematical Notes], 114