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PREFACE

Preface Issue 2-2014

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Most mathematicians have at least a basic knowledge of ordinary differential equations but I am rather sure that a lot of them including those, who work on differential equations, have not thought about delay differential equations, yet. At the same time there are many obvious examples of applications, where it takes some time for the state of a system to gain influence on its rate of change. Hans-Otto Walther begins his survey on "Topics in Delay Differential Equations" with the simple looking example $x'(t) = -\alpha x(t-1)$ and its variants to explain the most striking differences between differential equations with or without delay. After explaining a number of further examples and giving a brief overview of the basic theory the survey puts some emphasis on the dynamical behaviour of scalar equations with constant delay as well as on equations with state-dependent delay.

This is the second issue where we have, in collaboration with the Zentralblatt für Mathematik, a contribution to the category "Classics Revisited". Hervé Queffélec takes a new look at the paper on "The converse of Abel's theorem on power series" written by John E. Littlewood in 1911, at the age of 26. It generalises a previous result by Tauber and may be viewed as a seminal contribution to what we call nowadays "Tauberian theorems". It seems that this paper attracted Hardy's attention and initiated their long and fruitful collaboration.

The first of the book reviews in the current issue is concerned with "Large Scale Geometry" and its relations to topological and geometrical rigidity theory.

The other two reviews deal with books which are based on the (same) concept of "traces". However, this is developed in quite different directions: number, group and representation theory on the one side, functional analysis and spectral theory on the other side.

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SURVEY ARTICLE

Topics in Delay Differential Equations

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Abstract We introduce delay differential equations, give some motivation by applications, review basic facts about initial value problems from wellposedness to the variation-of-constants formula in the sun-star-framework, and discuss two topics in greater detail: (a) The dynamics generated by autonomous scalar equations with a single, constant time lag, from existence of periodic solutions to the fine structure of global attractors and chaotic motion, and (b) more recent results on equations with state-dependent delay (lack of smoothness, differentiable solution operators on suitable Banach manifolds, case studies). The final part addresses directions of future research.

Keywords Delay differential equation · Characteristic equation · Infinite-dimensional dynamical system · Semiflow · Periodic orbit · Bifurcation · Floquet multiplier · Global attractor · Morse-Smale decomposition · Hyperbolic set · Chaos · State-dependent delay · Solution manifold

Mathematics Subject Classification 34Kxx · 37Lxx

1 Introduction

The simplest differential equation with a delay, for a real function x, reads

$$x'(t) = -\alpha x(t-1), \tag{1.1}$$

with $\alpha \in \mathbb{R}$. This can be interpreted as modelling a system governed by feedback with a time lag. For $\alpha > 0$ feedback is negative with respect to the zero solution,

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and positive for $\alpha < 0$. The equation shows what makes delay differential equations (DDEs) different from ordinary differential equations (ODEs): In order to get a solution defined on some interval with left endpoint 0 initial data must be specified on the interval $[-1,0]$—if these are given by a continuous function then repeated integration over successive intervals $[n-1,n]$, $n \in \mathbb{N}$, results in a unique continuous solution $x : [-1,\infty) \to \mathbb{R}$ which is differentiable for $t > 0$, has a right derivative at $t = 0$, and satisfies Eq. (1.1) for $t \geq 0$. Trying exponentials $t \mapsto e^{zt}$ leads to the transcendental *characteristic equation*

$$z + \alpha e^{-z} = 0 \tag{1.2}$$

which has a countable set of complex conjugate solutions, with real parts accumulating at $-\infty$ and imaginary parts growing fast [163]. Non-real roots $z = u + iv$, $u = \Re z$ and $v = \Im z \neq 0$, of Eq. (1.2) yield oscillatory solutions

$$t \mapsto e^{ut}\big(c\cos(vt) + d\sin(vt)\big),$$

of Eq. (1.1), which is in obvious contrast to scalar autonomous ODEs, with all solutions monotone. At $\alpha = \alpha_k = (2k + \frac{1}{2})\pi$, $k \in \mathbb{N}_0$, pairs of complex conjugate roots cross into the right halfplane as α increases. The crossings occur at the points $\pm i\,\alpha_k$, and for $\alpha = \alpha_k$ the linear Eq. (1.1) has periodic solutions

$$t \mapsto c\cos(\alpha_k t) + d\sin(\alpha_k t)$$

whose minimal period $\frac{2\pi}{\alpha_k}$ equals 4 if $k = 0$ and is less than 1 for $k > 0$.

For $r > 0$ and $n \in \mathbb{N}$ let $C_{rn} = C([-r,0],\mathbb{R}^n)$ denote the Banach space of continuous functions $\phi : [-r,0] \to \mathbb{R}^n$, with the norm given by $|\phi| = \max_{-r \leq s \leq 0} |\phi(s)|$, and abbreviate $C_{11} = C$. Equation (1.1) is a special case of a linear autonomous *retarded functional differential equation* (RFDE)

$$x'(t) = Lx_t \tag{1.3}$$

with a continuous linear functional $L : C_{rn} \to \mathbb{R}^n$ and the *solution segment* x_t given by

$$x_t(s) = x(t+s).$$

Nonlinear autonomous RFDEs, have the form

$$x'(t) = f(x_t) \tag{1.4}$$

with a functional $f : C_{rn} \supset U \to \mathbb{R}^n$. A prototypic, simple-looking example is the equation

$$x'(t) = -\alpha x(t-1)\big[1 + x(t)\big] \tag{1.5}$$

with parameter $\alpha > 0$, which was first studied by E.M. Wright in a beautiful paper which appeared in 1955 [163]. Equation (1.5), or *Wright's equation*, arose in a heuristic proof of the prime number theorem, see [164], and is equivalent to the delayed logistic equation

$$n'(t) = b\,n(t)\left[\frac{K - n(t-1)}{K}\right], \quad b > 0 \text{ and } K > 0,$$

which had been proposed by the ecologist G.E. Hutchinson in 1948 [51] as a phenomenological model for single species populations whose densities are oscillatory in time. For solutions of Wright's equation which satisfy $x(t) > -1$ the transformation given by $y = \log(1 + x)$ yields

$$y'(t) = -g\big(y(t-1)\big) \tag{1.6}$$

with $g : \mathbb{R} \ni \xi \mapsto \alpha(e^{\xi} - 1) \in \mathbb{R}$. As for the linear Eqs. (1.1), (1.6)—with an arbitrary continuous real function g—models delayed feedback with a time lag. (Global) negative feedback with respect to zero is characterized by the condition

$$\xi g(\xi) > 0 \quad \text{for all } \xi \neq 0. \tag{NF}$$

Among others Wright showed that for $\alpha > \frac{\pi}{2}$ Eq. (1.5) has bounded solutions with zeros unbounded and amplitudes not decaying. This suggested to look for periodic solutions of Eq. (1.5), or more generally, of Eq. (1.6) in case of negative feedback, and of the equation

$$x'(t) = -h\big(x(t), x(t-1)\big), \tag{1.7}$$

with a continuous function $h : \mathbb{R}^2 \to \mathbb{R}$ which satisfies the delayed negative feedback condition $h(0, \xi)\xi > 0$ for $\xi \neq 0$. The search for periodic solutions of these equations began in the early 1960ies. We shall describe what has been achieved since then, from existence and global bifurcation of periodic orbits to the fine structure of global attractors, from results on uniqueness and non-uniqueness of periodic orbits to proofs that for certain functions g and h the solution behaviour is chaotic.

Another theme, which presently attracts much attention and is emphasized below, are variable delays and in particular state-dependent delays, for example, Eqs. (1.6), (1.7) with the time lag $r = 1$ replaced by a *delay functional* $d : C_{r1} \to [0, r]$ with $r > 0$. The topic is in fact old and started with a paper by Poisson from 1806, but more intense research activity in this area began not before the 1980ies.

Through the past 50 years functional differential equations grew to a visible part of dynamical systems theory, accompanied by an increasing number of models for real world systems with time lags. The man who was at the heart of this process is the late Jack Hale. Jack's dynamical systems perspective, his monographs, his broad research interests, and his personal generosity over the years inspired a large group of collaborators from all over the world. There is a good deal of other, deep and seminal work, which will be emphasized in the sections below. For this author it were probably Wright's paper [163], Jones's idea how to find periodic solutions [55], Nussbaum's results on existence and global bifurcation of periodic solutions [106, 108], and Yorke's daring phase plane approach [57, 58], all of them addressed in Sect. 4 below, which had a crucial influence.

The present survey briefly touches upon applications, sketches some basic theory of initial value problems for Eqs. (1.3) and (1.4), and focusses on the two themes mentioned above in Sects. 4 and 5. Even there the account is far from complete, and reflects the interest of the author. —The field as a whole contains several other large areas. RFDEs may be non-autonomous, of course. Moreover there are *equations with unbounded delay* like

$$x'(t) = \int_{-\infty}^{0} f\big(x(t+s)\big)d\mu(s)$$

for which the basic theory, say, from well-posedness of initial value problems to local invariant manifolds at equilibria, is not yet as complete as it is for Eq. (1.4). Another class are *neutral* RFDEs of the form

$$\frac{d}{dt}Dx_t = f(x_t)$$

with a *difference operator* $D : C_{rn} \to \mathbb{R}^n$, for example, $D(\phi) = \phi(0) - c\phi(-r)$, on the left hand side. A large part of one of the two most recent monographs on functional differential equations [22, 45], namely, the book [45] by Hale and Verduyn Lunel, is devoted to these neutral RFDEs. Furthermore, there are *mixed type* equations which contain both delayed and advanced arguments of the solutions. A theory for such equations is still in its infancy.

2 Input from Applications

In the former Soviet Union (SU) DDEs were studied at least since the 1940ies, motivated by problems of control, for example, stabilizing the position of a ship by pumping water from a tank on one side of the vessel to a tank on the other side, see [103] and also [29]. The first monograph on DDEs is due to Myshkis [104]. It was soon after the Sputnik shock of 1957 that in the USA the area of FDEs began to grow. Among the first monographs on DDEs which appeared outside the SU are the book by Bellman and Cooke [8], which employs Laplace transform techniques in the sections on linear autonomous DDEs, and Hale's *Functional Differential Equations* [43], which presents the basic theory more from a dynamical systems point of view, with semigroups and semiflows, and parallel to ODE theory as far as possible.

Wright's results as well as heuristic insight from ecology [21, 51] motivated the search for periodic solutions of nonlinear DDEs of the forms (1.6) and (1.7). A DDE of the type (1.6) also arises in a model related to physics, namely, in the control of high frequency oscillators by phase-locking, see [20, 40]. Later a further strong and lasting stimulus to study Eq. (1.7) came from the life sciences: Soon after complicated behaviour in dynamical systems had been popularized as *chaos* by Li and Yorke [81] in 1975, Lasota [79] and Mackey and Glass [82] independently proposed Eq. (1.7) with $-h(\xi,\eta) = -\mu\xi + p(\eta)$, $\mu > 0$ and p hump-shaped, e.g. $p(\eta) = \alpha(\eta)^8 e^{-\beta\eta}$, $\alpha > 0$ and $\beta > 0$, as a model for certain physiological control processes, in particular for the regulation of the densities of blood cell populations, and provided numerical evidence for complicated solution behaviour at certain values of the parameters.

Synchronous activity in a network of n neurons which is modelled by a system of the form

$$x_j'(t) = -\mu\, x_j(t) + \sum_{k=1}^{n} g\big(x_j(t), x_k(t-1)\big), \quad j = 1, \ldots, n,$$

leads to the scalar first order Eq. (1.7) with $h(\eta, \xi) = -\mu\eta - g(\xi)$ and, for example, $g(\xi) = a\tanh(b\,\xi)$ with $a > 0, b > 0$. In this case Eq. (1.7) describes the interaction of instantaneous negative feedback with monotone delayed positive feedback.

The basic models for population growth which take age structure into account are first order partial differential equations (PDEs) together with initial and boundary conditions. In [9] it is shown that special cases of the PDE models yield—not equations of the type (1.7) but *neutral* RFDEs with constant time lags.

Another impetus is from laser physics: The *Lang-Kobayashi equations* for a semiconductor laser with optical feedback [72] are an autonomous *system* of first order DDEs with a single, constant time lag, whose exploration is still in its infancy.

The search for travelling waves in *lattice dynamical systems*, with an ODE at each node, leads to mixed type functional differential equations for the wave profile [86–88].

The scalar nonautonomous linear *pantograph equation*

$$x'(t) = ax(\lambda t) + bx(t)$$

with its variable and unbounded delay $(1-\lambda)t$ (for $0 < \lambda < 1$, write $\lambda\, t = t - (1-\lambda)t$) arises in modelling mechanical properties of a current collection system of an electric locomotive [59, 115]. It also describes a stable age distribution in populations of dividing cells [46], and occurs elsewhere.

State-dependent delayed and advanced arguments are already present in the implicit functional differential equation studied by Poisson [119, 156], which describes solutions to a problem from plane geometry. Modelling automatic position control by echo results in DDEs with delays equal to the running time of a signal (travelling at constant speed); the running time depends on the position when the signal was emitted and on the position when later the echo of the signal is received [114, 151]. Vibrations of a tool cutting into a rotating workpiece obey a system of DDEs with a delay which is state-dependent [52, 118]. Special cases of PDE models for the growth of structured populations which involve threshold phenomena yield neutral DDEs with a state-dependent delay [6]. Even more complicated are the *Wheeler-Feynman equations* for the motion of charged point masses which interact through their electromagnetic fields [7, 30–35, 160, 161], that is, equations of the two-body-problem of electrodynamics. They are implicit, neutral, and contain state-dependent advanced and delayed arguments similar to those from position control by echo.

A huge collection of further DDEs arising in applications can be found in the book *Applied Delay Differential Equations* by Erneux [38].

3 Basics

Existence, uniqueness and smooth dependence on data are easy for RFDEs. For example, if $f : C_{rn} \supset U \to \mathbb{R}^n$ is locally Lipschitz continuous then the initial value problem (IVP)

$$x'(t) = f(x_t) \quad \text{for } t > 0, \qquad x_0 = \phi \in U \tag{3.1}$$

associated with Eq. (1.4) has a unique maximal solution $x = x^{\phi}$, $x : [-r, t_{\phi}) \to \mathbb{R}^n$, $0 < t_{\phi} \leq \infty$, in the sense that x is continuous with $x_0 = \phi$, has a differentiable restriction $x|[0, t_{\phi})$, satisfies Eq. (1.4) for $0 \leq t < t_{\phi}$ (with the right derivative at $t = 0$), and any other solution of the same IVP is a restriction of x.

Here local existence follows by a straightforward application of the contraction mapping principle to the equation

$$y(t) = \int_0^t f(y_s + \phi_{*,s}) ds$$

with $\phi_*(t) = \phi(0)$ on $[0, \infty)$, for $y : [-r, \tau] \to \mathbb{R}^n$ with $y_0 = 0$, $\tau > 0$ sufficiently small. In case of examples like Eq. (1.6), with g only continuous, even more elementary stepwise integration yields unique maximal solutions on $[-1, \infty)$: For $0 \leq t \leq 1$ integrate the ODE $y'(t) = -g(\phi(t-1))$, then proceed to $1 \leq t \leq 2, \ldots$.

How about backward solutions? Initial data which are not differentiable at points close to $0 \in \mathbb{R}$ cannot extend to solutions on intervals $[t_0 - r, 0]$ with $t_0 < 0$. If such extensions exist for differentiable initial data then they may not be unique: Suppose g in Eq. (1.6) is constant. Then all initial data $\phi \neq \psi$ with $\phi(0) = \psi(0)$ define solutions which coincide for $t \geq 0$.

For f as above the equations

$$\Omega = \left\{(t, \phi) \in [0, \infty) \times U : t < t_{\phi}\right\} \quad \text{and} \quad F(t, \phi) = x_t^{\phi}$$

define a continuous semiflow on U. The *solution operators*

$$F_t : \Omega_t \ni \phi \mapsto F(t, \phi) \in U,$$

for $t \geq 0$ with $\Omega_t = \{\phi \in U : (t, \phi) \in \Omega\} \neq \emptyset$, inherit the smoothness properties from f, and for f continuously differentiable the derivatives $DF_t(\phi) = D_2F(t, \phi)$ are given by

$$DF_t(\phi)\chi = v_t^{\phi,\chi}$$

with the (unique) solution $v = v^{\phi,\chi}$ of the IVP

$$v'(t) = Df\big(F(t, \phi)\big)v_t \quad \text{for } 0 < t < t_{\phi}, \qquad v_0 = \chi \in C_{rn},$$

for the *linear variational equation* along the *flowline* $[0, t_{\phi}) \ni t \mapsto F(t, \phi) \in U$.

Often the solution operators F_t with $t \geq r$ are compact, which is true for example in case f is bounded. Compact solution operators have no continuous inverse.

Derivatives of flowlines exist in general only for $t > r$ since necessarily,

$$D_1 F(t,\phi)1 = \frac{d}{dt}\left(x_t^{\phi}\right),$$

which implies differentiability of x^{ϕ} on $[t-r,t]$. If f is continuously differentiable then the restriction of the semiflow F to the set $\{(t,\phi) \in \Omega : r < t\}$ is continuously differentiable as well.

In case $f = L$ is linear and continuous, $L : C_{rn} \to \mathbb{R}^n$, the semiflow is a strongly continuous semigroup of continuous linear operators $T_t : C_{rn} \to C_{rn}$, $t \geq 0$, whose generator A is defined on

$$\left\{\phi \in C_{rn} : \phi \text{ continuously differentiable and } \phi'(0) = L\phi\right\}$$

and acts by differentiation. The spectrum of A is discrete and consists of eigenvalues of finite algebraic multiplicity, with at most a finite number of them in each halfplane given by $\Re z > b$, $b \in \mathbb{R}$. So the center and unstable spaces have finite dimension, and the stable space has finite codimension—analogously for the corresponding local invariant manifolds at stationary points of F in case f is continuously differentiable.

The eigenvalues of A are given by a transcendental *characteristic equation* which is obtained from an ansatz for exponential solutions, or from a computation of the resolvent of A. The distribution of the eigenvalues in the plane shows that in general the generator is not sectorial, and the semigroup is not analytic.

Incidentally, let us mention here an observation [135] which has no counterpart in linear autonomous ODEs: There are solutions $x : \mathbb{R} \to \mathbb{R}$ of Eq. (1.1) which are not analytic. For example, choose $\phi \in C \setminus \{0\}$ with derivatives of all orders and value zero close to -1 and close to 0, use Eq. (1.1) in order to extend ϕ backward to a solution $x : (-\infty, 0] \to \mathbb{R}$ of Eq. (1.1), and set $x(t) = x^{\phi}(t)$ for $t > 0$.

For the transfer of results on long-term solution behaviour from linear equations to nonlinear perturbations the variation-of-constants formula is instrumental. It is here where RFDEs become a bit peculiar: Consider a linear-inhomogeneous equation

$$x'(t) = Lx_t + h(t) \tag{3.2}$$

with $L : C_{rn} \to \mathbb{R}^n$ linear continuous and $h : [0,\infty) \to \mathbb{R}^n$ continuous. A variation-of-constants formula analogous to the familiar one for ODEs would require an application of solution operators T_{t-s}, $t \geq s$, to values of h—which does not make sense. Even segments h_t exist only for $t \geq r$. There are several ways to circumvent these obstacles and find a useful variation-of-constants formula. A rather elementary possibility is to extend the solution operators T_t to the vectorspace of bounded Borel maps $[-r,0] \to \mathbb{R}^n$. In this approach bounded pointwise convergence everywhere for maps on $[-r,0]$ plays a role, and the matrix-valued *fundamental solution* $X : [-r,\infty) \to \mathbb{R}^{n\times n}$ of the linear equation with $X(t) = 0$ on $[-r,0)$ and $X(0) = (\delta_{jk})$ is needed [137].

More general is the functional analytic framework of *sun-star-calculus* used in [22] which, together with the notion of weak-star integrals, yields the desired formula. Here the state space $X = C_{rn}$ is enlarged by a second dual as follows: For the semigroup $(T_t)_{t\geq 0}$ consider the Banach space $X^{\odot} \subset X^*$ of strong convergence, that is, the subspace of the dual X^* on which the adjoint operators T_t^* define a strongly continuous semigroup $(T_t^{\odot})_{t\ge 0}$. The space $X^{\odot}$ is pronounced X-sun, and depends on the original semigroup, of course. X is embedded into $X^{\odot *}$, and $T_t^{\odot *}$ extends T_t (modulo the embedding). The variation-of-constants formula which one finds for the segments of the solutions of Eq. (3.2) is an equation in $X^{\odot *}$ and contains a weak-star-integral of the form

$$\int_0^t T_{t-s}^{\odot *} \sum_{j=1}^{n} h_j(s) r_j^{\odot *} ds$$

where $r_j^{\odot *} = I(0, e_j)$ with an isomorphism $I : L^{\infty}([-r, 0], \mathbb{R}^n) \times \mathbb{R}^n \to X^{\odot *}$. The previous integral is in X (modulo embedding). Working with this variation-of-constants formula presents no further difficulties. Local invariant manifolds of the semiflow F at a stationary point x_0, with $x : \mathbb{R} \to \mathbb{R}^n$ a constant solution of Eq. (1.4), f continuously differentiable, $L = Df(x_0)$, can be constructed following the Lyapunov-Perron method, with the caveat that the norm on $X = C_{rn}$ is not differentiable. —For local stable and unstable manifolds at a stationary point of a semiflow like F, with continuously differentiable solution operators, one can avoid the variation-of-constants formula above, of course, since these manifolds coincide with their analogues for an arbitrary solution operator with $t > 0$, and the construction of the latter only requires the discrete variation-of-constants formula for maps, which is elementary. However, for local center manifolds and their generalizations (center-stable manifolds, etc.) the situation is different as center manifolds for solution operators are in general not positively invariant under the full semiflow. See [64] for a planar vectorfield showing this phenomenon.

4 Nonlinear Autonomous Equations: From Periodic Solutions to Motion on Global Attractors

We mentioned that in the early 1960ies the search for periodic solutions of Wright's equation and other simple-looking nonlinear autonomous equations of the form (1.6) and (1.7) began. The first result, on existence of a *slowly oscillating* periodic solution of Wright's equation for $\alpha > \frac{\pi}{2}$, is due to Jones [55]. Slowly oscillating means that zeros are spaced at distances larger than the delay; such solutions will play a major role in the sequel. Beginning with Jones' work the strategy to find periodic solutions was to look for a set of initial data to which flowlines return after an excursion into the ambient space, and to study the map R of first return, whose fixed points yield periodic solutions. The negative feedback condition (NF) for Eq. (1.6) in combination with the local hypothesis $g'(0) > 1$ makes it easy to find the desired map on the cone K of increasing functions $\phi \in C$ with $\phi(-1) = 0 < \phi(0)$ as these initial data define solutions $x : [-1, \infty) \to \mathbb{R}$ whose positive zeros z_j, $j \in \mathbb{N}$, satisfy $z_{j+1} > z_j + 1$, with the segment at $t = z_j + 1$ decreasing for j odd and increasing for j even. The segment at $t = z_2 + 1$ is increasing and back in the cone, see Fig. 1.

If in addition g is bounded from below or from above then the return map is continuous with relatively compact image. It may have no fixed point, though, which

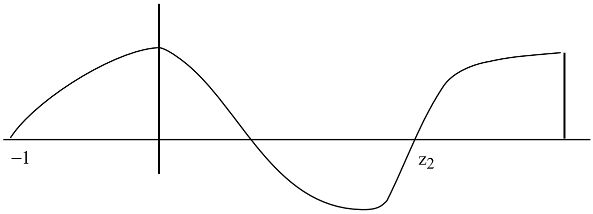


Fig. 1 A solution of Eq. (1.6) in case of negative feedback

happens to be true if for example g is smooth and satisfies $g(0) = 0$, $1 < g'(0)$, and $0 \le g'(\xi) \le \frac{3}{2}$ everywhere. The return map has a continuous extension to the closed convex cone $\overline{K} = K \cup \{0\}$ by the fixed point $0 \in C$—this trivial fixed point at the vertex of the cone corresponds to the constant solution $t \mapsto 0$. So another fixed point is wanted, which might exist due to instability of the linearized DDE, with characteristic roots in the open right halfplane, or equivalently, in case $g'(0) > \frac{\pi}{2}$. The technique of Wright [163] then yields unstable behaviour of the solutions to Eq. (1.6) which start from the cone K. For the return map this kind of unstable behaviour translates into the property of *ejectivity* of the fixed point 0, which requires a neighbourhood N of 0 so that for each $p \in N \setminus \{0\}$ there is an iterate outside N, $R^j(p) \notin N$, $j = j(p)$. The notion of ejectivity is due to Browder [10] who showed that ejectivity yields a second fixed point. Nussbaum proved that ejective fixed points in infinite-dimensional spaces have index 0 and used ejectivity in his remarkable results on existence of slowly oscillating periodic solutions [106], and on global bifurcation of slowly oscillating periodic solutions for families of DDEs

$$x'(t) = -\alpha g\big(x(t-1)\big) \tag{4.1}$$

with $g'(0) = 1$ and a parameter $\alpha > 0$ [108]. The global bifurcation result says that there is a continuum of pairs $(\phi, \alpha) \in C \times (0, \infty)$, $\phi = x_0$ the initial segment of a slowly oscillating periodic solution x of Eq. (4.1), which emanates from $(0, \frac{\pi}{2})$ and which contains points (ϕ, α) for every $\alpha > \frac{\pi}{2}$. The result is purely topological, according to the fact that the return map is not differentiable at the ejective fixed point.

Related early results on existence of slowly oscillating periodic solutions for Wright's equation and similar ones, all using return maps as above, are due to Grafton [41], Nussbaum [107], Chow [15], Hadeler and Tomiuk [42], and Alt [1]. Dunkel [36] and Pesin [116] constructed closed convex domains away from the trivial fixed point and positively invariant under the return map so that Schauder's theorem yields a periodic solution, under hypotheses stronger than mere instability of the linearization of the DDE at the equilibrium.

Ejectivity is a rather weak instability property which does not fully reflect the behaviour of iterates of the return map close to the ejective fixed point (so existence of non-ejective fixed points is a strong result). A closer look at small slowly oscillating solutions of Eq. (4.1) made it possible to find invariant sets for the return map bounded away from the fixed point 0 and to obtain the desired fixed points by means of Schauder's theorem if only the zero solution is linearly unstable, and the fact that in this case the index of the trivial fixed point (for the special map considered) is zero became immediate from additivity [139, 140].

More recently return maps for nonlinearities g in Eq. (1.6) which satisfy (NF) and are close to constants on long intervals were shown to be contractions on closed domains away from $0 \in C$, resulting in exponentially stable periodic orbits [146, 147]. The technique employed here extends to equations of the form (1.7) and to systems of DDEs, among others [83, 148, 165]. The underlying observation is that if g equals constants on long intervals then certain flowlines enter a positively invariant finite-dimensional set on which the semiflow can be computed.

Since the 1970ies also other non-local methods were developed in order to find slowly oscillating periodic solutions. For g in Eq. (1.6) odd Kaplan and Yorke [56] observed that a solution x with the symmetry $x(t) = -x(t+2)$ (and thereby, period 4) and $y = x(\cdot - 1)$ satisfy the planar Hamiltonian system

$$x' = -g(y),$$
$$y' = g(x),$$

and that conversely suitable periodic solutions of this Hamiltonian system yield symmetric periodic solutions of the DDE. For certain polynomials g Dormayer later even computed these symmetric periodic solutions [23]. For one-parameter families of Eq. (4.1), with g odd, positive on $(0, \infty)$, bounded, continuous, and with $g'(0) = 1$, the method from [56] yields an unbounded curve of (initial values for) slowly oscillating periodic solutions, all with period 4, which bifurcates at $\alpha = \frac{\pi}{2}$ from the zero solution. Let us call this curve the *KY-curve*. Its trace belongs to the continuum from Nussbaum's global bifurcation result, of course, and the projection onto the second factor covers the ray $(\frac{\pi}{2}, \infty)$.

For equations more general than Eq. (1.6) with g odd, and also of the form (1.7), with certain monotonicity properties, Kaplan and Yorke proved existence of slowly oscillating periodic solutions using planar evaluations

$$t \mapsto \big(x(t), x(t-1)\big) \quad \text{and} \quad t \mapsto \big(x(t), x'(t)\big)$$

along flowlines $t \mapsto x_t$ and Poincaré-Bendixson type arguments in the plane [57, 58]. Notice that in general the plane is *not* foliated by such evaluation curves along *all* flowlines of solutions to the DDE. In retrospective it may be said that the fact underlying the method of Kaplan and Yorke is that on certain manifolds of segments of slowly oscillating solutions the evaluation map $\phi \mapsto (\phi(0), \phi(-1))$ is a manifold chart.

R.A. Smith developed another approach to Poincaré-Bendixson theorems for flowlines in ω-limit sets of solutions to RFDEs (1.4) [127, 128], which yield periodic solutions of certain systems of DDEs and of single higher order differential equations with time lags.

For Eq. (1.6) with g odd, positive on $\mathbb{R}^+$, *monotone*, and satisfying a concavity condition Nussbaum [112] obtained the first result on uniqueness of slowly oscillating periodic solutions (up to translation, of course), using the technique developed in [57, 58]. For other odd g, positive and with a sharp hump on $\mathbb{R}^+$, Nussbaum found a second slowly oscillating periodic solution, with period much larger than 4 [112]. These results gave a first impression of how the dynamics generated by Eq. (1.6) depend in a subtle way on the function g. For the one-parameter-family of Eqs. (4.1) the

KY-curve may bifurcate to the left at $\alpha = \frac{\pi}{2}$, depending on the shape of g close to the origin [138]. This yields another multiplicity result, for symmetric slowly oscillating periodic solutions.

In [12, 13] Chapin and Nussbaum precisely described the form of long period solutions of Eq. (4.1) with g positive and hump-shaped on $\mathbb{R}^+$, for α large. Using similar techniques Xie [166–169] proved results on uniqueness and stability of periodic orbits for equations of the form (4.1). In particular he showed that for $\alpha > 5.67$ slowly oscillating periodic solutions of Wright's equation (1.5) are unique (up to translation), with the orbits in C stable and attracting.

For certain odd functions g which are positive and hump-shaped on $\mathbb{R}^+$ there exist secondary bifurcations of initial values of other slowly periodic solutions from the KY-curve [142]. The proof relies on a study of the Floquet multipliers of the periodic solutions x from the KY-curve, which are—as for ODEs—the eigenvalues of the linearized period map $D_2F(4, x_0)$. These Floquet multipliers satisfy a characteristic equation [142], which is related to a nonlocal boundary value problem for a system of linear periodic ODEs.

In [18] we saw that for monotone g as in Nussbaum's uniqueness result the Floquet multiplier $\lambda = 1$ of the slowly oscillating periodic solution is simple and all others belong to the open unit disk, which yields local exponential attraction towards the periodic orbit. Dormayer [24] extended the result from [138] on the direction of bifurcation of the KY-curve from $(0, \frac{\pi}{2})$ and used the characteristic equation for the Floquet multipliers in order to determine the stability of the bifurcating periodic orbits.

For different classes of odd functions g which are positive and hump-shaped on $\mathbb{R}^+$ he proved that there are different types of secondary bifurcations of periodic orbits from the KY-curve, and sequences of such bifurcations along the KY-curve, which are related to oscillations about $\lambda = 1$ of the leading Floquet multiplier along the KY-curve [25–27].

Later Floquet multipliers for periodic solutions of Eq. (1.6) which are not necessarily slowly oscillating but have periods and the delay rationally dependent were further investigated by means of characteristic equations in [28, 125, 126, 170, 171].

In the introduction we mentioned roots of the characteristic equation (1.2) crossing the imaginary axis at critical parameters $\alpha = \alpha_k$, $k \in \mathbb{N}_0$. For the nonlinear Eqs. (4.1) with g sufficiently smooth, $g(0) = 0$ and $g'(0) = 1$, the local Hopf bifurcation theorem for RFDEs (see e.g. [22, 45]) yields that at these critical parameters periodic solutions of Eq. (4.1) bifurcate off from the zero solution. For $k = 0$ their segments in the cone K belong to pairs in Nussbaum's global continuum (provided g satisfies the global conditions in [108]), and to the KY-curve if in addition g is odd. But for $k > 0$ the bifurcating periodic solutions are not slowly oscillating and have minimal periods less than 1. Let us call these periodic solutions *rapidly oscillating*. The different oscillation frequencies are related to the different roles of these periodic orbits in the global dynamics generated by the DDE. As already Myshkis [104] pointed out, solutions of DDEs like Eq. (1.1) have in common that their oscillation frequency when properly defined does not increase with the argument t. A closely related conjecture is that for Eq. (1.6) with a continuous function g satisfying condition (NF) all orbits of rapidly oscillating periodic solutions (except of the zero solution) are unstable. The main result in [101] implies that the conjecture is true for the slightly more general equation

$$x'(t) = -\mu x(t) - g\big(x(t-1)\big) \tag{4.2}$$

with $\mu \geq 0$, provided that in addition g is continuously differentiable and *monotone*, with $g'(\xi) > 0$ everywhere.

It came as a great surprise when Ivanov and Losson found an example of a rapidly oscillating periodic solution of Eq. (4.2) whose orbit is stable and attracting [53]. The function g in their example is a step function representing negative feedback and is, of course, not monotone. The proof relies on computer assistance. Later Schulze-Halberg obtained a similar computer-assisted result for the positive feedback case [122] with g smooth (and constant on long intervals). The step to proofs without computer aid, for both cases, was accomplished by Stoffer [131, 132]. All of these results require $\mu > 0$, and it remains an open problem whether the conjecture above, formulated for Eq. (1.6) only, is true or not.

Myshkis' observation that along solutions of certain linear scalar RFDEs the oscillation frequency tends to decrease became a powerful tool in the form of the *lap number*, a discrete Lyapunov functional which was introduced and used by Mallet-Paret [85] in his fundamental result on a *Morse-Smale decomposition* of the *global attractor* for the semiflow generated by equations of the form (1.7) which involve global delayed negative or positive feedback. By definition a global attractor A of a semiflow S on a complete metric space M is a compact invariant set which attracts all bounded sets in the sense that given a neighbourhood U of A and a bounded set $B \subset M$ there exists $t_U \geq 0$ such that $S(t, B) \subset U$ for all $t \geq t_U$. The definition implies uniqueness. For semiflows generated by RFDEs existence of the global attractor is often easily obtained, under assumptions on boundedness, and it consists of all segments of all bounded *entire* solutions $\mathbb{R} \to \mathbb{R}^n$. A Morse-Smale decomposition of a flow on a compact metric space is a finite ordered collection of disjoint compact invariant sets so that the remaining flowlines are top-down connections between these invariant sets. In Mallet-Paret's result the metric space M consists of the bounded entire solutions of Eq. (1.7), with the compact-open topology, the flow is given by translations, and the lap number $V : M \setminus \{0\} \to \mathbb{N}_0 \cup \{\infty\}$ counts sign changes on $(z-1, z]$ in case the solution has a first zero $z \geq 0$. In the same way one can define V on $C \setminus \{0\}$, of course. The point is that under global feedback conditions on h in Eq. (1.7) the lap number is non-increasing. If the stationary state $0 \in C$ is hyperbolic (no characteristic roots on the imaginary axis) then the sets of the Morse-Smale decomposition different from the singleton $\{0\}$ are subsets of level sets $V^{-1}(n)$ formed by the solutions with segments bounded away from $0 \in C$. More on connecting orbits is due to Fiedler and Mallet-Paret [39] and to McCord and Mischaikow [102], who employed the Conley index.

Other applications of the lap number concept concern a class of *systems* of $N + 1$ differential equations with nearest neighbour interaction and delays, namely, *monotone cyclic feedback systems with delays*, which in case $N > 0$ can be reduced to the form

$$\dot{x}^0(t) = g^0\big(t, x^0(t), x^1(t)\big),$$

$$\dot{x}^i(t) = g^i\big(t, x^{i-1}(t), x^i(t), x^{i+1}(t)\big), \quad 1 \le i \le N-1,$$
$$\dot{x}^N(t) = g^N\big(t, x^{N-1}(t), x^N(t), x^0(t-1)\big)$$

with a single delay and functions $g^0 : \mathbb{R}^3 \to \mathbb{R}$ and $g^i : \mathbb{R}^4 \to \mathbb{R}$, $1 \le i \le N$, satisfying the sign conditions

$$\begin{aligned}
g^0(t,0,\cdot) &\ge 0 \quad \text{on } \mathbb{R}_0^+ \quad \text{and} \quad g^0(t,0,\cdot) \le 0 \quad \text{on } \mathbb{R}_0^-,\\
g^i(t,\cdot,0,\cdot) &\ge 0 \quad \text{on } \mathbb{R}_0^+ \times \mathbb{R}_0^+ \quad \text{and}\\
g^i(t,\cdot,0,\cdot) &\le 0 \quad \text{on } \mathbb{R}_0^- \times \mathbb{R}_0^- \quad \text{for } 1 \le i \le N-1,\\
g^N(t,\cdot,0,\cdot) &\ge 0 \quad \text{on } \mathbb{R}_0^+ \times \delta^*\mathbb{R}_0^+ \quad \text{and} \quad g^N(t,\cdot,0,\cdot) \le 0 \quad \text{on } \mathbb{R}_0^- \times \delta^*\mathbb{R}_0^-
\end{aligned}$$

with $\delta^* \in \{\pm 1\}$. In the linear periodic case Mallet-Paret and Sell [99] extended the lap number to a functional on the states of such systems and used it for a proof that for each Floquet multiplier, that is, for each eigenvalue of the period map, the eigenspace is at most two-dimensional. The oscillation frequency of solutions with segments in the eigenspace decreases with increasing moduli of the Floquet multipliers. In the case $N = 0$ these results yield information about the Floquet multipliers of periodic solutions x to Eq. (1.7), which are determined by the variational equation

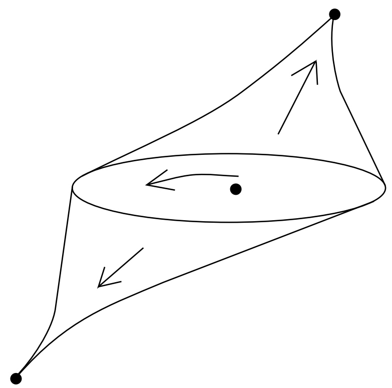
$$v'(t) = -\partial_1 h\big(x(t), x(t-1)\big)v(t) - \partial_2 h\big(x(t), x(t-1)\big)v(t-1).$$

For *nonlinear autonomous* monotone cyclic feedback systems with delays Mallet-Paret and Sell [100] used motion in the level sets of the discrete Lyapunov functional for a proof of a Poincaré-Bendixson result, namely, that ω-limit sets are either periodic orbits, or contain equilibria and heteroclinics. Moreover, on the ω-limit sets an evaluation map into the plane is homeomorphic. For periodic solutions which are rapidly oscillating (in terms of the Lyapunov functional, here) they also obtained results on instability.

Most of the results addressed in this section may be viewed as contributions to a description of motion in global attractors. In case of the single Eq. (4.2) with $\mu \ge 0$, g bounded from below or from above, and continuous with the negative feedback condition (NF) satisfied, we have detailed information about the long term dynamics as follows: The initial data $\phi \ne 0$ with at most one sign change form a set $S \subset C$ which is positively invariant under the semiflow F and contains all segments of slowly oscillating solutions, and there is a global attractor A_S for the semiflow restricted to the closure $\overline{S} = S \cup \{0\}$ of S. In [144, 159] we showed (without the lap number machinery) that if in addition g is continuously differentiable with $g'(\xi) > 0$ everywhere then A_S either is the singleton $\{0\}$, or a continuously differentiable graph over a compact subset of a plane in the space C which is diffeomorphic to the closed unit disk, with the manifold boundary of A_S the orbit of a slowly oscillating periodic solution. Flowlines in $A_S \setminus \{0\}$ wind around the origin and are periodic or heteroclinic connections. —The main result of [101], mentioned above as implying instability of rapidly oscillating periodic solutions, actually complements the result about A_S in saying that all those flowlines which do not approach A_S as $t \to \infty$ belong to a graph over a subset in a closed subspace of codimension 2 in C.

Without monotonicity of g motion in A_S is in general not planar and may be chaotic. The first results in this direction exploited that for g constant on long

Fig. 2 The spindle $\overline{W_u}$



intervals—as we said above—the semiflow F can be explicitly computed on certain invariant sets, and return maps are semiconjugate to chaotic interval maps, see Peters [117] for the case of step functions g and [3, 141] for smooth g which are constant on long intervals. In [77, 78] we proved existence of chaotic motion in A_S for Eq. (4.2) with $\mu = 0$ (that is, Eq. (1.6)) and smooth functions g which are not close step functions, satisfy (NF), and have several local extrema. An unstable periodic orbit with a flowline which is homoclinic to it yields a Poincaré return map with a *transversal homoclinic point*. Because of compactness the return map has no continuous inverse, which means that conjugacy to a symbol shift—the prototype of a chaotic homeomorphism—is possible only for the index shift induced by the return map on its entire trajectories close to the homoclinic loop. The proof of this conjugacy employs a generalization of the notion of a *hyperbolic set* and of the *Shadowing Lemma* from the case of diffeomorphisms in finite dimension to the case of C^1-maps in Banach spaces [129, 130], which had been instrumental also in another result on chaotic motion generated by a DDE [143]. More on hyperbolic sets for C^1-maps and on chaotic motion for DDEs is due to Lani-Wayda [73, 75, 76]. In [74] he obtained a result for Eq. (1.6) with a function g which has only one extremum, like the nonlinearity in the Mackey-Glass-Lasota equation—for the latter existence of chaotic motion is still an open problem.

We return to non-chaotic behaviour in nonlocal invariant sets. Consider Eq. (4.2), with $\mu > 0$, $g(0) = 0$ and $g'(\xi) < 0$ everywhere, modelling the interaction of instantaneous negative feedback with delayed monotone positive feedback. Suppose there are zeros $\xi_- < 0 < \xi_+$ of $g + \mu\,\mathrm{id}$, and $g(\xi) + \mu\xi \neq 0$ in $(\xi_-, 0) \cup (0, \xi_+)$. Then the closure $\overline{W_u}$ of the leading 3-dimensional unstable manifold at the origin can be written as a graph in C [2]. The monograph [69] describes the fine structure of $\overline{W_u}$: It has the form of a smooth solid spindle with the nonzero stationary points given by ξ_-, ξ_+ as tips, one on either side of a smooth invariant disk. The disk is bordered by an unstable periodic orbit, flowlines in the disk wind around the origin and connect it to the periodic orbit. All other flowlines are heteroclinic from the origin or from the periodic orbit to one of the tips, which are singularities [145], see Fig. 2.

While in general there is no global attractor for the equations studied in [69]—in case $\mu = 0$ the unstable manifold of $0 \in C$ contains flowlines of positive unbounded entire solutions—there exist μ and g, however, for which the set $\overline{W_u}$ is in fact the global attractor [68]. Cases with more periodic orbits in the global attractor were studied by Krisztin [62]. He and Vas also found μ and g as in [69] for which there is a periodic solution of large amplitude with orbit outside the closure of the unstable manifold at $0 \in C$, and not encountered before [67]. —Related work on manifolds of heteroclinic connections between periodic orbits, for systems of DDEs, is due Chen, Krisztin and Wu [14].

Let us end this section on Eq. (1.7) with results on analyticity, periods, singular perturbation, and periodic orbits related to a Takens-Bogdanov scenario. First, recall from Sect. 3 that there is an entire solution to the linear Eq. (1.1) which is not analytic—this solution is unbounded. In fact, Nussbaum [105] proved a result which implies that bounded entire solutions of Eq. (1.7) with h analytic are analytic.

In [109] he determined the range of periods along the continuum found in [108]. For the special case of Wright's equation this is the interval $(4, \infty)$. In [113] Nussbaum showed that Wright's equation has no slowly oscillating periodic solution of period 4 whatsoever.

In [89] Mallet-Paret and Nussbaum studied Eq. (4.2) in the form

$$\epsilon x'(t) = -x(t) + f\big(x(t-1)\big)$$

and proved results about convergence of slowly oscillating periodic solutions $x = x_\epsilon$ for $\epsilon \searrow 0$ to square waves (step functions) which correspond to periodic points of the function f. Related results were obtained by Ivanov and Sharkovsky [54].

The proofs of existence of slowly oscillating periodic solutions which exploit instability of the zero solution and which are mentioned above require a complex conjugate pair of characteristic roots of the linearized DDE with positive real part. The *exchange rate equation*

$$x'(t) = a\big[\big(x(t) - x(t-1)\big) + \big|x(t)\big|x(t)\big] \tag{4.3}$$

with $a > 0$ is an example where this fails: $z = 0$ is a zero of the characteristic function $z \mapsto z - a + a\,e^{-z}$ of the linearized DDE for all $a > 0$, double at $a = 1$ and simple for $a \neq 1$. At $a = 1$ a real zero crosses from the left into the right open halfplane, and all other zeros $z \neq 0$ have negative real parts. In particular, there is no Hopf bifurcation. In [11] we found periodic solutions of Eq. (4.3) with $a > 1$ whose orbits border the 2-dimensional center-unstable manifold at zero. In [152] it is shown that at $a = 1$ periodic orbits arise in a bifurcation from zero, with amplitudes small but periods close to infinity. This can be understood as part of a Takens-Bogdanov scenario, for an equation with two parameters and with the double characteristic root $0 \in \mathbb{C}$ of the linearized equation at a critical point in the parameter plane. The proof in [152] uses the evaluation $C \ni \phi \mapsto (\phi(0), \phi(-1)) \in \mathbb{R}^2$. Notice that the function h corresponding to Eq. (4.3), without a second order derivative at the origin, is not smooth enough for the application of familiar techniques in local bifurcation theory which employ center manifolds, truncation, and normal forms of ODEs.

Very recently Mallet-Paret and Nussbaum [97] developed a theory of tensor and exterior products of linear functional differential equations. This theory, in combination with the lap number, yields extensions of the results from [99] about Floquet multipliers, for periodic equations

$$x'(t) = \alpha(t)x(t) + \beta(t)x(t-1).$$

5 State-Dependent Delays

Delay in real world systems is often variable, depending on external or internal influence, or on a combination of both. Cases of internal influence are modelled by differential equations with state-dependent delay (DESDD). A toy example is

$$x'(t) = -\alpha x\big(t - d(x_t)\big) \tag{5.1}$$

with a non-constant *delay functional* $d : C \to [0, 1]$, and $0 \neq \alpha \in \mathbb{R}$. Equation (5.1) is nonlinear, and has the general form (1.4) of an RFDE with $f_C : C \to \mathbb{R}$ (in place of f) given by

$$f_C(\phi) = -\alpha\phi\big(-d(\phi)\big) = -\alpha ev \circ \big(id \times (-d)\big)(\phi),$$

which in general is not locally Lipschitz continuous as the evaluation map

$$ev : C \times [-1, 0] \ni (\phi, s) \mapsto \phi(s) \in \mathbb{R}$$

is not locally Lipschitz continuous. So the familiar existence and uniqueness theory for IVPs with RFDEs, as it is developed in monographs up to [22, 45], does not apply. Winston [162] gave an example where indeed continuous initial data do not uniquely determine solutions of a DESDD. A first possibility to avoid the obstacle is restriction to Lipschitz continuous initial data. This was—and still is—used in work on DESDDs which does not require more smoothness than continuous dependence of solutions on initial data. Linearization, however, remained a mystery in this framework [19], despite the observation that freezing the delay at a constant solution often yields a functional on C which can be differentiated and leads to a linear RFDE with constant delay. In case of the zero solution of Eq. (5.1) this heuristic approach results in the linear equation

$$x'(t) = -\alpha x\big(t - d(0)\big).$$

The situation becomes better—but also a bit unfamiliar—if we restrict interest to continuously differentiable initial data. For $r > 0$ and $n \in \mathbb{N}$ let $C^1_{rn} = C^1([-r, 0], \mathbb{R}^n)$ denote the Banach space of continuously differentiable maps $[-r, 0] \to \mathbb{R}^n$, with the norm given by $|\phi|_1 = |\phi| + |\phi'|$, and set $C^1 = C^1_{rn}$. The restricted evaluation map $ev_1 = ev|C^1 \times [-1, 0]$ is continuously differentiable with

$$Dev_1(\phi, s)(\chi, t) = \chi(s) + t\phi'(s). \tag{5.2}$$

It follows that, for the delay functional in Eq. (5.1) continuously differentiable, the restriction $f_1 = f_C|C^1$ of the functional from the example above is continuously differentiable. This suggests to consider Eq. (1.4) for an arbitrary continuously differentiable functional $f : U \to \mathbb{R}^n$ with U an open subset of the space C^1_{rn}. A solution

$x : [-r, t_e) \to \mathbb{R}^n$ of the associated IVP

$$x'(t) = f(x_t) \quad \text{for } t > 0, \qquad x_0 = \phi \in U,$$

would have all its segments x_t, $0 \leq t < t_e$, in the domain $U \subset C^1_{rn}$, hence be continuously differentiable, which in turn yields that the flowline $[0, t_e) \ni t \mapsto x_t \in C^1_{rn}$ is continuous. Therefore we may pass to the limit $t \searrow 0$ in Eq. (1.4) and obtain that $x_0 = \phi$ satisfies

$$\phi'(0) = f(\phi).$$

So the desired solutions require initial data in the closed set

$$X = \left\{\phi \in C^1 : \phi'(0) = f(\phi)\right\},$$

which is analogous to the domain of the attractor of the semiflow of Eq. (1.1) on the space C.

Notice that Eq. (5.2) implies for the example $f = f_1 = f_C|C^1$, with d in Eq. (5.1) continuously differentiable, that $Df_1(\phi)\chi$ does not depend on χ'. Therefore each derivative $Df_1(\phi)$, $\phi \in C^1$, extends to a linear map $D_e f_1(\phi) : C \to \mathbb{R}$. Moreover, the map $C^1 \times C \ni (\phi, \chi) \mapsto D_e f_1(\phi)\chi \in \mathbb{R}$ is continuous. In [149, 150] we showed that for general continuously differentiable functionals $f : C^1_{rn} \supset U \to \mathbb{R}^n$ with linear extensions $D_e f(\phi) : C_{rn} \to \mathbb{R}^n$, $\phi \in U$, so that the map

$$U \times C_{rn} \ni (\phi, \chi) \mapsto D_e f(\phi)\chi \in \mathbb{R}^n$$

is continuous, the set $X = \{\phi \in C^1_{rn} : \phi'(0) = f(\phi)\}$ is a continuously differentiable submanifold of codimension n in C^1_{rn}. Moreover, each initial value $\phi \in X$ defines a *maximal solution* $x : [-1, t_\phi) \to \mathbb{R}^n$, $0 < t_\phi \leq \infty$, which is continuously differentiable, satisfies Eq. (1.4) for $0 \leq t < t_\phi$, and $x_0 = \phi$, and any other continuously differentiable solution of the same IVP is a restriction of $x = x^\phi$. The relations

$$F(t, \phi) = x_t^\phi \quad \text{and} \quad \Omega = \left\{(t, \phi) \in [0, \infty) \times X : t < t_\phi\right\}$$

define a continuous semiflow on X, with all nonempty solution operators $F(t, \cdot)$ continuously differentiable, and the restriction of F to the subset of Ω given by $t > r$ is continuously differentiable.

The hypothesis on extensions is similar to the earlier concept of being *almost Frechet differentiable* introduced for functionals on C_{rn} by Mallet-Paret, Nussbaum and Paraskevopoulos [98]. For functionals involving state-dependent delays it is a rather mild condition whereas the slightly stronger property that the map $U \ni \phi \mapsto D_e f(\phi) \in L_c(C_{rn}, \mathbb{R}^n)$ be continuous in general does not hold.

At a constant solution, $t \mapsto 0$ for simplicity, the linearization of the semiflow F is the strongly continuous semigroup of the operators

$$D_2 F(t, 0) : T_0 X \to T_0 X, \quad t \geq 0,$$

which are given by $D_2 F(t, 0)\chi = v_t^\chi$ with the continuously differentiable solution $v = v^\chi$ of the IVP

$$v'(t) = Df(0)v_t, \qquad v_0 = \chi \in T_0 X = \left\{\chi \in C^1 : \chi'(0) = Df(0)\chi\right\}.$$

Moreover, the linearization is the restriction of the semigroup defined by the IVP of Eq. (1.3) with $L = D_e f(0)$, on the space C_{rn}, and the spectra of the generators of both semigroups coincide.

The former heuristic approach *freeze the delay at equilibrium, then linearize* is obviously restricted to examples where f is given by a term involving a delay functional. For such examples it fits into the framework presented here as it gives the correct formula for $Df(0)$ and $D_e f(0)$. In other words, the extended derivative $D_e f(0)$ generalizes what can be achieved by the heuristic approach to equations without anything visible to be frozen.

The continuously differentiable solution operators $F(t, \cdot)$ have continuously differentiable local stable and unstable manifolds at fixed points, which also serve as local stable and unstable manifolds at stationary points for the full semiflow [48]. The first result in this direction, on local unstable manifolds for a special class of DESDDs and without the theory sketched above, is due to Krishnan [61]. Optimal smoothness for local unstable manifolds, with higher order differentiability under appropriate hypotheses on f in Eq. (1.4), was obtained by Krisztin [63] who also established continuously differentiable local center manifolds [48, 65]. For center-stable manifolds, see [120]. Center-unstable manifolds were studied by Stumpf [134]. Let us mention already here that for infinite-dimensional invariant manifolds, like local stable and center-stable manifolds, higher order differentiability is an open problem. —In the setting of Lipschitz continuous data and solution segments Arino and Sanchez [5] proved the saddle point property, with Lipschitz continuous local stable and unstable manifolds and for a special class of DESDDs.

Local Hopf bifurcation theorems for DESDDs are due to Eichmann [37] and Sieber [124]. For special classes of DESDDs Hu and Wu [49, 50] obtained results on global Hopf bifurcation, by means of equivariant degree theory.

For another approach to smooth dependence on initial data, for a class of DESDDs and with Sobolev spaces, see Hartung [47].

We turn to case studies. Beginning in the 1960ies Driver investigated one-dimensional simplifications of the Wheeler-Feynman system for the two-body-problem of electrodynamics [30, 31, 33]. Bauer [7] proved an existence result for the original Wheeler-Feynman system. —Periodic solutions of DESDDs became a topic, as in case of constant time lags, with first contributions by Nussbaum [106] and Kuang and Smith [70, 71]. In [98] Mallet-Paret, Nussbaum and Parakevopoulos obtained a rather general result on existence of slowly oscillating periodic solutions using index computations and continuation. For variable delay the notion of being slowly oscillating must be adapted, of course. Further theorems on existence of slowly oscillating periodic solutions to DESDDs are due to Arino, Hadeler, and Hbid [4], and Magal and Arino [84]. All of them employ ejectivity of a trivial fixed point and circumvent linearization by means of the heuristic approach addressed above. In the existence proof from [154] ejectivity of a trivial fixed point is established by means of the solution manifold and (true) linearization of solution operators.

In a series of papers Mallet-Paret and Nussbaum investigated one-parameter families of equations of the form

$$\epsilon x'(t) = h\big(x(t), x\big(t - r\big(x(t)\big)\big)\big), \tag{5.3}$$

and determined the asymptotic shape of families of slowly oscillating periodic solutions in the singular limit $\epsilon \searrow 0$ [90, 91, 94]. This limiting shape is a compact subset in the plane, not necessarily a graph, and depends on h and on the delay function r. The methods used involve a new, topological theory of nonlinear eigenvalue problems for *max-plus operators* [92, 93]. Among the results are theorems on uniqueness and so-called *superstability* of periodic orbits for $\epsilon > 0$ sufficiently small [96].

Krisztin and Arino [66] obtained results on planar dynamics and periodic orbits, similar to those in [144], for an equation of the form

$$x'(t) = -\mu x(t) - f\big(x\big(t - r\big(x(t)\big)\big)\big).$$

In [151] we used smooth solution operators on the solution manifold and established an exponentially attracting periodic orbit for a system which describes position control by echo. Stumpf [133] proved a version of the result from [11] on a periodic orbit bordering a center-unstable manifold for a modification of the exchange rate equation with state-dependent delay. Kennedy [60] obtained multiple periodic orbits using direct computation of solutions for special equations which involve step functions, and continuation of the fixed point index.

In [153] we studied a further model for position control by echo and used local invariant manifolds in the search of initial data which yield solutions corresponding to soft landing.

We continue with a modelling issue: What do we actually have in mind when we speak of feedback acting with a state-dependent delay? Isn't that the following, in the simplest case: At present, the system is in a state $x(t) \in U \subset \mathbb{R}^n$. It changes this present state by means of a derivative given by a vectorfield $f : U \to \mathbb{R}^n$, but does so only at a later time $s > t$, where the time lag $s - t = \lambda \in [0, r]$ depends on the present state, $\lambda = \Lambda(x(t))$. This yields the differential equation

$$x'(s) = f\big(x(t)\big)$$

with transformed argument $s = t + \lambda$. Let us try to convert this into a RFDE: We have

$$x(t) = x\big(s + (t - s)\big) = x_s(t - s) = x_s(-\lambda)$$

and

$$\lambda = \Lambda\big(x(t)\big) = \Lambda\big(x_s(-\lambda)\big).$$

Using the evaluation map $ev : (\phi, u) \mapsto \phi(u)$ we get

$$\begin{aligned} x'(s) &= f\big(x(t)\big) = f\big(x_s(-\lambda)\big) \\ &= f\big(ev(x_s, -\lambda)\big), \\ \lambda &= \Lambda\big(ev(x_s, -\lambda)\big), \end{aligned}$$

which is a DDE with a delay which is implicitly given by an algebraic equation involving the state and its history. More generally, we are led to consider *algebraic-delay systems* of the form

$$x'(s) = F(x_s, \lambda),$$

$$0 = \Lambda(x_s, \lambda),$$

with maps

$$F : U^{[-r,0]} \times [0,r] \to \mathbb{R}^n \quad \text{and} \quad \Lambda : U^{[-r,0]} \times [0,r] \to \mathbb{R}.$$

The previous system obviously differs from a scalar equation of the form

$$x'(t) = f\big(x\big(t - d\big(x(t)\big)\big)\big) \tag{5.4}$$

with real functions f and $d : \mathbb{R} \to [0,r]$, which seems to offer itself as the simplest DDE with state-dependent delay. An interpretation of Eq. (5.3) as system behaviour clarifies what is different: Eq. (5.3) says that at time t the system considered determines a moment of time in the past, namely, $t - d(x(t))$, and reacts instantaneously with the derivative $x'(t) = f(x(t - d(x(t))))$ to the state it was in at that past moment.

A result on algebraic-delay systems in [155] emphasizes a phenomenon which eludes constant delay, namely, the distinction of solutions which correspond to reactions respecting the succession of stimuli, from other solutions which correspond to reactions in reversed order. For a differential equation involving a delay function $d : \mathbb{R} \to [0,\infty)$ the first and more familiar type of a solution x is characterized by an increasing *delayed argument function* $t \mapsto t - d(x(t))$ while for the second type the delayed argument function decreases. An example in [155] has both types of flowlines separated by a hypersurface in the boundary of the solution manifold; for some initial data in this separatrix there are multiple solutions, with their flowlines entering different components of the solution manifold. It also happens that for some solutions the delayed argument function is not monotone, for example, in periodic solutions to Eq. (5.3) with $r(\xi) = 1 + \xi^2$ [95].

We mentioned Poisson's work [119] of 1806. A part of this paper deals with solutions to an autonomous differential equation which is motivated by a problem from plane geometry and contains both delayed and advanced arguments. In addition the equation is implicit and may be called neutral. Recently we looked at this equation from a present day perspective [156]. Despite the fact that hypotheses analogous to the extension property from above are violated there is a continuously differentiable solution manifold, and one can compute a family of solutions which define a nice semiflow of continuously differentiable solution operators on the manifold. However, this semiflow does not include all flowlines. Uniqueness for the associated IVP is shown only outside a thin *singular set* in the solution manifold, and there exist initial data in the singular set which produce multiple solutions, some with flowlines in the singular set and others with flowlines leaving it.

From the work of Mallet-Paret and Nussbaum [90] we know that making a constant time lag in a hyperbolic linear equation of the form (1.7) state-dependent may introduce periodic orbits and so change the dynamics considerably. In [157] we show that in Eq. (1.1), for $\alpha > 0$ such that the equation is hyperbolic with the unstable space two-dimensional, one can replace the constant time lag 1 by a delay functional which equals 1 close to $0 \in C$ so that the new Eq. (5.1) has a solution which is homoclinic to 0, with the intersection of stable and unstable manifolds along the homoclinic flowline in X minimal. Close to the homoclinic loop one finds entire flowlines with complicated histories [158]. More on this, related to complicated solution behaviour discovered by Shilnikov [123] for vectorfields on $\mathbb{R}^4$, is in progress.

6 More Problems

A particular problem which up to now resisted all advances concerns Wright's equation (1.5). In [163] Wright proved that for $0 < \alpha \le \frac{37}{24}$ all solutions $x : [-1, \infty) \to \mathbb{R}$ of Eq. (1.5) which satisfy $x(0) > -1$ decay to 0 at infinity, and indicated that the technique of his proof extends to certain larger intervals $(0, \alpha_*] \subset (0, \frac{\pi}{2})$. (Solutions with $x(0) = -1$ are constant for $t \ge 0$ while solutions with $x(0) < -1$ tend to $-\infty$ for $t \to \infty$.) The set of all $\alpha \in (0, \frac{\pi}{2})$ with the said global attraction property of the zero solution is in fact open [136]. At $\alpha = \frac{\pi}{2}$ there is a supercritical Hopf bifurcation of stable and attracting orbits of slowly oscillating periodic solutions, see [16], and there is the continuum $S \subset C \times (0, \infty)$ found by Nussbaum [108] which emanates from $(0, \frac{\pi}{2})$ and contains points (ϕ, α) for every $\alpha > \frac{\pi}{2}$. Numerical studies strongly suggest that slowly oscillating periodic solutions of Wright's equation are unique (up to translation) for each $\alpha > \frac{\pi}{2}$. We mentioned that Xie [166, 169] proved this for $\alpha > 5.67$. Regala [121] and Lessard [80] obtained regularity properties of the branch S which is in fact a curve and has no folds over an interval $(\frac{\pi}{2} + 0.00073165, 2.3)$. However, a proof of uniqueness of slowly oscillating periodic solutions for $\frac{\pi}{2} < \alpha \le 5.67$ is still missing, as well as the complementary result that for *all* $\alpha \in (0, \frac{\pi}{2})$ all solutions $x : [-1, \infty) \to \mathbb{R}$ with $x(0) > -1$ tend to 0 as $t \to \infty$.

Section 4 above was almost entirely about the case of the scalar differential equation (1.7) with a single delayed argument. For scalar differential equations with more time lags, or with distributed delay, comparatively little about solution behaviour has been established beyond numerical results. In [110] Nussbaum studied periodic solutions of equations of the form

$$x'(t) = \alpha f\big(x(t-r_1)\big) + \beta f\big(x(t-r_2)\big).$$

Also, global Hopf bifurcation theorems for RFDE's by Nussbaum [111] and by Chow and Mallet-Paret [17] have been applied to equations with two time lags. More case studies, showing typical behaviours generated by the interaction of two (or more) time lags, would be most desirable.

Similar remarks can be made about systems. The cyclic monotone feedback systems explored by Mallet-Paret and Sell, with several time lags originally, are reduced to systems with a single time lag in only one of their equations, and through the strong results achieved shine the structures generated by the prototypic scalar equations of the form (1.7). There is a need for further case studies which exhibit more of the different solution behaviours caused by the interaction of the system components and the time lags.

For differential equations with variable, state-dependent delay, we have differentiable solution manifolds and operators. How about higher order smoothness? Under suitable hypotheses, there is a solution manifold in the space $C^2_{rn} = C^2([-r, 0], \mathbb{R}^n)$ which is *twice* continuously differentiable—but the solution operators which we get on this manifold are not better than continuously differentiable. What is the reason? Can one find solution operators with more derivatives on infinite-dimensional solution manifolds in other ambient spaces? —What can be said about the topology and geometry of the solution manifolds ? In some cases they are simply graphs over closed subspaces of codimension n in the ambient space C^1_{rn}, see for example a construction in [120].

The general question concerning state-dependent delays is, of course, how these shape the dynamics. In case of a vectorfield on the plane trajectories flow along its arrows. For a scalar differential equation with one constant time lag results obtained over the past 60 years begin to provide us with a repertoire of expectations about solution behaviour if we only see a particular example of Eq. (1.7). With variable delay, much more likely in nature, it is different, intuition for the effect of feedback through a state-dependent delay in a system seems—still—rather poor.

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CLASSICS REVISITED

J.E. Littlewood: "The Converse of Abel's Theorem on Power Series"

Lond. M. S. Proc. (2) 9 (1911), 434–448

Hervé Queffélec

Published online: 20 May 2014

In 1897, the Austrian mathematician A. *Tauber* published the short paper (5 pages) [2] which can be summarized as follows:

(1) If a series $\sum a_n$ converges, i.e., if $S_N := \sum_{n=0}^{N} a_n$ converges, then

$$\lim_{x\to 1^-} \sum_{n=0}^{\infty} a_n x^n =: \lim_{x\to 1^-} f(x) = \sum_{n=0}^{\infty} a_n$$

by a theorem of Abel.

(2) If a series $\sum a_n$ converges, then

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N} n a_n = 0$$

by a theorem of Kronecker.

(3) Both converses are false; the conclusions of either Abel's or Kronecker's theorems may hold, and the series $\sum a_n$ yet diverge. But if we assume *both* conclusions, then the series $\sum a_n$ does converge, and we thus have a necessary and sufficient condition for the convergence of a series of complex numbers.

The paper contains two theorems, labelled A and B. Theorem B is devoted to the special case $na_n = o(1)$ and then Theorem A is the general case, which is reduced after some effort to Theorem B. The proof of Theorem B contains a trick ($S_N -$

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$f(1 - 1/N) \to 0$) but is technically easy, and obviously extends (even if Tauber does not mention it) to the quite often encountered case $\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N} n|a_n| = 0$. Theorem A is slightly more difficult (typically the kind of proof you should prepare before a course if you do not want to be ridiculous). All in all, the paper is nice, even if it contains no example or application.

In 1911, motivated by a question of G.H. Hardy, who had just proved a similar result for the so-called Cesàro summation process, the English mathematician *J.E. Littlewood* published a longer paper (15 pages) entitled "The converse of Abel's theorem on power series", the paper under review, in which he replaces the second assumption of Theorem B of Tauber by the assumption $na_n = O(1)$. In his introduction, he quotes Tauber and qualifies his work as "remarkable", in the way a tennis player having just severely defeated you qualifies your back-hand as remarkable....

Indeed, the proof of the young Littlewood (26 years old at that time) turns out to be incredibly more difficult and elaborate than that of Tauber (and that of Hardy as well!) and somehow heralds the great analyst he will be. His dense paper, which should be read again by today's mathematicians, doubtlessly with great profit, contains at least three fascinating issues, namely:

1. An analysis of Tauber's proof, which in fact proves, according to Littlewood, that when $na_n = o(1)$, the respective cluster sets E_f and E_S of $f(x)$ as $x \to 1^-$ and of S_N as $N \to \infty$ are the same. And the detailed study of a non-trivial example ($a_n = n^{-1-i\alpha}$ with α a non-zero real number, observe that $n|a_n| = 1$), showing how different the situation can be under the conditions $na_n = o(1)$ or $na_n = O(1)$. Indeed, in that case, Littlewood shows that E_f and E_S are circles with the same center $\zeta(1 + i\alpha)$ and with respective radii r_α and R_α such that $r_\alpha < R_\alpha$.
2. The "tour de force" of the paper: the *positive* answer to Hardy's question in Theorem B. New ideas (use of a parameter and of a degree of freedom in an apparently quite rigid problem) appear. Indeed, setting $x = e^{-\varepsilon}$ and $S(t) = \sum_{n\le t} a_n$, Littlewood first reformulates the assumption in the form that
$$\lim_{\varepsilon\to 0^+} \varepsilon \int_0^\infty e^{-\varepsilon t} S(t)\, dt =: l$$
exists. Then, he forces the introduction of a parameter r by showing that
$$\lim_{\varepsilon\to 0} \varepsilon^{r+1} \int_0^\infty t^r e^{-\varepsilon t} S(t)\, dt = lr!$$
for each non-negative integer r. To that effect, the author needs a Theorem A on differentiable functions (essentially the fact that if $\Phi(x) \to s \in \mathbb{C}$ and $\Phi'(x), \Phi''(x)$ are bounded near infinity, then $\Phi'(x) \to 0$) which is indeed quite simple and was already known (Hadamard, Kneser). In any case, after some delicate estimates on integrals, Littlewood is able to derive that
$$\lim_{T\to\infty} \frac{1}{T} \int_0^T S(u) du = l$$
and then concludes with the help of the Hardy result already mentioned for Cesàro summation.
3. A proof of the *optimality* of the condition $na_n = O(1)$ in Theorem C, in the following form: If $\varphi_n \to +\infty$, there exists a series $\sum a_n$ such that

(1) $|a_n| \le \frac{\varphi_n}{n}, n \ge 1$.
(2) $\frac{S_0+\cdots+S_N}{N} \to 0$, implying $f(x) \to 0$ as $x \to 1^-$.
(3) S_N oscillates.
Indeed, Littlewood's example is (essentially) the following:

$$\Phi(n) = \sum_{j=1}^{n} \frac{\varphi_j}{j}, \quad a_n = e^{i\Phi(n)} - e^{i\Phi(n-1)}, \quad S_n = e^{i\Phi(n)} \quad \text{for } n \ge 1.$$

The second item is the non-trivial one (its "implication" is the extension by Frobenius of Abel's theorem), and requires fairly sharp estimates of independent interest on the exponential sums $S_0 + \cdots + S_N = \sum_{n=1}^{N} e^{i\Phi(n)}$. Littlewood achieves those estimates with a very simple proof, which announces the van der Corput and Kuzmin-Landau estimates in Number Theory, and precedes them by more than 20 years! Indeed, an analysis of the Kuzmin-Landau proof lets it appear as an improvement of Littlewood's original method.

Let us finish with a few comments:

(1) The example $a_n = (-1)^n e^{\sqrt{n}}$, which is Abel summable, but by no means Cesàro, or higher order Cesàro, summable (due to the severe increase of $|a_n|$) can heuristically explain why Littlewood's result is so much more difficult than Hardy's one. The assumption of Abel summability is very weak!
(2) Littlewood uses a much more general context than that of Taylor series, namely that of Dirichlet series $\sum a_n e^{-\lambda_n x}$, in Theorem B. In that general context, one has to assume that $a_n = O(\frac{\lambda_n - \lambda_{n-1}}{\lambda_n})$. This generality imposes some restrictions on the exponents λ_n, namely $\frac{\lambda_{n+1}}{\lambda_n} \to 1$, and will be the origin of the so-called problem of "high indices", later solved by Hardy and Littlewood. Actually $a_n = O(\frac{\lambda_n - \lambda_{n-1}}{\lambda_n})$ is optimal as well in the general framework of Dirichlet series.
(3) Theorem A of Littlewood can be completely dispensed with, even if heuristically that was the starting point of his discovery. The work of Karamata has shown how to introduce for free (changing ε into εr) the degree of freedom which Littlewood has to buy, in writing the assumption under the form

$$\lim_{\varepsilon \to 0^+} \varepsilon \int_0^\infty e^{-\varepsilon r t} S(t)\, dt = l \int_0^\infty e^{-rt}\, dt$$

for any $r > 0$ and then in linearizing:

$$\lim_{\varepsilon \to 0^+} \varepsilon \int_0^\infty e^{-\varepsilon t} P\big(e^{-\varepsilon t}\big) S(t)\, dt = l \int_0^\infty e^{-t} P\big(e^{-t}\big)\, dt$$

for every polynomial P.
(4) This sensational paper of Littlewood immediately attracted the interest of Hardy (34 years old at that time), and gave rise to the famous Hardy-Littlewood collaboration for more than thirty years. One important aspect of this collaboration was to give more credit to Tauber and to put his result into a general context, not only thinking of it as a criterion for convergence of series. As a result, this led to the general theory of "Tauberian theorems", like the Karamata, Wiener-Ikehara, Newman, Delange, Erdős-Feller-Pollard… theorems, and in particular

provided new and simple proofs of the Prime Number Theorem (in an arithmetic progression).

(5) Finally, J. Korevaar published nearly one century later the huge work [1] in the form of a 500 pages long research book, which summarizes the discoveries of the Tauberian theory as far as 2004. The posterity of Tauber's and Littlewood's early papers has thus proved to be remarkable!

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BOOK REVIEW

Piotr Nowak, Guoliang Yu: "Large Scale Geometry"
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Bernhard Hanke

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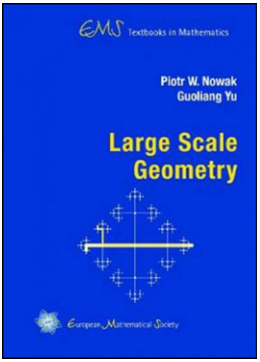

One of the most important invariants associated to a topological space X with base point x_0 is its *fundamental group* $\pi_1(X, x_0)$. It consists of equivalence classes of closed loops in X based at x_0, two loops being identified if they can be deformed into each other through loops based at x_0. Concatenation of loops defines the group structure on $\pi_1(X, x_0)$, which is usually not abelian.

The fundamental group plays an important role in the classification of topological spaces. For example, it can be used to distinguish closed surfaces of different genera. The famous Poincaré conjecture, proved by Perelman, states that closed 3-manifolds with trivial fundamental groups are homeomorphic to spheres.

To what extent are topological, smooth and Riemannian manifolds in arbitrary dimensions determined by their fundamental groups? Homotopy theory shows that *aspherical* manifolds, whose universal covering spaces are contractible by definition, are determined by their fundamental groups up to homotopy equivalence. This is a much weaker notion than homeomorphism, diffeomorphism and isometry, the natural equivalence relations for topological, smooth and Riemannian manifolds.

Hyperbolic manifolds, i.e. Riemannian manifolds of constant sectional curvature -1, form a beautiful and rich class of aspherical manifolds. The famous rigidity theorem of Mostow (1968) states that closed hyperbolic manifolds of dimension $n \geq 3$ are isometric, if their fundamental groups are isomorphic. This is an instance

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of *rigidity*, because a relatively rough invariant, the isomorphism type of the fundamental group, encodes the complete geometric information on the underlying object.

For the proof of Mostow rigidity one lifts the given homotopy equivalence to universal covering spaces, both of which are isometric to the hyperbolic space $\mathbb{H}^n$. This lift is a *quasi-isometry*: It differs from an actual isometry of $\mathbb{H}^n$ by a uniform bound. It can therefore be extended to a quasi-conformal map on the ideal boundary $\partial\mathbb{H}^n$, which can be identified with $\mathbb{S}^{n-1}$. For $n \geq 3$ it can be shown that this map is in fact conformal and induced by an isometry of $\mathbb{H}^n$. This in turn induces an isometry of the original hyperbolic manifolds. This proof illuminates the role of asymptotic properties of hyperbolic space and its group of isometries.

Mostow rigidity is related to the famous *Borel conjecture* stating that closed aspherical manifolds with isomorphic fundamental groups are homeomorphic. This conjecture is known to hold for manifolds of dimension at least 5 and with Gromov hyperbolic fundamental groups (Bartels-Lück 2012), a property which is defined by referring to large scale geometric properties of groups equipped with word length metrics.

Another instance of rigidity is the *Novikov conjecture*, which predicts homotopy invariance of higher signatures. It can be expressed as an injectivity statement of the assembly map relating the topological K-theory of the fundamental group to the K-theory of its group C^*-algebra. By the descent principle in coarse geometry this property is connected to the large scale geometry of the fundamental group. The Novikov conjecture is known to hold, if the fundamental group admits a coarse embedding into Hilbert space (Yu 2000), a property which is again of a large scale geometric nature.

It is remarkable that in these examples the fundamental group enters not only as an algebraic, but also as a geometric object: The *word length metric* on a group defines the distance of two group elements g and h as the minimal length of a word expressing $g^{-1}h$ as a product in a given set of generators and their inverses. For finitely generated groups the resulting metric is uniquely determined up to *quasi-isometry*, i.e. word metrics associated to different finite sets of generators of a given group are bi-Lipschitz equivalent up to a globally bounded difference. With respect to the equivalence relation of quasi-isometry, which can be defined for arbitrary metric spaces, the real line with the usual metric is identified with the integers regarded as a discrete subspace. Intuitively the relevant geometric information is the one which remains visible when seen from far apart, or at "large scales". This concept, which had appeared before in Mostow's book *Strong rigidity of locally symmetric spaces* (1973) and in Gromov's work in geometric group theory, among others, was later generalized and put into an axiomatic setting in John Roe's *coarse geometry*, revealing important connections between large scale geometric properties, index theory and non-commutative geometry.

The monograph under review offers a systematic introduction to the subject of large scale geometry, with an emphasis on geometric group theory and functional analytic methods, some of which are relevant for Yu's theorem mentioned before.

The book is divided into eight chapters. The first one outlines general concepts like metric spaces, word metrics on groups, quasi-isometries, the Švarc-Milnor lemma and coarse equivalences. It ends with a short introduction to Gromov hyperbolic spaces and groups, which share essential features with the hyperbolic space $\mathbb{H}^n$ on a large scale.

The following five chapters discuss several asymptotic properties of metric spaces and groups. Asymptotic dimension (Gromov 1993) and decomposition complexity (Guentner-Tessera-Yu 2012), which are the theme of Chap. 2, play important roles in recent progress concerning the Novikov- and stable Borel conjectures.

Chapter 3 introduces amenability, a fundamental concept going back to von Neumann (1929), and growth conditions for groups. The latter occur in the polynomial growth theorem of Gromov (1981), one of the cornerstones of geometric group theory.

Property A (Yu 2000), a non-equivariant analogue of amenability, is defined and discussed in Chap. 4. This notion is put into the context of coarse embeddability into Hilbert space (Gromov 1991) in Chap. 5, which also discusses expanders and their use for the construction of metric spaces not coarsely embeddable into Hilbert space.

An important class of groups which embed coarsely into Hilbert space are a-T-menable groups (Haagerup 1979; Gromov 1991). This concept is explained in Chap. 6, which deals with group actions on Banach spaces. A-T-menability is presented as an equivariant analogue of coarse embeddability into Hilbert space, in a similar way as amenability can be viewed as an equivariant analogue of property A. This chapter also introduces Kazhdan's property (T), its spectral properties and applications to the construction of expanders.

Chapter 7 is devoted to concepts of coarse algebraic topology. It starts with an introduction to coarse homology (Roe 1993) and its bounded refinement, uniformly finite homology (Block-Weinberger 1992). The equivalence of the vanishing of 0-th uniformly finite homology with non-amenability, the characterization of quasi-isometries that are close to bi-Lipschitz maps in terms of uniformly finite homology and the use of uniformly finite homology for a construction of aperiodic tilings are presented. The chapter ends with a short introduction to coarse generalized homology theories and an application of coarse homology to the determination of upper bounds on the asymptotic dimension of bounded geometry metric spaces.

The last chapter contains a rather brief account of applications to topological rigidity, largeness properties of non-compact manifolds, index theory, the Baum-Connes and Novikov conjectures, existence of metrics of uniformly positive scalar curvature, and the zero-in-the-spectrum conjecture.

Each of the first seven chapters ends with a set of exercises and with some very informative "Notes and remarks" on the history of the subject, important results and open problems.

After Gromov's papers and books on geometric group theory, Roe's texts on coarse geometry and Higson-Roe's book on analytic K-homology this monograph serves as an up-to-date introductory guide to the active research field of large scale geometry. The emphasis lies on a presentation of concepts and their interrelation, illustrated by plenty of examples, rather than on a complete exposition of the theory. Many important results, classical and recent, are mentioned, but not discussed in detail, such as, remarkably, Yu's proof of the Novikov conjecture for groups that coarsely embed into Hilbert space. For a more thorough study of these topics, that motivate the concepts treated in the monograph, one must refer to other sources including the research literature.

In summary Nowak-Yu's "Large scale geometry" serves both as a nice complement to Roe's "Lectures on coarse geometry" (2003) and as a valuable survey of some of the more modern aspects of the field.

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BOOK REVIEW

James Arthur: "The Endoscopic Classification of Representations. Orthogonal and Symplectic Groups"

Amer. Math. Soc., Colloquium Publications 61 (2013), 590 pp.

Ulrich Görtz

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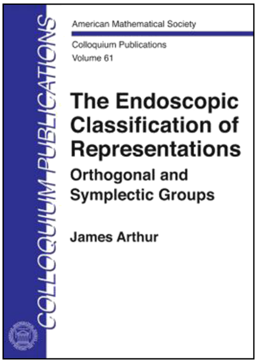

The theory of automorphic representations is an important topic in representation theory and number theory, with connections to other areas of mathematics such as harmonic analysis and algebraic geometry. In the book under review, James Arthur proves a number of far-reaching theorems about such representations for the classical groups mentioned in the title: (quasi-split) special orthogonal and symplectic groups.

To put things into context, let us start by discussing the *trace formula*. The simplest instance of the trace formula is the identity

$$\sum_{i=1}^{n} a_{ii} = \sum_{i=1}^{n} \lambda_i$$

which expresses the trace of a $(n \times n)$-matrix $A = (a_{ij})_{i,j}$ over some algebraically closed field as the sum of its diagonal entries on the one hand, and as the sum of its eigenvalues $\lambda_1, \ldots, \lambda_n$, on the other hand. The left hand side of this formula is called the *geometric side*, the right hand side is called the *spectral side* of the trace formula. See [5] for an accessible introduction to the trace formula which takes this equality as the starting point.

Selberg has proved a trace formula in the situation where H is a locally compact unimodular topological group, and where $\Gamma \subset H$ is a discrete subgroup such that the quotient $\Gamma\backslash H$ is compact. As a rough approximation, the geometric side of the trace

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formula is a sum over conjugacy classes in Γ, and the spectral side is a sum over the set of irreducible representations of H occurring in $L^2(\Gamma\backslash H)$. Both sides express, in different ways, the trace of operators on the space $L^2(\Gamma\backslash H)$. If H is a finite group, then the trace formula is basically the Frobenius reciprocity theorem.

If $H = \mathbb{R}$ and $\Gamma = \mathbb{Z}$, then the trace formula reduces to the Poisson summation formula: For $g \in C_c^\infty(\mathbb{R})$,

$$\sum_{u\in\mathbb{Z}} g(u) = \sum_{\lambda\in 2\pi i\mathbb{Z}} \int_{\mathbb{R}} g(y)e^{-\lambda y}dy.$$

From the point of view of the general trace formula, we view the left hand side as a sum over the conjugacy classes in the abelian group $\mathbb{Z}$, and the right hand side as a sum over irreducible representations, i.e., characters, of $\mathbb{R}$, according to the decomposition $L^2(\mathbb{Z}\backslash\mathbb{R}) = \bigoplus_{\lambda\in 2\pi i\mathbb{Z}} \mathbb{C}_\lambda$ (of representations of the group $\mathbb{R}$), where $\mathbb{C}_\lambda$ is a 1-dimensional $\mathbb{C}$-vector space on which $\mathbb{R}$ acts through the character $x \mapsto e^{\lambda x}$. Both sides express the trace of the operator

$$\varphi \mapsto \left(x \mapsto \int_{\mathbb{R}} g(y)\varphi(x+y)dy\right) \quad \text{on } L^2(\mathbb{Z}\backslash\mathbb{R}).$$

Let X be a compact Riemann surface of genus > 1; its universal covering is the complex upper half plane $\mathbb{H} = SL_2(\mathbb{R})/SO_2(\mathbb{R})$. Using the trace formula for $H = SL_2(\mathbb{R})$ and $\Gamma = \pi_1(X)$, the fundamental group of X (a *Fuchsian group*), Selberg was able to obtain applications to the geometry and analysis of X. For instance, he found an asymptotic formula for the number of closed geodesics in X of length $< N$ with a sharp error term.

The trace formula gained further attention when Langlands found new possible applications related to number theory, and was later generalized by Arthur to the Arthur–Selberg trace formula which applies to the situation $\Gamma := G(F) \subset G(\mathbb{A}_F) =: H$, where F is a global field, G is a reductive group over F, and $\mathbb{A}_F$ is the ring of adèles of F. The ring $\mathbb{A}_F$ is a locally compact topological ring which is built from the completions of F at all archimedean and non-archimedean places (as a restricted product with respect to the rings of integers of the non-archimedean completions); for example $\mathbb{A}_\mathbb{Q} = \mathbb{R} \times (\prod_p \mathbb{Z}_p) \otimes_\mathbb{Z} \mathbb{Q}$, where the product ranges over all prime numbers p, and $\mathbb{Z}_p$ denotes the ring of p-adic integers. Via the adèles, arithmetic aspects enter the game.

Since the quotient $\Gamma\backslash H$ is usually not compact, the proof is much more difficult. It is contained in an impressive series of papers by Arthur. There are several variants of the trace formula which have been established (in particular the stable trace formula, see below), and there are further variants which are still conjectural, in particular the stabilization of the twisted trace formula for general linear and even special orthogonal groups. This stabilization, together with some other technical results which can be expected to be obtained in its proof, is put as a hypothesis in the volume under review, i.e., the results obtained here are currently still conditional.

The trace formula has turned out to be extremely fruitful towards arithmetic applications, notably within the Langlands program, a web of far-reaching conjectures formulated by Robert Langlands around 1970 which became a central topic in number theory, representation theory and arithmetic geometry in the last few decades.

Put in very simple terms, the main theme is a correspondence between algebraic/geometric objects (such as representations of a Galois group of a number field or a local field which could in a typical case arise from the cohomology of some algebraic variety) and analytic objects (such as modular forms, i.e., holomorphic functions on the complex upper half plane satisfying certain periodicity and growth conditions, or more generally automorphic forms and automorphic representations). An instance of such a correspondence is predicted to exist for any given reductive algebraic group. Its existence is not at all obvious. Even the simplest instance, which is associated with the algebraic group GL_1, amounts to highly non-trivial theorems: global and local class field theory, respectively, and Artin's reciprocity map.

In the case of GL_2, the objects on the analytic side are basically modular forms. The Langlands conjecture for $GL_2(\mathbb{Q})$ is still open. The theorem of Wiles, Taylor and others, that every elliptic curve over $\mathbb{Q}$ is modular, i.e., that its L-function which is built from the numbers of points of the elliptic curve over finite fields, is actually the L-function of a modular form, can be seen as one piece of evidence for it.

The automorphic representations of a reductive algebraic group G over F are, roughly speaking, the irreducible representations of $G(\mathbb{A})$ that occur in the regular representation $L^2(G(F)\backslash G(\mathbb{A}_F))$. Such an automorphic representation π can be decomposed as a restricted tensor product of local representations π_v of $G(F_v)$, where v runs through the places of F. Almost all of the π_v (at finite places v) are *unramified*. They can be constructed easily from characters by a parabolic induction process, and as a consequence, unramified representations can be parameterized in a simple way. It is one of the key observations of Langlands that instead of using characters, it is better to work in a dual setting. More precisely, Langlands introduced the *dual group* $\widehat{G}$ of G, an algebraic group over $\mathbb{C}$ whose root datum is dual to that of G and which has a maximal torus which can naturally be identified with the dual torus (over $\mathbb{C}$) of the fixed maximal torus in G. Rephrasing the classification of unramified representations alluded to above, one obtains that they are classified by semisimple conjugacy classes in $\widehat{G}$. In fact, unless the group G is split over F, the situation is a little more complicated, because the Galois action has to be taken into account, and the dual group hence should be replaced by a semi-direct product ${}^LG := \widehat{G} \rtimes W_{F_v}$, where W_{F_v} denotes the Weil group of F_v (which can be thought of as a variant of the absolute Galois group of F). In other words, we are looking at certain homomorphisms $\mathbb{Z} \to {}^LG$ up to $\widehat{G}$-conjugacy; such homomorphisms are called *Langlands parameters*.

The advantage of this point of view is that, at least conjecturally, a similar classification of representations which are not necessarily unramified can be given in similar terms. It is natural to view the source $\mathbb{Z}$ of the above homomorphisms as the infinite cyclic group generated by a Frobenius automorphism over F_v, and then to enlarge this group either to the absolute Galois group of F_v, or—as turns out to be a better choice—to the Weil group W_{F_v} or rather the product $L_{F_v} := W_{F_v} \times SU_2$.

In fact, for $G = GL_N$, the *local Langlands correspondence* (proved by Harris and Taylor; further proofs have been given by Henniart and by Scholze) asserts (up to "technical details" which we omit here) a "natural" bijection between parameters $L_{F_v} \to {}^LG$ and equivalence classes of irreducible admissible representations of $G(F_v)$. The bijection is characterized by certain compatibilities between L-functions and ε-factors on both sides.

For groups other than GL_N, the word *correspondence* has to be taken more literally: Instead of a bijection between parameters and representations, one only expects a correspondence which attaches to each Langlands parameter a *packet*, i.e., a finite set of representations.

In the global case, one can still aim at a *Langlands correspondence*, i.e., a classification of automorphic representations along these lines. In particular, Langlands conjectured that there exists a group L_F which serves as the right source for Langlands parameters in this case. This conjecture was later refined by Kottwitz. However, so far no candidate for this group is known. At the moment, it seems to be out of reach, being one of the deepest conjectures of the whole Langlands program.

The book under review aims at a classification of automorphic representations of (quasi-split) special orthogonal and symplectic groups. The point of view taken here is that automorphic representations of GL_N are known. This is justified by the local Langlands correspondence for GL_N mentioned above, and by the global theorems of Jacquet and Shalika, and by Moeglin and Waldspurger. So Arthur's goal is a classification of the automorphic representations of classical groups in terms of automorphic representations of GL_N. His main tool are several variants of the trace formula. The key point which allows him to compare these different variants and to extract information of this comparison, is the phenomenon of endoscopy, whence he speaks of an *endoscopic classification*.

To give a very rough idea of endoscopy, recall that the geometric side of the trace formula is a sum indexed by the set of conjugacy classes in $G(F)$. However, to compare trace formulas for different groups, this is not an ideal starting point, in particular when one thinks about groups which become isomorphic after base change to an algebraic closure $\overline{F}$ of F: In general elements in $G(F)$ which are conjugate in $G(\overline{F})$ (such elements are called *stably conjugate*), need not be conjugate in $G(F)$. While this is true for $G = GL_N$, it fails already for $SL_2(\mathbb{Q})$. It is desirable to *stabilize* the trace formula in the sense that the geometric side is replaced by a sum over *stable* conjugacy classes.

The price one has to pay when one wants to work with stable conjugacy classes is that besides G (and its Levi subgroups) further groups naturally arise in the trace formula giving rise to "error terms" that make up for the use of stable, instead of usual, conjugacy classes. These further groups are the so-called endoscopic groups for G. They are defined in a rather technical way in terms of the dual group of G, and are usually not subgroups of G.

A key observation on which Arthur's book is based, is that conversely, one can take advantage of the "correction terms" arising from endoscopic groups. For example, special orthogonal and symplectic groups arise as endoscopic groups for GL_N, so terms of their stable trace formula also occur in the stable trace formula for GL_N. Hence a comparison of trace formulas, at least in principle, can give information about automorphic representations of classical groups in terms of automorphic representations of GL_N. To actually turn this idea into precise theorems is a rather involved enterprise which takes up about 400 pages of the book, not counting the vast amount of previous work on this topic, by Arthur and many others.

Let us sketch the main results of the book. The groups considered here are quasi-split special orthogonal and symplectic groups. The main local result says that the local Langlands correspondence is true for G of the form $SO(2n+1)$ or $Sp(n)$, and a slightly weaker result for $SO(2n)$.

The main global result is a decomposition of the discrete part of $L^2(G(F)\backslash G(\mathbb{A}_F))$,

$$L^2_{\text{disc}}\big(G(F)\backslash G(\mathbb{A}_F)\big) = \bigoplus_{\psi\in\tilde{\Psi}_2(G)} m_\psi \bigg(\bigoplus_{\pi\in\tilde{\Pi}_\psi(\varepsilon_\psi)} \pi \bigg),$$

where $\tilde{\Psi}_2(G)$ should be thought of the appropriate set of Langlands parameters, m_ψ is a multiplicity (which is either 1 or 2), and $\tilde{\Pi}_\psi(\varepsilon_\psi)$ is a subset of the global packet attached to ψ.

Among the direct applications in the global situation are the following results: Rankin–Selberg L-functions for quasi-split orthogonal and symplectic groups have analytic continuation and satisfy a functional equation. The group G has no "embedded eigenvalues"; for GL_N this was proved by Jacquet and Shalika.

Because in this review we can only give a very sketchy, and not fully correct impression of the theorems and methods of Arthur's book, we point the reader to the following more detailed survey articles. The Foreword and Chap. 1 of the book under review are a good starting point. In addition, there are several introductory papers by Arthur on this and related topics, e.g., [2–4], as well as his 260-pp. long survey [1]. See also Labesse's paper [6].

The structure of the book is as follows. The first chapter starts out by giving motivation for the results and the methods, in particular discussing the role of the hypothetical Langlands group L_F, and how to get around the problem that its existence is not known. Its final section contains the formal statements of the three main theorems. The proofs of these theorems are intertwined and take up most of the rest of the volume, being concluded only in Chap. 8. Along the way, local and global endoscopy are studied (Chaps. 2 and 3, resp.). In Chap. 4, the comparison of different trace formulas is started. In Chaps. 6 and 7, the author analyzes generic and non-generic local parameters, resp. In Chap. 5, certain key cases of global parameters are studied (square integrable, elliptic, ...). In Chap. 8, the proof is finished off, and complemented by some reflections on the result and further refinements. While the above-mentioned theorems deal with the case of quasi-split groups, in Chap. 9, analogous statements are stated for their inner forms. Their proofs will appear elsewhere.

The three main theorems were already described in [1] §30, and have been the topic of lecture courses by Arthur in 1994/95 and in 2000, at the Institute for Advanced Study and the University of Paris VII.

The book is carefully written, with a lot of attention to details. The structure of the proof is discussed at several points of the book which provides a helpful guide to the reader. In addition to the Index, there is a Notational Index listing all symbols used in the book. Even though it is mainly directed at specialists working on automorphic representations and the Langlands program, with the extensive Foreword and the motivational sections in Chap. 1, it also provides interesting material for non-experts.

This book is a milestone in the theory of automorphic forms and the Langlands program. Its results have a large impact on current and future developments, and its methods can be expected to yield applications to a much larger class of groups. Everybody whose work is related to automorphic representations will profit from reading it.

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BOOK REVIEW

Steven Lord, Fedor Sukochev, Dmitriy Zanin: "Singular Traces, Theory and Applications"

de Gruyter, 2012, 468 pp.

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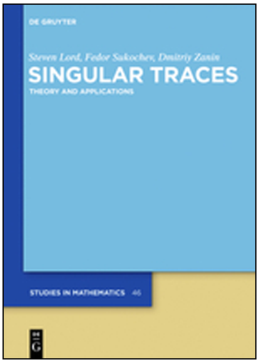

This monograph provides an extensive presentation of singular traces reaching out to the latest results on the topic. There are very few books available on traces and this may be the first one on *singular traces*. Since the book touches on noncommutative geometry and on pseudo-differential operators—the latter being the topic of other recent books—it will surely prove useful to various communities in mathematics for whom traces are an essential tool. It is organised in a reader friendly way which makes it accessible to a lay person as the reader is gently guided through some unavoidable arduous material inherent to the topic.

Singular traces, the objects of study of this monograph, are to be opposed to *normal traces*. Roughly speaking,[1] a trace is normal when it is characterised by its purely algebraic properties; a prototype for a normal trace is the operator trace Tr that generalises the ordinary matrix trace to "infinite dimensional matrices". It is defined on a small class of operators acting on a separable Hilbert space H, called *trace-class operators* that form the ideal $\mathcal{L}_1(H)$ in the algebra $\mathcal{L}(H)$ of bounded operators on H. A bounded linear operator A on H lies in the trace-class if for some (and hence all) orthonormal bases $\{e_k\}_{k\in\mathbb{N}}$ of H the sum of positive terms[2] $\|A\|_1 = \mathrm{Tr}(|A|) :=$

[1] A more precise definition is given below.

[2] Here the square root of the self-adjoint operator A^*A is defined using functional calculus similarly to the way one defines the square root of a positive definite matrix.

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$\sum_k \langle (A^*A)^{1/2} e_k, e_k\rangle$ is finite. In this case, the sum $\mathrm{Tr}(A) := \sum_k \langle Ae_k, e_k\rangle$ is finite and defines the trace of A independently of the choice of orthonormal basis. On the even smaller set of finite-rank operators the operator trace yields back the matrix trace.

In contrast, singular traces are of a purely infinite dimensional nature and vanish on trace-class operators; they are traces on ideals of compact operators on Hilbert spaces that vanish on the sub-ideal of finite rank operators. Singular traces have applications to several fields of mathematics and theoretical physics including noncommutative symmetric spaces, noncommutative integration theory, the geometry of Banach spaces, pseudo-differential operators, index theory and geometric analysis, fractal geometry and quantum field theory. On operators on manifolds, singular traces are expected to be *local* in that they can be expressed as integrals of differential forms. They can thereby provide useful topological information on the underlying manifold or physical information in the form of local anomalies. Singular traces are also important in their own right for the understanding of singular traces contributes to the understanding of general non-normal traces.

Driven by the urge to characterise normal traces on positive bounded operators, Jacques Dixmier in 1966 was the first to construct a singular trace which lives on a certain ideal of compact operators and which since then carries his name. The *Dixmier trace* lives on a space of linear operators on a Hilbert space larger than the space of trace-class operators. For example, a compact self-adjoint operator with eigenvalues $1, \frac{1}{2}, \frac{1}{3}, \ldots$ is not trace-class but has Dixmier trace equal to 1; the Dixmier trace therefore assigns to the harmonic series $1 + \frac{1}{2} + \frac{1}{3} + \cdots$ the residue of the ζ-function $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$ at $s = 1$. This is not a mere coincidence since Dixmier traces relate to complex residues of ζ-functions via the *noncommutative residue* introduced by Mariusz Wodzicki around 1986. As Alain Connes first observed in 1988, the noncommutative residue and the Dixmier trace actually coincide on an appropriate class of pseudodifferential operators. This showed that singular traces are not "pathological monsters"[3] as Jacques Dixmier originally feared they might be; as mentioned above they are now used in various fields of mathematics and even in physics.

Numbers are easier to grasp than operators and traces on operators can best be understood from functionals on the set of their eigenvalues. Rather than the operator A acting between two Hilbert spaces itself, it is convenient to consider the non-negative self-adjoint operator A^*A where A^* denotes the adjoint of A and to study functionals on its set of eigenvalues. The n-th largest eigenvalue (with multiplicity counted) $\mu_n(A)$ of square roots of the eigenvalues of the non-negative self-adjoint operator A^*A is called the n-th *singular value* of A. Dixmier's construction may be described in terms of singular values. To build a Dixmier trace, one chooses a normalising sequence with a suitable asymptotic behaviour, divides the partial sums of the singular values by the given normalising sequence. The last step which is more technical, consists in applying to the resulting sequence a dilation invariant extended limit on the algebra ℓ_∞ of all bounded sequences (see Eq. (5)). This procedure gives a linear functional on positive operators which defines a trace.

[3] This expression is borrowed from a letter by J. Dixmier to the conference "Singular Traces and Their Applications" Luminy 2012, a quote to be found in a note of the monograph.

Dixmier traces are not the only singular traces on ideals of compact operators; on the Lorentz ideal (see Eq. (4)) on which Dixmier traces are finite, there exist singular traces which unlike the Dixmier trace, are not continuous and hence differ from the latter. Moreover, on that ideal Dixmier traces do not span the set of all continuous traces. A central result of the monograph says that operator ideals in a class which includes the Lorentz operator ideal have one of the following mutually exclusive features; they have either no continuous trace or a unique one (up to a constant) or an infinite number of traces.

The monograph comprises twelve chapters organised in four parts. The first part (Chaps. 1 and 2) serves as an introduction to singular traces on symmetric spaces accessible to graduate level readers. The second part (Chaps. 3–6) discusses symmetric operator spaces of general semifinite (atomic or atomless) von Neumann algebras and shows the existence of Dixmier traces on Lorentz operator ideals. The third part (Chaps. 7–9) discusses formulas for Dixmier traces on Lorentz operator ideals including heat-kernel formulas and the ζ-function residues. The fourth and last part (Chaps. 10–12) provides a novel treatment of the noncommutative residue in noncommutative geometry using the concept of modulated operators. Then follows an appendix which comprises useful matrix and operator results. Each chapter comes with an introduction presenting the concepts and essential results of the chapter, complemented with extended notes and commented references, which makes the task for the non-expert reader very smooth.

The singular value set $\mu(A) = \{\mu_n(A), n \in \mathbb{N}\}$, which we saw is an essential ingredient in building the Dixmier trace, leads to a central protagonist of this book, the *Calkin correspondence* (1941)

$$\mu : \mathcal{J} \longrightarrow J$$
$$A \longmapsto \mu(A)$$

between two-sided ideals $\mathcal{J}$ of compact operators in the set $\mathcal{L}(H)$ of bounded linear operators on a complex separable Hilbert space H and sequence spaces J generated by singular values $\mu(A)$ of such an operator A. This correspondence reviewed in Chap. 1, provides a way to construct two-sided ideals of compact operators from sequence spaces and hence paves the path for the construction of traces on compact operator ideals.

The operator trace Tr defined on trace-class operators extends to a normal semi-finite trace on $\mathcal{L}(H)$, the semi-finiteness corresponding to the fact that any projection p in H can be approximated by a sequence of finite-rank projections p_i whose traces $\mathrm{Tr}(p_i)$ are therefore finite. The algebra $\mathcal{L}(H)$ equipped with Tr is actually the prototype of a von *Neumann algebra* $\mathcal{M}$ equipped with a *faithful normal semifinite trace* τ. A *trace* on a von Neumann algebra $\mathcal{M}$ is a weight, namely a map $\tau : \mathcal{M}_+ \to [0, +\infty]$ defined on the cone of its positive elements (i.e. those of the form B^*B) which has the property $\tau(B^*B) = \tau(BB^*)$ for any B in $\mathcal{M}$. It is *normal* if moreover $A = \sup_\alpha A_\alpha \Longrightarrow \tau(A) = \lim_\alpha \tau(A_\alpha)$.

The Calkin correspondence between two-sided ideals of $\mathcal{L}(H)$ and their Calkin sequence spaces extends to the context of von Neumann algebras equipped with a faithful normal semifinite trace. For this purpose, one introduces (see Chap. 2) a generalised *singular value function*

$$\begin{aligned}\mu : \mathcal{M} &\longrightarrow \mathrm{Map}\big((0,+\infty), \mathbb{R}_{\geq 0}\big)\\ A &\longmapsto \mu(t, A).\end{aligned}$$

This generalised singular map actually extends to the larger *Calkin space* $\mathcal{S}(\mathcal{M}, \tau)$ of τ-measurable operators. In order to avoid going into the technicalities of the definition of a measurable operator, we can legitimately think of $\mathcal{S}(\mathcal{M}, \tau)$ as $\mathcal{M}$ for in the case $\mathcal{M} = \mathcal{L}(H)$ it coincides with $\mathcal{M}$; in that case the singular function is a step function which for non negative integers n, is constant on $[n, n+1[$ with value $\mu(n, A)$ given by the n-th singular value of the operator A. A Calkin operator space is a linear subspace $\mathcal{J}$ of $\mathcal{S}(\mathcal{M}, \tau)$ such that for any B in $\mathcal{S}(\mathcal{M}, \tau)$ the following condition holds:

$$\big(A \in \mathcal{J} \wedge \mu(B) \leq \mu(A)\big) \Longrightarrow B \in \mathcal{J}. \tag{1}$$

Other central protagonists (see Chap. 3) are the symmetric operator and symmetric function (resp. sequence) spaces. In the case when $\mathcal{M}$ is a factor, a subspace $\mathcal{E}(\mathcal{M}, \tau)$ of $\mathcal{S}(\mathcal{M}, \tau)$ equipped with a norm $\|\cdot\|_{\mathcal{E}}$ is a *symmetric operator space* if it satisfies a strengthened version of condition Eq. (1) involving an extra requirement on the norm of B on the right hand side, or equivalently, if it is a Banach space and satisfies the following "symmetric" property:

$$\|BAC\|_{\mathcal{E}} \leq \|B\|_{\mathcal{E}} \|A\| \|C\|_{\mathcal{E}} \quad \forall A \in \mathcal{M},\ \forall B, C \in \mathcal{E}.$$

Symmetric function (resp. sequence) spaces are symmetric operator spaces for a von Neumann algebra $\mathcal{L}_\infty(0,1)$, $\mathcal{L}_\infty(0,\infty)$ or ℓ_∞. *Symmetric functionals* are continuous traces on a symmetric operator space $(\mathcal{E}, \tau)$; these turn out to be the only symmetric functionals if $\mathcal{M}$ is an atomless (or atomic) factor. The Calkin correspondence extends (see Chap. 4) to symmetric functionals on symmetric operator and symmetric function spaces for a von Neumann algebra $(\mathcal{M}, \tau)$ equipped with a faithful normal semi-finite trace τ.

In the context of symmetric functionals on function (resp. sequence) spaces, issues relative to functionals on operators can be translated to issues relative to functionals on their singular value spaces. Indeed, every symmetric functional $f \in E^*$ lifts to a symmetric functional $\phi \in \mathcal{E}(\mathcal{M}, \tau)^*$ such that

$$\phi(A) = f\big(\mu(A)\big) \quad \forall A \geq 0. \tag{2}$$

Conversely, in the atomic or atomless case, for every functional $\phi \in \mathcal{E}(\mathcal{M}, \tau)^*$ there is a functional $f \in E^*$ such that (2) holds. Consequently, the study of symmetric functionals on function (resp. sequence) spaces answers questions such as what kind of semi-finite trace the space $(\mathcal{E}, \tau)$ might admit, distinguishing between normal and singular traces. The main result of Chap. 4 yields a classification of traces on a fully symmetric operator space $\mathcal{E}(\mathcal{M}, \tau)$ equipped with a Fatou norm (the unit ball is closed with respect to convergence in the strong topology). Such a space either does not admit a nontrivial continuous trace, or if it does, either it has infinitely many nontrivial continuous singular traces or it has (up to a constant factor) a unique nontrivial continuous trace and the latter extends to the faithful normal semifinite trace τ.

For the symmetric normed ideal $\mathcal{E}$ in $\mathcal{L}(H)$ of compact operators (acting on a separable Hilbert space) with Calkin space E, the *Lidskii formula* for continuous traces provides a translation to functionals on *eigenvalue value spaces* of issues relative to functionals on operator spaces. To make this more precise, let us consider the map $a \longmapsto \mathrm{diag}(a)$, which to a sequence a, assigns the diagonal operator $\mathrm{diag}(a)$. It induces a map $\mathcal{E}^* \to E^*$; $\phi \mapsto \phi \circ \mathrm{diag}$. The Lidskii formula for continuous traces discussed in Chap. 5 gives its explicit converse

$$\phi(A) = f\big(\lambda(A)\big) \tag{3}$$

in terms of the eigenvalue sequence $\lambda(A)$ of the compact operator A.

However, the Lidskii formula only provides existence of a symmetric functional f satisfying Eq. (3) but it does not tell us how to construct it. Chapter 6 gives Dixmier's general construction of all fully symmetric traces on a fully symmetric ideal of compact operators excepting the trace class operators. Given an increasing positive concave function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$, it is proved that the Lorentz ideal

$$\mathcal{M}_\psi := \left\{ A \in \mathcal{L}(H);\ \|A\|_{\mathcal{M}_\psi} := \sup_{n\geq 0} \frac{1}{\psi(n+1)} \sum_{k=0}^{n} \mu(k, A) < \infty \right\} \tag{4}$$

either admits no continuous trace or if $\liminf_{t\to\infty} \frac{\psi(2t)}{\psi(t)} = 1$, it possesses an infinite number of continuous traces. In this case every normalised fully symmetric trace on $\mathcal{M}_\psi$ is a singular continuous trace called a Dixmier trace; for nonnegative A in $\mathcal{M}_\psi$ it is defined as

$$\mathrm{Tr}_\omega(A) := \omega\left(\frac{1}{\psi(n+1)} \sum_{k=0}^{n} \mu(k, A) \right), \tag{5}$$

where ω is some dilation invariant extended limit on ℓ_∞. Moreover it is shown that on a fully symmetric ideal $\mathcal{E} \neq \mathcal{L}_1(H)$ of $\mathcal{L}(H)$, the set of Dixmier traces which is non void if there is some continuous trace, is weak* dense in the set of fully symmetric traces. Chapter 7 then provides various Lidskii type formulas for Dixmier traces, by lifting to the appropriate ideal of $\mathcal{L}(H)$ a commutative version of the formula established on the corresponding function space. For atomless or atomic von Neumann algebras equipped with a faithful normal semifinite trace, Chap. 8 gives a description of Dixmier traces by means of a heat-kernel asymptotic formula and discusses its relation to the generalised ζ-function residue used in noncommutative geometry. Chapter 9 then characterises which operators are measurable, namely for which operators A in a Lorentz ideal $\mathcal{M}_\psi$, the Diximer trace $\tau_\omega(A) = \omega(\frac{1}{\psi(t)} \int_0^t \mu(s, A) ds)$ is independent of the extended limit ω (e.g. a dilation invariant extended limit) on $\mathcal{L}_\infty(0, \infty)$ used to define it. The last part of the book, dedicated to applications, offers a generalisation of Alain Connes' trace theorem which relates the Dixmier trace to the Wodzicki residue, originally for an appropriate class of pseudodifferential operators, namely compactly supported operators of order minus the dimension of the underlying closed manifold. In Chap. 11 this relation is generalised to a larger class of integral operators, namely compactly supported Laplacian modulated operators first introduced by Nigel Kalton. This generalised trace theorem is then illustrated by the case of Hodge–Laplacian modulated operators on closed Riemannian manifolds.

Using the properties of modulated operators shown in Chap. 11, a vector-valued noncommutative residue is then introduced in Chap. 12 which it is defined on operators modulated by an appropriate power of the underlying Dirac type operator. This leads to a noncommutative version of the trace which relates this generalised Wodzicki residue to a Dixmier trace. Examples and properties of the noncommutative residue are then discussed, and the last chapter of the book ends with possible future applications of singular traces in relation to high temperature limits seen as classical limits.