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PREFACE

Preface Issue 4-2014

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The last few years have been a good time for solving long standing geometrical-topological conjectures. This issue reports on the solution of the Willmore conjecture—the "best" topological torus is a "real" torus with ratio of radii equal to $\sqrt{2}$—and one of Thurston's conjectures—every hyperbolic 3-manifold can be fibered over a circle, up to passing to a finite cover.

Starting in the 1960s, Thomas Willmore studied the integral of the squared mean curvature of surfaces in $\mathbb{R}^3$ as the simplest but most interesting frame invariant elastic bending energy. This energy had shown up already in the early 19th century—too early for a rigorous investigation. Willmore asked: What is the shape of a compact surface of fixed genus minimising the Willmore energy in this class? (In the 1990s, existence of minimisers was proved by Leon Simon, with a contribution by Matthias Bauer and Ernst Kuwert.) Willmore already knew that the genus-0-minimiser is a sphere. Assuming that symmetric surfaces require less energy than asymmetric ones (which has not been proved, yet) he studied families of geometric tori with the smaller radius 1 fixed and found that the larger radius $\sqrt{2}$ would yield the minimum in this very special family. He conjectured that this particular torus would be the genus-1-minimiser. Almost 50 years later Fernando Marques and André Neves found and published a 100-page-proof. In the present survey article, disregarding technical details, they explain their approach and their fundamental ideas and provide also an excellent historical and mathematical background.

After Thurston's Geometrization Conjecture was finally solved about 10 years ago by Grigori Perelman, one of the most important questions in three-dimensional topology has been to understand the hyperbolic 3-manifolds. William Thurston's 1982—

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paper “Three-dimensional manifolds, Kleinian groups and hyperbolic geometry” (see the “Classics Revisited”—contribution of this year’s first issue) lists 24 at that time open problems in 3-manifolds-theory. Problem No. 18 reads as follows: “Does every finite volume hyperbolic 3-manifold have a finite cover which fibers over the circle?” This conjecture was indeed proved 2012 by Ian Agol and Dani Wise. Stefan Friedl explains their work and the requisite background in his very comprehensible survey article “Thurston’s Vision and the Virtual Fibering Theorem for 3-Manifolds.” As Stefan Friedl explains, after Agol & Wise, only “Thurston’s last challenge” concerning ratios of volumes of hyperbolic 3-manifolds remains to be solved.

In this issue the “Classics Revisited”-contribution (in collaboration with the Zentralblatt für Mathematik) is written by Jean Mawhin. It concerns the “Cours d’Analyse Infinitésimale” of Charles-Jean de La Vallée Poussin, who is still famous for his proof (independent of Hadamard’s) of the prime number theorem. These textbooks were used in Belgium for an exceptionally long time. Reviewers praise the originality, clearness and elegance of the exposition and its merits as a handbook in analysis.

In the “Book Reviews”—section Günter M. Ziegler gives us a critical view of Alexander Soifer’s “The Mathematical Coloring Book”. Johannes Huebschmann evaluates the book on “Poisson Structures” written by Camille Laurent-Gengoux, Anne Picherau, and Pol Vanhaecke.

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SURVEY ARTICLE

The Willmore Conjecture

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Abstract The Willmore conjecture, proposed in 1965, concerns the quest to find the best torus of all. This problem has inspired a lot of mathematics over the years, helping bringing together ideas from subjects like conformal geometry, partial differential equations, algebraic geometry and geometric measure theory.

In this article we survey the history of the conjecture and our recent solution through the min-max approach. We finish with a discussion of some of the many open questions that remain in the field.

Keywords Minimal surfaces · Willmore surfaces · Min-max theory · Bending energy

Mathematics Subject Classification 53A10 · 53C42

1 Introduction

A central theme in Mathematics and particularly in Geometry has been the search for the optimal representative within a certain class of objects. Partially motivated by this principle, Thomas Willmore started in the 1960s the quest for the optimal immersion of a compact surface in three-space. This optimal shape was to be found,

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presumably, by minimizing a natural energy over all compact surfaces in $\mathbb{R}^3$ of a given topological type. In this survey article we discuss the history and our recent solution of the Willmore conjecture, the problem of determining the best torus among all.

We begin by defining the energy. Recall that the local geometry of a surface in $\mathbb{R}^3$ around a point p is described by the principal curvatures k_1 and k_2, the maximum and minimum curvatures among all intersections of the surface with perpendicular planes passing through p. The classical notions of curvature of a surface are then the *mean curvature* $H = (k_1 + k_2)/2$ and the *Gauss curvature* $K = k_1 k_2$. With the aforementioned question in mind, Willmore associated to every compact surface $\Sigma \subset \mathbb{R}^3$ a quantity now known as the *Willmore energy*:

$$W(\Sigma) = \int_\Sigma H^2 d\mu = \int_\Sigma \left(\frac{k_1 + k_2}{2}\right)^2 d\mu,$$

where $d\mu$ stands for the area form of Σ.

The Willmore energy is remarkably symmetric. It is invariant under rigid motions and scalings, but less obvious is the fact that it is invariant also under the inversion map $x \mapsto x/|x|^2$. Hence $W(F(\Sigma)) = W(\Sigma)$ for any conformal transformation F of three-space. It is interesting that Willmore himself became aware of this fact only after reading the paper of White [66], many years after he started working on the subject. As we will explain later, this conformal invariance was actually known already in the 1920s.

In applied sciences this energy had already been introduced long ago, to study vibrating properties of thin plates. Starting in the 1810s, Sophie Germain [19] proposed, as the elastic or bending energy of a thin plate, the integral with respect to the surface area of an even, symmetric function of the principal curvatures. The Willmore energy is the simplest possible example (excluding the area functional). Similar quantities were considered by Poisson [53] around the same time.

The Willmore energy had also appeared in Mathematics in the 1920s through Blaschke [5] and his student Thomsen [62], of whose works Willmore was not initially aware. The school of Blaschke was working under the influence of Felix Klein's Erlangen Program, and they wanted to understand the invariants of surface theory in the presence of an action of a group of transformations. This of course included the Möbius group—the conformal group acting on Euclidean space with a point added at infinity.

A natural map studied by Thomsen in the context of Conformal Geometry, named the *conformal Gauss map* by Bryant (who rediscovered it in [8]), associates to every point of an oriented compact surface in $\mathbb{R}^3$ its central sphere, the unique oriented sphere having the same normal and mean curvature as the surface at the point. This is a conformal analogue of the Gauss map, that associates to every point of an oriented surface its unit normal vector. If we include planes as spheres of mean curvature zero, the space of oriented spheres in $\mathbb{R}^3$, denoted here by $\mathcal{Q}$, can be identified with the 4-dimensional Lorentzian unit sphere in the five-dimensional Minkowski space. The basic principle states that applying a conformal map to the surface corresponds to applying an isometry of $\mathcal{Q}$ to its conformal Gauss map. Since the area of the image of a closed surface under the conformal Gauss map is equal to the Willmore energy of the surface minus the topological constant $2\pi\chi(\Sigma)$, the conformal invariance of $W(\Sigma)$ becomes apparent this way.

Finally, the Willmore energy has continued to appear in applied fields, like in biology to study the elasticity of cell membranes (it is the highest order term in the Helfrich model [21]), in computer graphics in order to study surface fairing [38], or in geometric modeling [7].

Back to the quest for the best immersion, Willmore showed that round spheres have the least possible Willmore energy among all compact surfaces in three-space. More precisely, he proved that every compact surface $\Sigma \subset \mathbb{R}^3$ satisfies

$$W(\Sigma) \geq 4\pi,$$

with equality only for round spheres.

Here is a geometric way to see this. First note that when a plane is translated from very far away and touches the surface for the first time, it will do so tangentially in a point where the principal curvatures share the same sign. At such points the Gauss curvature $K = k_1 k_2$ must necessarily be nonnegative. Therefore the image of the set of points where $K \geq 0$ under the Gauss map N must be the whole S^2. Since $K = \det(dN)$, the area formula tells us that $\int_{\{K\geq 0\}} K\,d\mu \geq \text{area}(S^2) = 4\pi$. The result of Willmore follows because $H^2 \geq K$, with equality at umbilical points ($k_1 = k_2$), and the only totally umbilical surfaces of $\mathbb{R}^3$ are the round spheres.

Willmore continued the study of his energy and, having found the compact surface with least possible energy, tried to find the minimizing shape among the class of tori [67]. It is interesting to note that no obvious candidate stands out a priori. To develop intuition about the problem, Willmore considered a special type of torus: he fixed a circle C of radius R in a plane and looked at the tube Σ_r of constant radius $r < R$ around C. When r is small, Σ_r is very thin and the energy $W(\Sigma_r)$ is very large. If we keep increasing the value of r, the size of the middle hole of the torus decreases and eventually the hole disappears for $r = R$. The energy $W(\Sigma_r)$ becomes arbitrarily large when r approaches R. Therefore the function $r \mapsto W(\Sigma_r)$ must have an absolute minimum in the interval $(0, R)$, which Willmore computed to be $2\pi^2$.

Up to scaling, the optimal tube in this class has $R = \sqrt{2}$ and $r = 1$:

$$\Sigma_{\sqrt{2}} = \left\{\left((\sqrt{2}+\cos u)\cos v, (\sqrt{2}+\cos u)\sin v, \sin u\right) \in \mathbb{R}^3 : u, v \in \mathbb{R}\right\}.$$

In light of his findings, Willmore conjectured that this torus of revolution should minimize the Willmore energy among all tori in three-space:

Conjecture (Willmore [67]) *Every compact surface Σ of genus one in $\mathbb{R}^3$ must satisfy*

$$W(\Sigma) \geq 2\pi^2.$$

It seems at first rather daring to have made such a conjecture after having it tested only for a very particular one-parameter family of tori. On the other hand, the torus that Willmore found is very special and had already appeared in geometry in disguised form. It turns out that there exists a stereographic projection from the unit

3-sphere $S^3 \subset \mathbb{R}^4$ minus a point onto Euclidean space $\mathbb{R}^3$ that maps the Clifford torus $\hat{\Sigma} = S^1(\frac{1}{\sqrt{2}}) \times S^1(\frac{1}{\sqrt{2}})$ onto $\Sigma_{\sqrt{2}}$. We will say more about the Clifford torus later.

In [39], we proved:

Theorem 1.1 *Every embedded compact surface Σ in $\mathbb{R}^3$ with positive genus satisfies*

$$W(\Sigma) \geq 2\pi^2.$$

Up to rigid motions, the equality holds only for stereographic projections of the Clifford torus (like $\Sigma_{\sqrt{2}}$).

The rigidity statement characterizing the equality case in Theorem 1.1 is optimal because stereographic projections are conformal and, as we have mentioned, the Willmore energy is conformally invariant.

Since Li and Yau [37] had proven in the 1980s that compact surfaces with self-intersections have Willmore energy greater than or equal to 8π, our result implies:

Corollary 1.2 *The Willmore conjecture holds.*

2 Some Particular Cases and Related Results

In this section we survey some of the results and techniques that have been used in understanding the Willmore conjecture. Our emphasis will be in giving a glimpse of the several different ideas and approaches that have been used, instead of giving an exhaustive account of every result proven. The reader can find more about the subject in [22, 48, 51, 69, 70].

The richness of the problem derives partially from the fact that the Willmore energy is invariant under conformal maps. Since stereographic projections are conformal, one immediate consequence is that the conjecture can be restated for surfaces in the unit 3-sphere S^3. Indeed, if Σ is a compact surface in S^3 and $\tilde{\Sigma}$ denotes its image in $\mathbb{R}^3$ under a stereographic projection, then one has

$$W(\tilde{\Sigma}) = \int_\Sigma 1 + \left(\frac{k_1 + k_2}{2}\right)^2 d\mu,$$

where k_1, k_2 are the principal curvatures of Σ with respect to the standard metric on S^3. For this reason we take the right-hand side of the equation above as the definition of the Willmore energy of $\Sigma \subset S^3$:

$$\mathcal{W}(\Sigma) = \int_\Sigma (1 + H^2)\, d\mu.$$

The Willmore conjecture can then be restated equivalently in the following form:

Conjecture (Willmore [67]) *Every compact surface $\Sigma \subset S^3$ of genus one must satisfy*

$$\mathcal{W}(\Sigma) \geq 2\pi^2.$$

We need to introduce an important class of surfaces that plays a central role in understanding the Willmore energy, as we will see later. This is the class of *minimal surfaces*, defined variationally as surfaces that are stationary configurations for the area functional, i.e., those for which the first derivative of the area is zero with respect to any variation.

These surfaces are characterized by the property that their mean curvature H vanishes identically. Hence it follows immediately from the expression for $\mathcal{W}(\Sigma)$ that the Willmore energy of a minimal surface in S^3 is equal to its area. The equators (or great spheres) are the simplest examples of minimal surfaces in S^3, with area 4π, while the Clifford torus is a minimal surface in S^3 with area $2\pi^2$. Note that this is compatible with the fact that $\Sigma_{\sqrt{2}}$ is a stereographic projection of the Clifford torus and $W(\Sigma_{\sqrt{2}}) = 2\pi^2$. There are infinitely many known compact minimal surfaces in S^3. For instance Lawson [36] in the 1970s found embedded orientable minimal surfaces in S^3 of any genus.

Given that the Willmore conjecture was initially tested only for a very particular set of tori in $\mathbb{R}^3$, the first wave of results consisted in testing the conjecture on larger classes. Willmore himself in 1971 [68], and independently Shiohama and Takagi [59], verified the conjecture for tubes of constant radius around a space curve γ in $\mathbb{R}^3$. An explicit computation gives that such a torus must satisfy

$$W(\Sigma) \geq \pi \int_\gamma |k|\, ds,$$

where $|k| \geq 0$ is the curvature of the space curve γ. Hence the result follows from Fenchel's Theorem [16], which establishes $\int_\gamma |k|\, ds \geq 2\pi$.

In 1973, Chen [10] checked that every intrinsically flat torus in S^3 has Willmore energy greater than or equal to $2\pi^2$, with equality only for the Clifford torus. The inverse image under the Hopf map $S^3 \to S^2$ of a closed curve in S^2 is a flat torus in S^3 and thus such examples abound. Chen's Theorem follows from integral geometric arguments that we quickly describe. Given a surface Σ in $\mathbb{R}^{k+2}$ with unit normal bundle B, we have the generalized Gauss map

$$\mathcal{G} : B \to S^{k+1} = \left\{ v \in \mathbb{R}^{k+2} : |v| = 1 \right\}.$$

The *total curvature* $\tau_k(\Sigma)$ is defined as the total $(k+1)$-volume parametrized by $\mathcal{G}$ divided by the volume of S^{k+1}. Chern-Lashof [12] showed that the total curvature is equal to the average number of critical points of a linear function on Σ. Therefore every torus has total curvature greater than or equal to 4. Through purely local computations, Chen showed that if a torus $\Sigma \subset S^3 \subset \mathbb{R}^4$ is flat, then

$$\mathcal{W}(\Sigma) \geq \frac{\pi^2}{2} \tau_2(\Sigma),$$

that combined with the Chern-Lashof inequality implies the result. Later he extended this result to include flat tori in the unit n-sphere S^n [11]. See [2] and [63] for other related results.

In 1976, Langevin and Rosenberg [35] showed that the Willmore energy of a knotted torus in $\mathbb{R}^3$ is at least 8π. The key estimate was to show that if the torus is knotted then every linear function has at least 8 critical points. Hence

$$\frac{1}{4\pi}\int_{\Sigma} |K| d\mu = \frac{1}{2}\tau_1(\Sigma) \geq 4,$$

where the first identity follows from the definition of total curvature and the fact that $K = \det(dN)$, while the inequality follows from the Chern-Lashof result. Therefore, since $\int_{\Sigma} K d\mu = 0$ by the Gauss–Bonnet Theorem, we have

$$W(\Sigma) \geq \int_{\{K\geq 0\}} \left(\frac{k_1+k_2}{2}\right)^2 d\mu \geq \int_{\{K\geq 0\}} K d\mu = \frac{1}{2}\int_{\Sigma} |K| d\mu \geq 8\pi.$$

This result is reminiscent of the Fary-Milnor Theorem [15, 42], which states that the total curvature $\int_C |k|\, ds$ of a knotted closed curve C in $\mathbb{R}^3$ must exceed 4π.

In 1978, Weiner [65] checked that the Clifford torus is a critical point with non-negative second variation of the Willmore energy. If this were not the case there would be some small perturbation of the Clifford torus with strictly smaller energy, contradicting the conjecture.

In 1982, Li and Yau [37] were the first to exploit in a crucial way the conformal invariance of the problem. They introduced the important notion of *conformal volume* of an immersion $\phi : \Sigma \to S^n$:

$$V_c(n,\phi) = \sup_{g\in Conf(S^n)} \text{area}(g\circ\phi),$$

and obtained various striking results. Their ideas, that we describe below, had a lasting impact in Geometry. The results are valid when the ambient space is the unit n-sphere but we restrict ourselves to S^3 for simplicity.

The conformal group of S^3, modulo isometries, is parametrized by the unit 4-ball B^4: for each $v \in B^4$ we associate the conformal map

$$F_v : S^3 \to S^3, \quad F_v(x) = \frac{(1-|v|^2)}{|x-v|^2}(x-v) - v. \tag{1}$$

The map F_0 is the identity and, for $v \neq 0$, F_v is a conformal dilation with fixed points $v/|v|$ and $-v/|v|$.

For illustrative purposes, we note that if B is a geodesic ball in S^3 and $p \in S^3$, then $F_{tp}(B)$ is a geodesic ball that, as $t < 1$ tends to 1, could have three types of behavior. It converges to the whole 3-sphere if p is inside B, to the antipodal point $-p$ if p is outside the closure of B, or to the hemisphere touching ∂B tangentially at p in case p is in the boundary of B.

Using these transformations, Li and Yau showed that if a surface Σ contains a k-point p, i.e., if nearby p the surface looks like k small discs containing p, then

$\mathcal{W}(\Sigma) \geq 4\pi k$. The proof goes as follows: as $t < 1$ tends to 1, $F_{tp}(\Sigma)$ converges to a union of k great spheres (boundaries of hemispheres). Since

$$\mathcal{W}(\Sigma) = \mathcal{W}\big(F_{tp}(\Sigma)\big) \geq \text{area}\big(F_{tp}(\Sigma)\big),$$

by taking the limit as $t \to 1$ we get

$$\mathcal{W}(\Sigma) \geq \text{area}(k \text{ great spheres}) = 4\pi k.$$

One consequence of this result is that the energy of every compact surface is at least 4π (a result we already mentioned). It also follows that the energy of any compact surface that is not embedded, hence contains a double-point at least, must be greater than or equal to 8π. In particular, one can restrict to the class of embedded tori in order to prove the Willmore conjecture.

Another consequence is that every immersed projective plane in $\mathbb{R}^3$ must have Willmore energy greater than or equal to 12π. This is because such projective planes always contain a triple-point at least. Bryant [9] and Kusner [26] found projective planes in $\mathbb{R}^3$ with Willmore energy exactly equal to 12π. All such projective planes were classified in [9].

In that same paper, Li and Yau also found a region R (see below for an explicit description) in the space of all conformal structures so that if the conformal class of a torus $\Sigma \subset S^3$ lies in R, then $\mathcal{W}(\Sigma) \geq 2\pi^2$. The conformal class of the Clifford torus (square lattice) is in the boundary of R. This result relates in an ingenious way the conformal invariance of the Dirichlet energy $\int_\Sigma |\nabla f|^2 \, d\mu$ in dimension two with the conformal invariance of the Willmore energy. We discuss briefly the main ideas.

Given a torus $\Sigma \subset S^3$, the Uniformization Theorem tells us that Σ is diffeomorphic to $\mathbb{R}^2/\Gamma$, where Γ is some lattice in $\mathbb{R}^2$, and that the induced metric g on Σ is conformal to the Euclidean metric g_0 on $\mathbb{R}^2/\Gamma$. The lattice Γ can be chosen to be generated by the vectors $(1, 0)$ and (x, y), with $0 \leq x \leq 1/2$, $y \geq 0$ and $x^2 + y^2 \geq 1$.

Recall that the first nontrivial eigenvalue of the Laplacian with respect to g_0 is given by

$$\lambda_1 = \inf_{\int_\Sigma f \, d\mu_{g_0} = 0, f \neq 0} \frac{\int_\Sigma |\nabla_{g_0} f|^2 \, d\mu_{g_0}}{\int_\Sigma f^2 \, d\mu_{g_0}}. \tag{2}$$

Li and Yau first prove, through a degree argument, that Σ can be balanced by applying a conformal transformation, i.e. that there exists $v_0 \in B^4$ such that $\int_\Sigma x_i \circ F_{v_0} \, d\mu_{g_0} = 0$ for every $i = 1, \ldots, 4$. By evaluating the quotient in the right hand side of Eq. (2) with $f = x_i \circ F_{v_0}$, summing over $i = 1, \ldots, 4$ and using the conformal invariance of the Dirichlet energy in dimension two, they show that

$$\begin{aligned}\lambda_1 \, \text{area}\big(\mathbb{R}^2/\Gamma, g_0\big) &= \lambda_1 \sum_{i=1}^4 \int_\Sigma (x_i \circ F_{v_0})^2 \, d\mu_{g_0} \leq \sum_{i=1}^4 \int_\Sigma \big|\nabla_{g_0}(x_i \circ F_{v_0})\big|^2 \, d\mu_{g_0} \\ &= \sum_{i=1}^4 \int_\Sigma \big|\nabla_g (x_i \circ F_{v_0})\big|^2 \, d\mu_g = 2 \, \text{area}\big(F_{v_0}(\Sigma)\big).\end{aligned}$$

Using the conformal invariance of the Willmore energy we have,

$$\operatorname{area}\big(F_{v_0}(\Sigma)\big) \leq \mathcal{W}\big(F_{v_0}(\Sigma)\big) = \mathcal{W}(\Sigma),$$

where the first inequality comes from the expression of the Willmore energy in S^3. Putting these two inequalities together Li and Yau obtained that

$$\lambda_1 \operatorname{area}\big(\mathbb{R}^2/\Gamma, g_0\big) \leq 2\mathcal{W}(\Sigma).$$

The left-hand side of the above inequality can be computed for every conformal class of the torus. It turns out that if Γ is in the set R of lattices such that the generators $(1,0)$ and (x,y) satisfy the additional assumption $y \leq 1$, besides $0 \leq x \leq 1/2$, $y \geq 0$ and $x^2 + y^2 \geq 1$, then

$$4\pi^2 \leq \lambda_1 \operatorname{area}\big(\mathbb{R}^2/\Gamma, g_0\big).$$

This finishes Li-Yau's proof that $\mathcal{W}(\Sigma) \geq 2\pi^2$ when the conformal class of Σ lies in R. In 1986, Montiel and Ros [44] found a larger set of lattices, still containing the square lattice on the boundary, for which the Willmore conjecture holds.

In 1984, Langer and Singer [33] showed that the energy of every torus of revolution (with possibly noncircular section) in space is greater than or equal to $2\pi^2$, with equality only for the Clifford torus and dilations of it. The basic fact, observed independently by Bryant and Pinkall, is that if γ is a closed curve in the upper half-plane $P = \{(x,y,0) : y > 0\}$, and Σ is the torus obtained by revolving γ around the x-axis, then

$$W(\Sigma) = \frac{\pi}{2}\int_\gamma k_{-1}^2\, ds,$$

where k_{-1} is the geodesic curvature of γ computed with respect to the *hyperbolic metric* on P. Langer and Singer showed that every regular closed curve in the hyperbolic plane satisfies $\int_\gamma k_{-1}^2 ds \geq 4\pi$, and the inequality follows. Later, Hertrich-Jeromin and Pinkall [23] extended this computation to a larger class of tori (*Kanaltori* in German), for which the Willmore energy is still given by a line integral.

The variational problem associated to the Willmore energy is extremely interesting, and many solutions are known that are not of minimizing type. Surfaces that constitute critical points of the Willmore energy are called *Willmore surfaces* (they were previously called *conformal minimal surfaces* by Blaschke). The Euler-Lagrange equation for this variational problem, attributed by Thomsen [62] to Schadow, is of fourth order and is the same for surfaces in $\mathbb{R}^3$ or S^3:

$$\Delta H + \frac{(k_1 - k_2)^2}{2} H = 0.$$

In particular, minimal surfaces are always Willmore and therefore Lawson's minimal surfaces in S^3 provide examples of compact embedded surfaces of any genus that are stationary for the Willmore functional.

A classification of all Willmore spheres was achieved in a remarkable work of Bryant [8, 9], who exploited with significant creativity the conformal invariance of the problem. He discovered that their Willmore energies are always of the form $4\pi k$, with $k \in \mathbb{N} \setminus \{2, 3, 5, 7\}$ (see also [49]), and that the only embedded ones are the round spheres.

This result had a great impact, so we briefly describe it. Motivated by his finding that the conformal Gauss map of a Willmore surface is a conformal harmonic map, Bryant constructed a quartic holomorphic differential on any such surface. This holomorphic differential must vanish on a topological sphere. Bryant used this fact to show that a Willmore sphere must be the conformal inversion of some minimal surface in $\mathbb{R}^3$ with finite total curvature and embedded ends. It follows from the theory of these minimal surfaces that the Willmore energy of the sphere has to be a multiple of 4π. Finally, Bryant reduced the problem of finding all possible such minimal immersions to a problem in algebraic geometry concerning zeros and poles of meromorphic maps on S^2, from which he derived his classification. Ejiri, Montiel, and Musso [14, 43, 46] extended this work and classified Willmore spheres in S^4.

Up to this point, every known Willmore surface was the conformal image of some minimal surface in $\mathbb{R}^3$ or S^3. Pinkall [50] found in 1985 the first examples of embedded Willmore tori that are not of this type. His idea was to look at the inverse image under the Hopf map π of closed curves γ in S^2. The Willmore energy is given by

$$\mathcal{W}\big(\pi^{-1}(\gamma)\big) = \pi \int_\gamma 1 + k^2\, ds,$$

where k is the geodesic curvature of γ. Langer and Singer [34] had shown that there are infinitely many simple closed curves that are critical points of the functional in the right-hand side (these are called elastic curves). Pinkall argued that the inverse image of an elastic curve is a Willmore torus, and that among those the only one that is conformal to a minimal surface in S^3 is the Clifford torus (the inverse image of an equator in S^2). Finally, if any of the Pinkall tori were to be the conformal image of some minimal surface S in $\mathbb{R}^3$, the embeddedness of the torus would imply that S had to be a nonplanar minimal surface asymptotic to a plane. These surfaces do not exist. Later, Ferus and Pedit [17] found more examples of Willmore tori.

In 1991, the biologists Bensimon and Mutz [47] (see also [41]), gave experimental evidence to the Willmore conjecture with the aid of a microscope while studying the physics of membranes! They produced toroidal vesicles in a laboratory and observed that their shape, which according to the Helfrich model should approach the minimizer for the Wilmore energy, was the one predicted by Willmore or one of its conformal images.

The existence of a torus that minimizes the Willmore energy among all tori was proven by Simon [60] in 1993. This result was obtained through a technical tour de force and many of the ideas involved are now widely used in Geometric Analysis. Very briefly, Simon picked a sequence of tori whose energies converge to the least possible value and showed the existence of a limit in some weak measure theoretic sense. Exploiting with great effectiveness the fact that the tori in the sequence are embedded (otherwise the energy would be at least 8π), he obtained that the weak limit must be a smooth embedded surface. A serious difficulty in accomplishing this comes from the conformal invariance of the problem. For instance, one could start with a minimizing torus and apply a sequence of conformal maps so that the images look like some round sphere with increasingly smaller handles attached. In the limit one would obtain a round sphere instead of a torus. To overcome this, Simon showed that every torus can be corrected by applying a carefully chosen conformal map so that it becomes far away in Hausdorff distance from all round spheres. This way he is sure that in his minimization process he will get a limiting surface that is a torus.

More generally, let β_g denote the infimum of the Willmore energy among all orientable compact surfaces of genus g. It was independently observed by Kusner and Pinkall (see [25, 27, 60]) that a Lawson minimal surface of genus g, for every g, has area strictly smaller than 8π. This implies $\beta_g < 8\pi$. Later Kuwert, Li and Schätzle [31] showed that β_g tends to 8π as g tends to infinity.

It is natural to ask whether there exists a genus g surface with energy β_g. Note that for every partition $g = g_1 + \cdots + g_k$ by integers $g_i \geq 1$, one has

$$\beta_g \leq \beta_{g_1} + \cdots + \beta_{g_k} - 4(k-1)\pi.$$

In order to see this take Σ_i to be a surface of genus g_i with energy arbitrarily close to β_{g_i}, $i = 1, \ldots, k$. Apply a conformal map to Σ_i to get a surface $\tilde{\Sigma}_i$ that can be decomposed into two regions: one that looks like a round sphere of radius one minus a small spherical cap and another one that contains g_i small handles. Remove the handle regions for $i \geq 2$ and sew them into $\tilde{\Sigma}_1$ to get a surface with g handles and energy close to $\beta_{g_1} + \cdots + \beta_{g_k} - 4(k-1)\pi$. Simon [60] showed that β_g is indeed attained provided there is no partition $g = g_1 + \cdots + g_k$ by integers $g_i \geq 1$ with $k \geq 2$ such that each β_{g_i} is attained and

$$\beta_g = \beta_{g_1} + \cdots + \beta_{g_k} - 4(k-1)\pi. \tag{3}$$

Bauer and Kuwert [4], inspired by Kusner [28], showed that such partitions do not exist, hence β_g can be realized by a surface of genus g for all g. More precisely, they show that if M with genus h and N with genus s realize β_h and β_s, respectively, then a careful connected sum near non-umbilic points of M and N produces a surface with genus $g = h + s$ and Willmore energy *strictly* smaller than $\beta_h + \beta_s - 4\pi$. This suffices to show that (3) never occurs.

Incidentally, one immediate consequence of our Theorem 1.1 is that $\beta_g \geq 2\pi^2$ for all $g \geq 1$. Since we also have $\beta_g < 8\pi$, it is not difficult to see that partitions as above can never occur.

In 1999, Ros [57] proved the Willmore conjecture for tori in S^3 that are preserved by the antipodal map. (This also follows from the work of Topping [63] on integral geometry.) More precisely, Ros showed that every orientable surface in the projective space $\mathbb{RP}^3$ has Willmore energy greater than or equal to π^2. His approach was quite elegant and based on the fact, proven by Ritoré and Ros [54], that of all surfaces which divide $\mathbb{RP}^3$ into two pieces of the same volume, the Clifford torus is the one with least area (which in this case is π^2). The result follows because he also proved that the surface induced in $\mathbb{RP}^3$ by an antipodally-symmetric surface of odd genus in S^3 is necessarily orientable.

We now describe the method of Ros because it was inspirational in our approach. An orientable surface Σ in $\mathbb{RP}^3$ must be the boundary of a region Ω, which we can choose so that the volume of Ω is not bigger than half the volume of $\mathbb{RP}^3$. Next, Ros looked at the region Ω_t of points that are at a distance at most t from Ω. For very large t, Ω_t will be the whole $\mathbb{RP}^3$ and thus there must exist some t_0 so that the volume of Ω_{t_0} is equal to half the total volume. The result we mentioned in the previous paragraph implies that area$(\partial\Omega_{t_0}) \geq \pi^2$. Finally, Ros observed that

$$\mathcal{W}(\Sigma) \geq \text{area}(\partial\Omega_t) \quad \text{for all } t \geq 0 \tag{4}$$

and thus $\mathcal{W}(\Sigma) \geq \pi^2$, as he wanted to show. The above inequality is a consequence of more general inequalities due to Heintze and Karcher [20], and it plays a crucial role in our method as well.

We briefly sketch its proof. If N is the normal vector of Σ that points outside Ω, we can consider the map

$$\psi_t : \Sigma \to \mathbb{RP}^3, \quad \psi_t\big([x]\big) = \big[\cos t x + \sin t N(x)\big].$$

This map is well defined independently of the representative, x or $-x$, we choose for $[x]$. Since $\partial\Omega_t \subset \psi_t(\Sigma)$, we have that

$$\text{area}(\partial\Omega_t) \leq \int_{\{\text{Jac}\psi_t \geq 0\}} \text{Jac}\,\psi_t d\mu.$$

Denoting by k_1, k_2 the principal curvatures of Σ with principal directions e_1, e_2, respectively, so that $D_{e_i} N = -k_i e_i$, $i = 1, 2$, we have

$$\begin{aligned}
\text{Jac}\,\psi_t &= (\cos t - \sin t k_1)(\cos t - \sin t k_2) \\
&= \cos^2 t + \sin^2 t k_1 k_2 - \sin t \cos t (k_1 + k_2) \\
&\leq \cos^2 t + \frac{1}{4}\sin^2 t (k_1 + k_2)^2 - \sin t \cos t (k_1 + k_2) \\
&= \left(\cos t - \frac{k_1 + k_2}{2}\sin t\right)^2 \leq 1 + \left(\frac{k_1 + k_2}{2}\right)^2.
\end{aligned}$$

Therefore

$$\text{area}(\partial\Omega_t) \leq \int_{\{\text{Jac}\,\psi_t \geq 0\}} \text{Jac}\,\psi_t d\mu \leq \int_{\{\text{Jac}\psi_t \geq 0\}} 1 + \left(\frac{k_1 + k_2}{2}\right)^2 d\mu \leq \mathcal{W}(\Sigma).$$

One year later, Ros [58] used again the area inequality (4) to show that any odd genus surface in S^3 invariant under the mapping $(x_1, x_2, x_3, x_4) \mapsto (-x_1, -x_2, -x_3, x_4)$ must have Willmore energy greater than or equal to $2\pi^2$.

Finally, we mention that the understanding of the analytical aspects of the Willmore equation has been greatly improved in recent years thanks primarily to the works of Kuwert, Schätzle (e.g. [29]) and Rivière (e.g. [55]). In [29], Kuwert and Schätzle analyze isolated singularities of Willmore surfaces in codimension one and prove, as a consequence, that the Willmore flow (the L^2 negative gradient flow of the Willmore energy) of a sphere with energy less than 8π exists for all time and becomes round. This uses a blow-up analysis of possible singular behavior of the flow and Bryant's classification of Willmore spheres in codimension one. In [55], Rivière derives a general weak formulation of the Willmore Euler-Lagrange equation in divergence form, in any codimension, and proves regularity of weak solutions. He also extends the analysis of point singularities of [29] to the higher codimension case. Here is a list of some other current topics of interest, on which there is a lot of activity today: compactness results (modulo the Möbius group) and blow-up analysis of Willmore surfaces, existence of minimizers of the Willmore functional under different constraints (fixed topology, conformal class, isoperimetric ratio), study of Willmore-type functionals in Riemannian manifolds among others. This is a fast paced area with many developments currently taking place and so, instead of listing them exhaustively, we point the reader to the survey article on the Willmore functional of Kuwert and Schätzle [30], and to the lecture notes of Rivière [56] where an introduction to the analysis of conformally invariant variational problems is provided.

3 Min-Max Approach

We now describe the approach we used to solve the Willmore conjecture. As in the previous sections, we appeal to intuition so that we can emphasize the geometric nature of the arguments.

One purpose of the min-max technique is to find unstable critical points of a given functional, i.e., critical points that are not of minimum type. For example, consider the surface $M = \{(x, y, z) \in \mathbb{R}^3 : z = x^2 - y^2\}$ and the height function $f(x, y, z) = z$. Then $(x, y, z) = (0, 0, 0)$ is a critical point of f of saddle type. It turns out that it is possible to detect this critical point by a variational approach. The idea is to fix a continuous path $\gamma : [0, 1] \to M$ connecting $(0, 1, -1)$ to $(0, -1, -1)$ and set $[\gamma]$ to be the collection of all continuous paths $\sigma : [0, 1] \to M$ that are homotopic to γ with fixed endpoints. We define

$$L = \inf_{\sigma\in[\gamma]} \max_{0\le t\le 1} f\big(\sigma(t)\big).$$

The projection on the xy-plane of any path in $[\gamma]$ has to intersect the diagonal line $\{(t, t, 0) \in \mathbb{R}^3 : t \in \mathbb{R}\}$, where f vanishes. Hence $L \ge 0$. But considering $\sigma(t) = (0, 1 - 2t, -(1 - 2t)^2)$, $0 \le t \le 1$, we see that

$$0 = f(0, 0, 0) = f\big(\sigma(1/2)\big) = \max_{0\le t\le 1} f\big(\sigma(t)\big).$$

Hence $L = 0$. The tangential projection of $\nabla f_{|(0,0,0)}$ on M has to vanish because otherwise we would be able to perturb the path σ near the origin and obtain another path $\bar\sigma$ in $[\gamma]$ with $\max_{0\le t\le 1} f(\bar\sigma(t)) < f(0, 0, 0) = 0$. This is impossible because $L = 0$, and so we have found a critical point of f restricted to M via a variational method.

Note that the same reasoning implies that the index of the critical point has to be less than or equal to one because otherwise the origin would be a strict local maximum for f. Again we would be able to perturb σ to obtain another $\bar\sigma$ in $[\gamma]$ with $\max_{0\le t\le 1} f(\bar\sigma(t)) < f(0, 0, 0) = 0$.

Almgren [1] in the 1960s developed a min-max theory for the area functional. His motivation was to produce minimal surfaces (which are critical points for the area functional, as we have mentioned before). The techniques he used come from the field of Geometric Measure Theory, but we will try to keep the discussion with as little technical jargon as possible. This min-max theory applies to more general ambient manifolds, but we restrict to the case of the round 3-sphere for simplicity.

Denote by $\mathcal{Z}_2(S^3)$ the space of integral 2-currents with boundary zero, which can be thought of intuitively as the space of oriented Lipschitz closed surfaces in S^3, with integer multiplicities. For instance, the boundary ∂U of any open set U of finite perimeter is a well-defined element of $\mathcal{Z}_2(S^3)$. This space is endowed with the flat topology, according to which two surfaces are close to each other if the volume of the region in between them is very small.

Let $I^k = [0,1]^k$ be the unit k-dimensional cube. We will consider continuous functions $\Phi : I^k \to \mathcal{Z}_2(S^3)$, and denote by $[\Phi]$ the set of all continuous maps $\Psi : I^k \to \mathcal{Z}_2(S^3)$ that are homotopic to Φ through homotopies that fix the maps on ∂I^k. We then define

$$\mathbf{L}\big([\Phi]\big) = \inf_{\Psi \in [\Phi]} \sup_{x \in I^k} \operatorname{area}\big(\Psi(x)\big).$$

The prototype theorem was proven by Pitts [52] (see also [13] for a nice exposition), a student of Almgren at the time, and states the following.

Theorem 3.1 (Min-max Theorem) *Suppose*

$$\mathbf{L}\big([\Phi]\big) > \sup_{x \in \partial I^k} \operatorname{area}\big(\Phi(x)\big).$$

Then there exists a disjoint collection of smooth, closed, embedded minimal surfaces $\Sigma_1, \dots, \Sigma_N$ *in* S^3 *such that*

$$\mathbf{L}\big([\Phi]\big) = \sum_{i=1}^{N} m_i \operatorname{area}(\Sigma_i),$$

for some positive integer multiplicities $m_1, \dots, m_N$.

The condition $\mathbf{L}([\Phi]) > \sup_{x \in \partial I^k} \operatorname{area}(\Phi(x))$ in the above theorem is important and reflects the fact that $[\Phi]$ is capturing some nontrivial topology of $\mathcal{Z}_2(S^3)$. One should also expect that the Morse index of Σ, i.e., the maximum number of linearly independent deformations whose linear combinations all decrease the area of Σ, should be no higher than k. This is because the definition of $\mathbf{L}([\Phi])$ suggests that Σ is maximized in at most k directions. Due to the technical nature of the subject, this expectation remains to be proven.

It is instructive to study a simple example. The equator $S^3 \cap \{x_4 = 0\}$ in S^3 is a minimal surface and its Morse index is one. One way to see that its Morse index is greater than or equal to one is to note that if we move the equator up with constant speed, the area decreases. On the other hand, any deformation of the equator that fixes the enclosed volume cannot decrease the area because the equator is the least area surface that divides S^3 into two pieces of the same volume. Therefore, the Morse index of the equator is exactly one.

To produce the equator using the Almgren-Pitts min-max theory we consider the map

$$\Phi_1 : [0,1] \to \mathcal{Z}_2\big(S^3\big), \quad \Phi_1(t) = S^3 \cap \{x_4 = 2t-1\}$$

and the corresponding homotopy class $[\Phi_1]$. Given $\Psi \in [\Phi_1]$, there is some $0 \le t_0 \le 1$ such that $\Psi(t_0)$ divides S^3 in two pieces of identical volume. Hence the area of $\Psi(t_0)$ must be bigger than or equal to the area of an equator, and this implies $L([\Phi_1]) \ge 4\pi$. Moreover,

$$L\big([\Phi_1]\big) \le \sup_{0\le t\le 1} \operatorname{area}\big(\Phi_1(t)\big) = 4\pi$$

and thus $L([\Phi_1]) = 4\pi$.

The construction of the equator by the Almgren-Pitts min-max theory is so natural that we became interested in the following question:

Can we produce the Clifford torus using a min-max method?

This question is specially suggestive given the following result of Urbano:

Theorem 3.2 ([64], 1990) *Let Σ be a compact minimal surface of S^3 with Morse index no bigger than* 5. *Then either Σ is an equator* (*of Morse index one*) *or a Clifford torus* (*of Morse index* 5).

The proof consists in a very elegant and short argument that we now quickly describe. Denote by $\mathcal{L}$ the second variation operator associated to the area functional, i.e., if N is the unit normal vector to Σ, then for every $f \in C^\infty(\Sigma)$ we have

$$\frac{d^2}{(dt)^2}\bigg|_{t=0} \operatorname{area}\big(P_t(\Sigma)\big) = -\int_\Sigma f\mathcal{L}f\, d\Sigma,$$

where $\{P_t\}_{-\varepsilon<t<\varepsilon}$ is a one parameter family of diffeomorphisms generated by a vector field X that satisfies $X = fN$ along Σ.

Let $e_1, \dots, e_4$ denote the coordinate vectors in $\mathbb{R}^4$. An explicit computation shows that the functions $\langle N, e_i\rangle$, $i = 1, \dots, 4$, are eigenfunctions of $\mathcal{L}$ on Σ with eigenvalue $\lambda = -2$. One can also see that they are linearly independent, unless the minimal surface Σ is an equator. Moreover, it is well known that the first eigenfunction has multiplicity one and so it cannot belong to the span of $\{\langle N, e_1\rangle, \dots, \langle N, e_4\rangle\}$ (unless again the minimal surface Σ is an equator). Hence, if Σ is not an equator then the Morse index of Σ is at least 5. In case it is equal to 5, Urbano used the Gauss–Bonnet Theorem to show that Σ has to be the Clifford torus.

Motivated by this, we defined a 5-parameter family of surfaces that we explain now after introducing some notation. Let F_v be the conformal maps defined in (1). Given an embedded surface $S = \partial\Omega$, where Ω is a region of S^3, we denote by S_t the surface at distance t from S, which means that S_t is given by

$$\partial\big\{x \in S^3 : d(x,\Omega) \le t\big\} \quad \text{if } 0 \le t \le \pi$$

and by

$$\partial\big(S^3 \setminus \{x \in S^3 : d\big(x, S^3 \setminus \Omega\big) \le -t\}\big) \quad \text{if } -\pi \le t < 0.$$

Note that S_t is not necessarily a smooth surface (due to focal points), but it is nonetheless the boundary of an open set with finite perimeter hence constitutes a well defined element of $\mathcal{Z}_2(S^3)$.

Given an embedded compact surface $\Sigma \subset S^3$, we defined in [39] the *canonical family* $\{\Sigma_{(v,t)}\}_{(v,t)\in B^4\times[-\pi,\pi]}$ of Σ by

$$\Sigma_{(v,t)} = \big(F_v(\Sigma)\big)_t \in \mathcal{Z}_2\big(S^3\big).$$

If Σ is the Clifford torus, then the infinitesimal deformations of $\Sigma_{(v,t)}$ near $(v,t) = (0,0)$ correspond to the 5 linearly independent directions described in the proof of Urbano's Theorem. In light of this, we decided to apply the min-max method to the homotopy class of such canonical families in order to produce the Clifford torus.

The canonical family also has the great property that

$$\text{area}(\Sigma_{(v,t)}) \le \mathcal{W}\big(F_v(\Sigma)\big) = \mathcal{W}(\Sigma), \quad \text{for all } (v,t) \in B^4 \times [-\pi,\pi].$$

The above inequality is just like inequality (4), while the identity is a consequence of the conformal invariance of the Willmore energy.

From these ingredients we devised a strategy to prove the Willmore conjecture. If the homotopy class Π, determined by the canonical family associated to a surface with positive genus, indeed produced the Clifford torus via min-max then we would have from the above inequality that

$$2\pi^2 = L(\Pi) \le \sup_{(v,t)\in B^4\times[-\pi,\pi]} \text{area}(\Sigma_{(v,t)}) \le \mathcal{W}(\Sigma).$$

At this point the question of whether or not the canonical family could produce the Clifford torus by a min-max method was upgraded from an issue on which we had an academic interest to a question which we *really* wanted to answer.

Hence it became important to understand the geometric and topological properties of the canonical family, especially the behavior of $\Sigma_{(v,t)}$ as (v,t) approaches the boundary of the parameter space $B^4 \times [-\pi,\pi]$. The fact that the diameter of S^3 is equal to π implies that $\Sigma_{(v,\pm\pi)} = 0$ for all $v \in B^4$. Hence we are left to analyze what happens when v approaches $S^3 = \partial B^4$.

Assume $v \in B^4$ converges to $p \in S^3$. If p does not belong to Σ, then it should be clear that $F_v(\Sigma)$ is pushed into $\{-p\}$ as v tends to p, and that area$(F_v(\Sigma))$ converges to zero in this process. When p lies in Σ the situation is more subtle. Indeed, if v approaches p radially, i.e., $v = sp$ with $0 < s < 1$, then $F_{sp}(\Sigma)$ converges, as s tends to 1, to the unique great sphere tangent to Σ at p. Therefore the family of continuous functions in S^3 given by $f_s(p) = \text{area}(\Sigma_{sp})$ converges pointwise, as $s \to 1$, to a discontinuous function that is zero outside Σ and 4π along Σ.

Therefore, for any $0 < \alpha < 4\pi$ and $p \in \Sigma$, there must exist a sequence $\{v_i\}_{i\in\mathbb{N}}$ in B^4 converging to p so that area(Σ_{v_i}) converges to α and thus it is natural to expect that the convergence of $F_v(\Sigma)$ depends on how v approaches $p \in \Sigma$. A careful analysis revealed that, depending on the angle at which v tends to p, $F_v(\Sigma)$ converges to a round sphere tangent to Σ at p, with radius and center depending on the angle of convergence.

Initially we were somewhat puzzled by this behavior, but then we realized that, even if this parametrization became discontinuous near the boundary of the parameter space, the closure of the family $\{\Sigma_{(v,t)}\}_{(v,t)\in B^4\times[-\pi,\pi]}$ in $\mathcal{Z}_2(S^3)$ constituted a nice continuous 5-cycle relative to the space of round spheres $\mathcal{G}$. In other words, the discontinuity of the canonical family was being caused by the parametrization of the conformal maps of S^3 chosen in (1).

To address this issue we performed a blow-up procedure along the surface Σ and we were able to reparametrize the canonical family by a continuous map $\Phi : I^5 \to \mathcal{Z}_2(S^3)$. The image $\Phi(I^5)$ is equal to the closure of $\{\Sigma_{(v,t)}\}_{(v,t)\in B^4\times[-\pi,\pi]}$ in $\mathcal{Z}_2(S^3)$. Moreover, we have:

(A) $\sup_{x\in I^5} \text{area}(\Phi(x)) = \sup_{(v,t)\in B^4\times[-\pi,\pi]} \text{area}(\Sigma_{(v,t)}) \le \mathcal{W}(\Sigma)$;
(B) $\Phi(x,0) = \Phi(x,1) = 0$ for any $x \in I^4$;
(C) for any $x \in \partial I^4$ there exists $Q(x) \in S^3$ such that $\Phi(x,t)$ is a sphere of radius πt centered at $Q(x)$ for every $t \in I$.

The explicit expression for the center map $Q : \partial I^4 \to S^3$ mentioned in property (C) can be found in [39].

Finally, we discovered a key topological fact:

(D) the degree of the center map $Q : S^3 \to S^3$ is equal to the genus of Σ. Hence it is nonzero by assumption.

This point is absolutely crucial because it shows that the topology of the surface Σ, i.e. the information of its genus, determines topological properties of the map $\Phi : I^5 \to \mathcal{Z}_2(S^3)$. Note that if the surface Σ we start with is a topological sphere, our approach could not work because the Willmore conjecture fails in this case. The genus of the surface Σ enters in our approach via property (D).

In [39], we proved the following result:

Theorem 3.3 ($2\pi^2$ Theorem) *Consider a continuous map $\Phi : I^5 \to \mathcal{Z}_2(S^3)$ satisfying properties (B), (C), and (D). Then*

$$\sup_{x\in I^5} \text{area}\big(\Phi(x)\big) \ge 2\pi^2.$$

We now briefly sketch some of the main ideas behind the proof of the $2\pi^2$ Theorem. The first thing to show is that

$$L\big([\Phi]\big) > 4\pi = \sup_{x\in\partial I^5} \text{area}\big(\Phi(x)\big). \tag{5}$$

The proof is by contradiction, hence assume $L([\Phi]) = 4\pi$. Let $\mathcal{R}$ denote the space of all oriented great spheres. This space is canonically homeomorphic to S^3 by identifying a great sphere with its center. With this notation, we have from condition (C) that

$$\Phi\big(\partial I^4 \times \{1/2\}\big) \subset \mathcal{R}.$$

Moreover, the degree of the map $\Phi : \partial I^4 \times \{1/2\} \to \mathcal{R} \approx S^3$ is equal to $\deg(Q)$ by property (D), and thus it is nonzero. For simplicity, suppose we can find $\Psi \in [\Phi]$ so that

$$\sup_{x \in I^5} \operatorname{area}\big(\Psi(x)\big) = \mathbf{L}\big([\Phi]\big) = 4\pi.$$

The basic fact is that, given any continuous path $\gamma : [0,1] \to I^5$ connecting $I^4 \times \{0\}$ to $I^4 \times \{1\}$, the map $\Psi \circ \gamma : [0,1] \to \mathcal{Z}_2(S^3)$ is a one-parameter sweep-out of S^3 with

$$\sup_{t \in [0,1]} \operatorname{area}\big((\Psi \circ \gamma)(t)\big) \leq 4\pi.$$

But 4π is the optimal area for the one-parameter min-max in S^3. Hence

$$\sup_{t \in [0,1]} \operatorname{area}\big((\Psi \circ \gamma)(t)\big) = 4\pi,$$

and there must exist some $t_0 \in (0,1)$ such that $\Psi(\gamma(t_0))$ is a great sphere, i.e., such that $\Psi(\gamma(t_0)) \in \mathcal{R}$.

Using this, we argue in [39] that there should be a 4-dimensional submanifold R in I^5, separating the top from the bottom of the cube, such that $\Psi(R) \subset \mathcal{R}$ and $\partial R = \partial I^4 \times \{1/2\}$. Hence

$$\Psi_*[\partial R] = \partial\big[\Psi(R)\big] = 0 \quad \text{in } H_3(\mathcal{R}, \mathbb{Z}).$$

On the other hand, $\Psi = \Phi$ on $\partial R = \partial I^4 \times \{1/2\}$ and so

$$\Psi_*[\partial R] = \Phi_*\big[\partial I^4 \times \{1/2\}\big] = \deg(Q)[\mathcal{R}] \neq 0.$$

This is a contradiction, hence $L([\Phi]) > 4\pi$.

Because of this strict inequality, we can invoke the min-max Theorem and obtain a closed minimal surface $\hat{\Sigma} \subset S^3$, possibly disconnected and with integer multiplicities, such that $L([\Phi]) = \operatorname{area}(\hat{\Sigma})$. Since the area of any compact minimal surface in S^3 is at least 4π, we assume that $\hat{\Sigma}$ is connected with multiplicity one. Otherwise $L([\Phi]) = \operatorname{area}(\hat{\Sigma}) \geq 8\pi > 2\pi^2$ and we would be done.

It is natural to expect that $\hat{\Sigma}$ has Morse index at most five because Φ is defined on a 5-cube. Urbano's Theorem would imply in this case that $\hat{\Sigma}$ should be either an equator or a Clifford torus. But since $L([\Phi]) = \operatorname{area}(\hat{\Sigma}) > 4\pi$, the surface $\hat{\Sigma}$ would have to be a Clifford torus, and then

$$2\pi^2 = \operatorname{area}(\hat{\Sigma}) = L\big([\Phi]\big) \leq \sup_{x \in I^5} \operatorname{area}\big(\Phi(x)\big).$$

Because the Morse index estimate in the Almgren-Pitts theory is not available, we had to exploit the extra structure coming from the canonical family to get an Morse index estimate. See [39] for more details.

Proof of Theorem 1.1 We can now put everything together and explain how to prove the inequality in Theorem 1.1. Through a stereographic projection, we can think of

Σ as a compact surface with positive genus in S^3. The canonical family associated to Σ gives us a map $\Phi : I^5 \to \mathcal{Z}_2(S^3)$ satisfying properties (A), (B), (C), and (D). We use the $2\pi^2$ Theorem and property (A) to conclude that

$$2\pi^2 \leq \sup_{x \in I^5} \operatorname{area}\big(\Phi(x)\big) \leq \mathcal{W}(\Sigma).$$

In the equality case $\mathcal{W}(\Sigma) = 2\pi^2$, there must exist some (v_0, t_0) such that $\operatorname{area}(\Sigma_{(v_0,t_0)}) = \mathcal{W}(\Sigma_{v_0}) = 2\pi^2$. With some extra work (see [39]) we prove that $t_0 = 0$ and that Σ_{v_0} is the Clifford torus. Since $\Sigma = F_{v_0}^{-1}(\Sigma_{v_0})$ and F_{v_0} is conformal, the rigidity case also follows. □

4 Beyond the Willmore Conjecture

The study of the Willmore energy is a beautiful subject which, as we could see in Sects. 2 and 3, has brought together ideas from conformal geometry, geometric analysis, algebraic geometry, partial differential equations and geometric measure theory. Nonetheless many fundamental questions remain unanswered. We finish by discussing some of them.

A basic question is to determine the minimizing shape among surfaces of genus g in $\mathbb{R}^3$, for $g \geq 2$. A conjecture of Kusner [28] states that the minimizer is $\xi_{1,g}$, a genus g minimal surface found by Lawson [36]. Some numerical evidence for this conjecture was provided in [24]. It would be interesting to determine the Morse index of $\xi_{1,2}$.

Another natural question is whether the Clifford torus also minimizes the Willmore energy among tori in $\mathbb{R}^4$. Li and Yau [37] showed that the Willmore energy of any immersed $\mathbb{RP}^2$ in $\mathbb{R}^4$ is greater than or equal to 6π. This value is optimal because of the Veronese surface. It would be nice to have an analogue of Urbano's Theorem in this setting.

For surfaces in $\mathbb{CP}^2$, Montiel and Urbano [45] showed that the quantity $\mathcal{W}(\Sigma) = \int_\Sigma 2 + |H|^2 d\mu$ (the Willmore energy) is conformally invariant. They conjectured that the Clifford torus minimizes the Willmore energy among all tori. The Willmore energy of the Clifford torus in $\mathbb{CP}^2$ is equal to $8\pi^2/3\sqrt{3}$. For comparison, the Willmore energy of a complex projective line is 2π and there are totally geodesic projective planes with Willmore energy equal to 4π. Montiel and Urbano also showed that the Willmore energy of complex tori is greater than or equal to $6\pi > 8\pi^2/3\sqrt{3}$, which gives some evidence towards their conjecture. If answered positively, this conjecture would help in finding the nontrivial Special Lagrangian cone in $\mathbb{C}^3$ with least possible density.

Another interesting problem [27] is to determine the infimum of the Willmore energy in $\mathbb{R}^3$ or $\mathbb{R}^4$ among all non-orientable surfaces of a given genus or among all surfaces in a given regular homotopy class (for instance, in the regular homotopy class of a twisted torus). As far as we know, it is not even known whether the minimum is attained. In a related problem, Kusner [28] conjectured that a surface in $\mathbb{R}^4$ with Willmore energy smaller than 6π has to be a sphere.

The study of the gradient flow of the Willmore energy, referred to as the Willmore flow, also suggests many unanswered questions. For instance, it is not known whether the flow develops finite time singularities but there is numerical evidence [40] showing that they can occur. In [6] an analytical study was done and it is shown that singularities happen in finite or infinite time. From Bryant's classification [8], we know that the lowest energy of a non-umbilical immersed Willmore sphere is 16π. Hence one would expect that Willmore spheres with energy 16π could be perturbed so that the flow exists for all time and converges to a round sphere. Lamm and Nguyen classified the singularity models that could potentially arise in this situation [32].

There is an interesting relation with the process of turning a sphere inside out. The existence of these deformations, called sphere eversions, was discovered by Smale long ago [61]. The point is that, by a result of Banchoff and Max [3], every sphere eversion must pass through a sphere that has at least one quadruple point. The Willmore energy of such sphere is by Li-Yau's Theorem greater than or equal to 16π. The existence of a Willmore flow line connecting a Willmore sphere of energy 16π to a round sphere would therefore provide an optimal sphere eversion. This possibility was proposed by Kusner and confirmed experimentally in [18].

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SURVEY ARTICLE

Thurston's Vision and the Virtual Fibering Theorem for 3-Manifolds

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Abstract The vision and results of William Thurston (1946–2012) have shaped the theory of 3-dimensional manifolds for the last four decades. The high point was Perelman's proof of Thurston's Geometrization Conjecture which reduced 3-manifold topology for the most part to the study of hyperbolic 3-manifolds. In 1982 Thurston gave a list of 24 questions and challenges on hyperbolic 3-manifolds. The most daring one came to be known as the Virtual Fibering Conjecture. We will give some background for the conjecture and we will explain its precise content. We will then report on the recent proof of the conjecture by Ian Agol and Dani Wise.

Keywords Geometrization Theorem · Virtual Fibering Theorem · Fibered 3-manifolds

Mathematics Subject Classification (2010) Primary 57M10 · Secondary 57M05 · 57M50

1 Introduction

The development of the theory of 3-dimensional manifolds, henceforth referred to as 3-manifolds, does not start out with a theorem, but with the formulation of the following conjecture by Henri Poincaré [42, 58] in 1904: The 3-dimensional sphere is the only simply connected, closed 3-manifold. We will give a definition of 'simply connected' in Sect. 3.1, but at this stage it suffices that in essence the Poincaré Conjecture gives an elegant, intrinsic and purely topological characterization of the 3-dimensional sphere.

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Progress towards resolving the Poincaré Conjecture was virtually non-existent for over 50 years. In fact, the first genuinely non-trivial result in 3-manifold topology was proved only in 1957 by Christos 'Papa' Papakyriakopoulos [38]. His proof of 'Dehn's Lemma'[1] removed a major stumbling block which had held back the development of 3-manifold topology for many decades. This result was in particular instrumental in the work of Wolfgang Haken [24] and Friedhelm Waldhausen [59] who resolved many classification problems for what became known as 'Haken manifolds'. We will give a definition of a 'Haken manifold' in Sect. 5.2. For the time being it is enough to know that many natural examples of 3-manifolds, e.g. complements of non-trivial knots in S^3, are Haken manifolds.

The modern theory of 3-manifolds is for the most part due to the results and the vision of William Thurston [55, 56]. In the 1970s Thurston developed the point of view that 3-manifolds should be viewed as geometric objects. In particular he formulated the Geometrization Conjecture which loosely speaking states that for the most part the study of 3-manifolds can be reduced to the study of hyperbolic 3-manifolds. As we will see in Sect. 6, the Geometrization Conjecture can be viewed as a far reaching generalization of the aforementioned Poincaré Conjecture. In an amazing tour de force Thurston proved the Geometrization Conjecture for all Haken manifolds. This proof was announced in 1979, but due to the complexity of the argument it took another 20 years before all details had finally been established rigorously and had appeared in print.

The full Geometrization Conjecture, and thus in particular the Poincaré Conjecture, was finally proved by Grisha Perelman [35, 39–41]. The question thus became, what can we say about hyperbolic 3-manifolds? What do hyperbolic 3-manifolds look like? To this effect Thurston [56] posed 24 questions and challenges which have been guiding 3-manifold topologists over the last 30 years.

Arguably the most famous of these questions is the following that we quote verbatim:

> '*Does every hyperbolic* 3*-manifold have a finite-sheeted cover which fibers over the circle? This dubious-sounding question seems to have a definite chance for a positive answer.*'

We will give a definition of 'fibers over the circle' in Sect. 4.2 and we will give a definition of 'finite-sheeted cover' in Sect. 7. In a slightly simplified formulation the question asks whether every hyperbolic 3-manifolds can be given a particularly simple description after doing a certain basic modification. According to [19] the 'question was upgraded in 1984' to the 'Virtual Fibering Conjecture', i.e. it was conjectured by Thurston that the question should be answered in the affirmative.

The recent article by Otal [37] discussing Thurston's famous article [56] shows in particular that many of the aforementioned 24 questions and challenges were answered in the years after Thurston formulated them, and that each time his vision was

[1]Dehn's Lemma says that 'if c is an embedded curve on the boundary of a 3-manifold N such that c bounds an immersed disk in N, then it already bounds an embedded disk in ∂N'. This statement goes back to Max Dehn [12] in 1910, but Hellmuth Kneser [28, p. 260] found a gap in the proof provided by Dehn. It then took another 30 years to find a correct proof.

vindicated. But for many years there had been only very scant evidence towards the Virtual Fibering Conjecture. It seems fair to say that nobody really had an idea for how to address the conjecture. The situation changed dramatically within the last couple of years with the revolutionary work of Ian Agol [2, 3] and Dani Wise [61–63]. As a consequence, in April 2012, just before William Thurston's untimely death, the Virtual Fibering Conjecture was finally proved. The work of Agol and Wise is easily the greatest step forward in 3-manifold topology since Perelman's proof of the Geometrization Theorem. In some sense it is arguably an even more astounding achievement: everybody 'knew' that the Geometrization Conjecture just had to be true, but researchers had rather mixed opinions on Thurston's Virtual Fibering Conjecture.

Prerequisites and Further Reading Our goal has been to write a paper which on the one hand is accessible for mathematicians who might have only a modest background in topology, but which on the other hand is also interesting for researchers in the field. This dual goal creates some unavoidable tensions. For example, our attempt at making the paper as accessible as possible leads at times to consciously vague and imprecise formulations. We refer the reader to the indicated references for precise statements. On the other hand at times we need to use technical terms, but hopefully one can follow the flow of the story, even if one treats some terms as black boxes.

We refer to [4], [7] and [9] for more technical and detailed accounts of the work of Agol and Wise.

Organization This paper is organized as follows. In Sect. 2 we will first give an introduction to manifolds and we will revisit the classification of surfaces, i.e. of 2-manifolds, and of geometric structures on surfaces. In Sect. 3 we will introduce the notion of a simply connected space and of the fundamental group of a space, and we will have a quick peek at manifolds of dimension greater than three. Afterwards we finally settle for the 3-dimensional case. In Sect. 4 we will provide ourselves with some examples of 3-manifolds to work with and in Sect. 5 we will introduce two special types of 3-manifolds. In Sect. 6 we will discuss the Geometrization Theorem, its relation to the Poincaré Conjecture, the partial proof by Thurston and the full proof by Perelman. In Sect. 7 we will explain in detail the statement of the Virtual Fibering Conjecture and we will report on its proof by Ian Agol and Dani Wise. We conclude this note with an exposition in Sect. 8 of the last of Thurston's challenges that is still open.

Conventions On several occasions we will use a mathematical term in quotes. This means that we will not give a definition, and the term can be treated as a black box. In some cases, e.g. when we use the term 'non-positively curved cube complex', the name will hopefully give at least a vague feeling of what it stands for.

2 Surfaces

2.1 The Definition of a Manifold

Loosely speaking, an *n-dimensional manifold* is an object which at any given point looks like we are in $\mathbb{R}^n$. Some low-dimensional examples are given as follows:

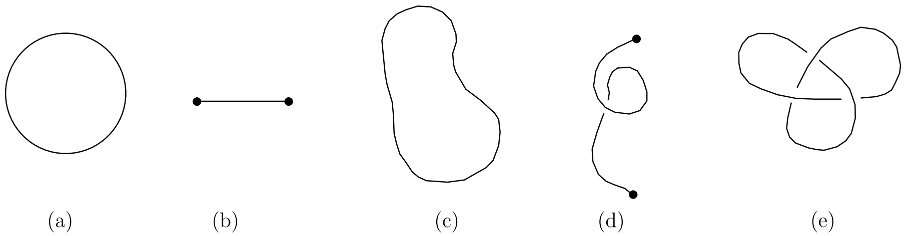

Fig. 1 Examples of 1-manifolds

(1) a point is a 0-dimensional manifold,
(2) a curve is a 1-dimensional manifold,
(3) a surface, e.g. the surface of a ball, of a donut, of a pretzel or of the earth, is a 2-dimensional manifold,
(4) the physical universe, as we personally experience it, is a 3-dimensional manifold, and
(5) spacetime is a 4-dimensional manifold.

Following the usual terminology we will subsequently refer to an n-dimensional manifold as an *n-manifold*. Furthermore, to be on the safe side, throughout this note we will make the technical assumption that all manifolds are connected, orientable, differentiable and 'compact', unless we say explicitly otherwise. We will not attempt to give the definition of 'compact'. Loosely speaking it means that we are only interested in 'finite' manifolds; for example the euclidean space $\mathbb{R}^n$ is not compact.

We are interested in the study of the intrinsic shapes of manifolds. If two manifolds have the same intrinsic shape then they are called *homeomorphic*. For example, let us consider the 1-manifolds sketched in Fig. 1. The 1-manifolds (a), (c) and (e) have the property that 'walking along the 1-manifold one eventually ends up at the starting point'. On the other hand (b) and (d) behave qualitatively very differently: walking along either of the 1-manifolds we eventually come to an end. The mathematically precise way of saying this is that the 1-manifolds (a), (c) and (e) are homeomorphic to each other and that the 1-manifolds (b) and (d) are homeomorphic to each other, but none of the former manifolds is homeomorphic to either of the latter.

In the following we say that a manifold is *closed* if it has no boundary, i.e. walking on the manifold one never reaches an end. For example, the manifolds (a), (c) and (e) are closed, the other ones are not. This notion also makes sense in other dimensions, for example the surface of the earth is closed. On the other hand an annulus or the Möbius band are not closed.

With this notion we can now state the classification of 1-manifolds: Any closed 1-manifold is homeomorphic to (a) and any non-closed 1-manifold is homeomorphic to (b).[2]

[2]Here recall that throughout the paper we restrict ourselves to compact manifolds; that is why $\mathbb{R}$ is missing from our list of 1-manifolds.

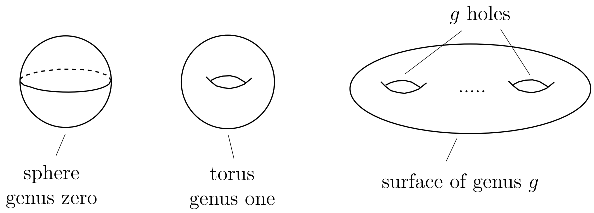

Fig. 2 Examples of surfaces

2.2 The Classification of 2-Manifolds

After stating the classification of 1-manifolds we now move up one dimension. As usual we will refer to a 2-manifold as a surface. Just like for curves, the classification of surfaces up to homeomorphism is quite straightforward to state. First of all, any closed surface is homeomorphic to the standard surface of genus g that is shown in Fig. 2. Furthermore, g is uniquely determined. For instance, the surface of a pretzel is a surface of genus three.

The classification of surfaces with boundary is only slightly more complicated: any surface with boundary is homeomorphic to the result of removing k disjoint 'open' disks from a surface of genus g. Again g and k are uniquely determined. For example, a disk is homeomorphic to the surface one obtains from deleting one 'open' disk from the 2-sphere. Furthermore, one obtains the annulus by deleting two 'open' disks from the 2-sphere.

This classification of (orientable) surfaces seems obvious. After all, what other (orientable) surfaces should there be? But a quick look at non-orientable surfaces, such as the Möbius band, shows that perhaps we can easily become a victim of our intuition. Even among more experienced mathematicians, how many readers feel comfortable in stating the classification of non-orientable surfaces?

2.3 Geometric Structures on Surfaces

Before we proceed to other dimensions we want to consider surfaces as geometric objects.[3] This is a fascinating story in its own right and it will also help us later on in our study of 3-manifolds.

The most familiar geometries are of course euclidean geometry and spherical geometry, but it has been known since the early 1800s that these two geometries are naturally complemented by hyperbolic geometry. In Fig. 3 we show the 2-sphere, the euclidean plane and the Poincaré disk model for hyperbolic geometry.[4] In each picture we sketch a triangle formed by geodesics. In the euclidean plane the angle sum of a triangle is of course equal to π. The fact that the sphere is positively curved implies that the angle sum of a triangle is always greater than π. On the other hand

[3] This topic was also discussed by Klaus Ecker [13] in an earlier Jahresbericht.

[4] In the Poincaré disk model for hyperbolic geometry the space is given by an open disk in $\mathbb{R}^2$ and the geodesics are given by the segments of Euclidean circles which intersect the given disk orthogonally. This model of hyperbolic geometry inspired M.C. Escher to create his famous woodcuts Circle Limit I, II, III and IV.

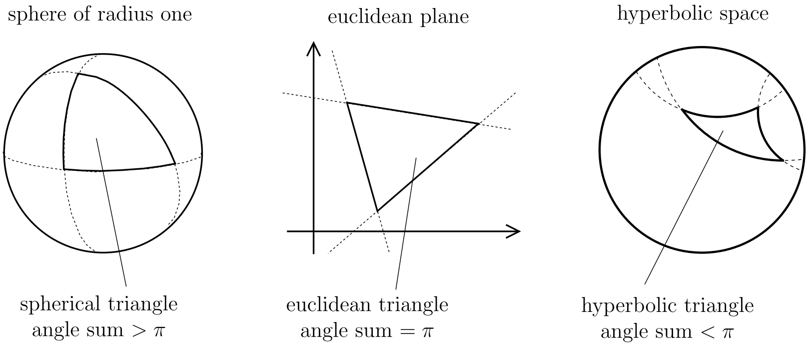

Fig. 3 The spherical, euclidean and hyperbolic geometry

the angle sum of a triangle in the hyperbolic space is always less than π, reflecting the fact that hyperbolic space is negatively curved.

We can now state one of the most beautiful findings of 19th century mathematics: Every surface is either spherical, euclidean or hyperbolic. This means that every surface can be equipped with a 'metric' such that at each point the surface looks like either the 2-sphere, or the euclidean plane or hyperbolic space.

In the following we will quickly outline where these metrics come from. Almost by definition the 2-sphere has a spherical metric. Moving on, the fact that the annulus admits a euclidean metric follows from the observation that we can roll a piece of paper into a tube (which is homeomorphic to an annulus) without stretching or wrinkling. Now we turn to the 2-dimensional torus. We will in fact present not just one, but three arguments that the torus supports a euclidean metric.

(1) The first approach generalizes what we did for the annulus. The torus can be built out of the tube by gluing the two ends together. If we do the bending and gluing in $\mathbb{R}^3$, then we have to deform the tube, and the resulting torus is no longer euclidean. On the other hand, if we do the bending and gluing in $\mathbb{R}^4$, then this can be done without stretching and the resulting torus has a euclidean metric.
(2) As we illustrate in Fig. 4, we can build the torus out of the humble unit square in $\mathbb{R}^2$ by gluing the opposite sides together. While gluing we have to ensure that the metrics match up. The gluing can be performed via isometries and a moment's thought shows that the result is indeed a euclidean metric on the torus.
(3) Finally, the most concise but also the most abstract way of seeing an euclidean structure on the torus is by realizing the torus as the quotient space $\mathbb{R}^2/\mathbb{Z}^2$.

Now we turn to a surface of genus $g \geq 2$. We will try to see how far we can get with the second approach that we took for the torus. We obtained the torus by gluing the opposite sides of a square, i.e. of a regular 4-gon. Similarly one can obtain the surface of genus g by gluing the sides of a regular $4g$-gon in an appropriate way. In order to simplify our discussion and our pictures henceforth we restrict ourselves to the case $g = 2$. We consider the regular octagon on the left of Fig. 5 and we glue the sides with the same symbol to each other in such a way that the orientations match.

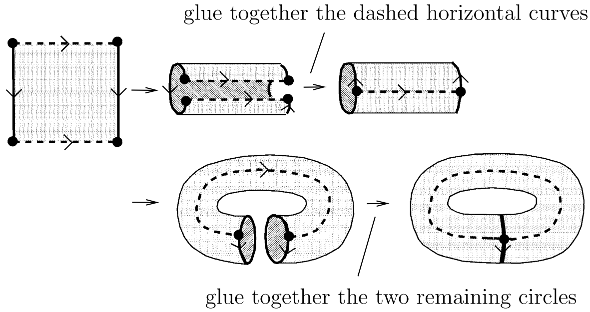

Fig. 4 By gluing together opposing sides of a square we obtain a torus

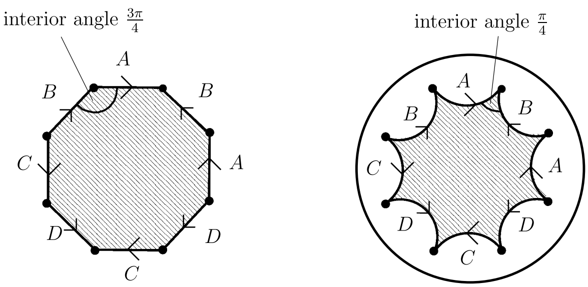

Fig. 5 Construction of a hyperbolic metric on a surface of genus 2

With very well-trained eyes one can spot that we just obtained a surface of genus 2. (For less well-trained eyes an illustration is given in [14, Chap. 7.1].)

Again we can do the gluing via isometries, but now there is an obstacle. In order to see the problem, note that all the eight vertices of the octagon get glued together and form one point on the surface. The eight interior angles of the octagon form one full angle around the new point. But the interior angles add up to $8 \cdot \frac{3\pi}{4} = 6\pi$, which is much more than a full circle. Thus we see that this attempt of finding a euclidean metric on the surface of genus 2 has failed. In fact there is a deeper reason why this approach does not work: it is an immediate consequence of the Gauss–Bonnet theorem that the surface of genus 2 cannot admit a euclidean metric.

We will now modify the approach. Instead of a euclidean octagon we will use a hyperbolic octagon. As we already pointed out, the salient feature of hyperbolic geometry is that the angle sum of n-gons is smaller than for euclidean n-gons. In fact there exists a regular hyperbolic octagon such that the interior angle at each vertex is $\frac{\pi}{4}$, see Fig. 5 on the right. Now we use reflections and translations in the hyperbolic plane to perform the same type of gluings as before. Again we obtain a surface of genus 2, but this time the interior angles add up to $8 \cdot \frac{\pi}{4} = 2\pi$, so we obtain a hyperbolic metric on the surface.

In fact, playing with the construction, using irregular hyperbolic octagons, one can produce many more hyperbolic metrics which are pairwise non-isometric. (For example they can be distinguished by the length of the shortest closed geodesic.) We refer to [52] for a much more detailed discussion of metrics on surfaces.

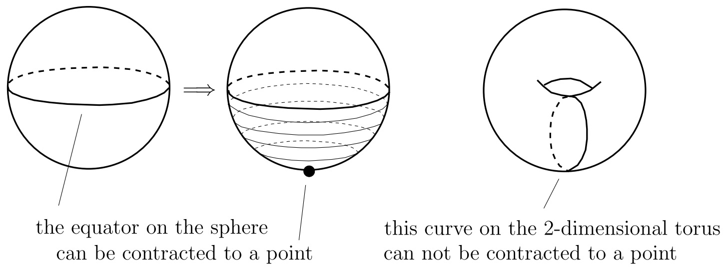

Fig. 6 Loops on the sphere and on the torus

3 Manifolds of Dimension Greater than Three

Now we want to have a quick peek at manifolds of dimension greater than three. At this point it is helpful to introduce the notion of a 'simply connected' manifold, which we already used in the formulation of the Poincaré Conjecture.

3.1 Simply Connected Manifolds and the Fundamental Group

Loosely speaking, a space is said to be *simply connected* if every lasso in the space can be pulled tight. A little more precisely, a space is called simply connected if every loop in the space can be contracted to a point. In Fig. 6 we show on the left that the equator on the sphere can be contracted to a point. In fact the 2-sphere, and also all spheres of dimension greater than two are simply connected.

In Fig. 6 on the right we show a loop on the 2-dimensional torus which cannot be contracted to a point. The torus, and in fact any surface of genus greater than zero, is not simply connected. Together with the classification of closed surfaces in Sect. 2.2 this gives us the Poincaré Conjecture in dimension 2: the 2-sphere is the only simply connected, closed 2-manifold.

Given a connected space X, the *fundamental group* $\pi_1(X)$ of a space is a group which 'measures' how far X is from being simply connected. More precisely, the fundamental group $\pi_1(X)$ is trivial if and only if the space X is simply connected. For example, the fundamental group of the torus is $\mathbb{Z}^2$ and the fundamental group of a surface of genus greater than one is an infinite non-abelian group. In fact the fundamental group of any hyperbolic manifold, of any dimension, is infinite and non-abelian.

3.2 Manifolds of Dimension Greater than Three

Now we will have a quick look at manifolds whose dimensions are greater than four. Somewhat surprisingly, the 'extra room' one has in the high-dimensional setup makes some of the classification problems much easier. For example the natural generalization of the Poincaré Conjecture to higher dimensions was proved in dimensions

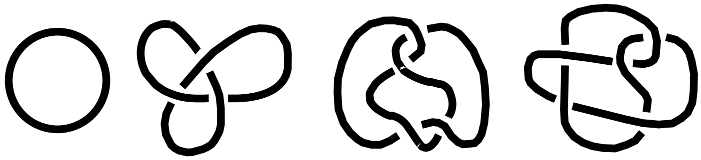

Fig. 7 Examples of knots

greater than four by Stephen Smale [51] in the early 1960s. This predates Perelman's proof of the original Poincaré Conjecture by a wide margin.[5]

Despite this success, once one considers non-simply connected manifolds, a very different picture emerges. It is relatively easy to show, see e.g. [11, Theorem 5.1.1], that given any $n \geq 4$ one has enough flexibility in constructing n-manifolds to realize any 'finitely presented' group as the fundamental group of a closed n-manifold. This saddles all problems from group theory onto topology. For example, Sergei Adyan [1] showed that 'finitely presented' groups cannot be classified, which then implies that it is impossible to classify closed n-manifolds. Here 'impossible to classify' is meant in the strongest terms: not only are we at this moment not able to classify those manifolds, in fact there cannot exist an algorithm which determines whether or not two given closed n-manifolds are homeomorphic. We refer to [53, Sect. 9.4] for a detailed discussion.

4 Examples of 3-Manifolds

Before we delve into the theory of 3-manifolds it is convenient to equip ourselves with some examples of 3-manifolds. In order to avoid pathologies we henceforth only consider 3-manifolds which are either closed or such that the boundary consists of a union of tori.

4.1 Knot Complements

The easiest example of a 3-manifold is of course the 3-sphere S^3, which is by definition the sphere of radius one in $\mathbb{R}^4$. We obtain many more examples of 3-manifolds by taking the complement of a knot in the 3-sphere. Here, loosely speaking, a knot is a tied up piece of rope as shown in Fig. 7. (More technically speaking, in this paper a knot is an embedded open solid torus in S^3.) As we will see later on, this deceptively easy way of constructing 3-manifolds is in fact a surprisingly rich source of examples.

It is an amusing visual exercise to convince oneself that the complement of the trivial knot in S^3 is a solid torus.

[5]The Poincaré Conjecture in dimension four was proved by Michael Freedman [15] in 1982. More precisely, he showed that any simply connected closed, topological 4-manifold is *homeomorphic* to S^4. It is not known, whether any simply connected, closed, differential 4-manifold is *diffeomorphic* to S^4. Resolving that question is often considered as the hardest problem in low-dimensional topology.

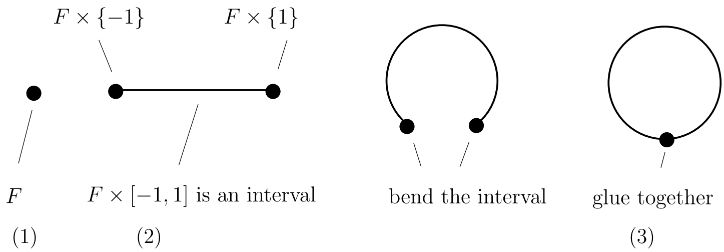

Fig. 8 Building a circle as a fibered manifold

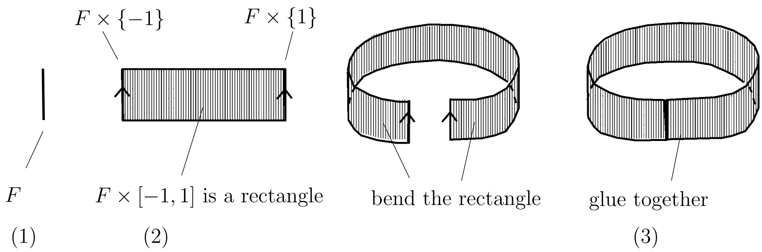

Fig. 9 Building an annulus as a fibered manifold

4.2 Fibered Manifolds

Now we turn to a general three-step procedure for building an $(n + 1)$-manifold out of an n-manifold:

(1) pick an n-manifold F,
(2) consider the product $F \times [-1, 1]$,
(3) glue the manifold $F \times \{-1\}$ on the left to the manifold $F \times \{1\}$ on the right.

The result is an $(n+1)$-dimensional manifold. This new manifold can be viewed as a disjoint union of copies of the manifold F. In fact there is a 'circle's worth of copies' of F. We therefore say that the resulting $(n + 1)$-manifold *fibers over the circle* and we refer to each copy of F as a *fiber*. We sometimes simplify the language and we just say that the $(n + 1)$-manifold is *fibered*.

Let us look at several low-dimensional examples to get a feeling for the definition of a fibered manifold. If we take F to be a 0-dimensional manifold, i.e. a point, then the resulting fibered 1-manifold is a circle; see Fig. 8.

We move on to the next dimension and we take F to be an interval. In Fig. 9 we see that we obtain the annulus by gluing the interval on the left to the interval on the right in the 'obvious way'.

While gluing the 'left interval' to the 'right interval' we notice that we can also glue in a different way: instead of gluing as in Fig. 9 we can also first perform a twist and then glue. As we can see in Fig. 10 the result is a Möbius band.

Thus we see that in Step (3) above we have a choice for how to glue the left to the right. This can be formalized as follows: given a homeomorphism $f : F \to F$ the

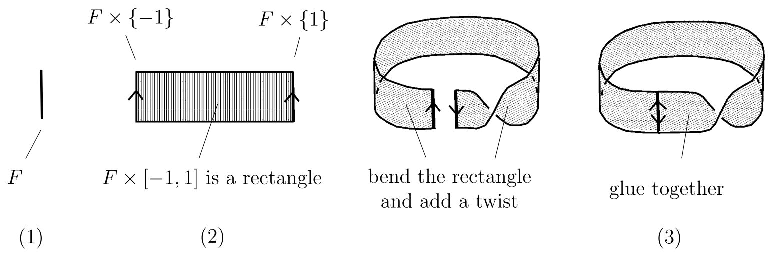

Fig. 10 Building a Möbius band as a fibered manifold

mapping torus

$$F \times [-1,1]/(x,-1) \sim \big(f(x),1\big)$$

is an $(n+1)$-manifold. For example, if $F = S^1 := \{z \in \mathbb{C} : |z| = 1\}$ and $f = \mathrm{id}$, then the corresponding fibered 2-manifold is the torus. On the other hand, if $F = S^1$ and $f(z) := \bar{z}$ where $z \in S^1$, then the resulting fibered 2-manifold is the Klein bottle. It is an entertaining exercise to try to visualize the latter construction.

Now we used up all 1-manifolds and all self-homeomorphisms of 1-manifolds. This implies, that the list of fibered 2-manifolds we just constructed is complete. Summarizing, even if we do not demand orientability, there are only four fibered 2-manifolds, namely the annulus, the Möbius band, the torus, and the Klein bottle.

The picture is quite different once we increase the dimension by one. In fact every surface of genus greater than zero has many 'different' self-homeomorphisms, and thus gives rise to infinitely many distinct fibered 3-manifolds.

Returning to the knot complements, very well-trained eyes can spot that the complements of the first three knots of Fig. 6 are fibered (see e.g. [47, Chap. 10.I] for the trefoil), whereas the complement of the knot 5_2 is not fibered. Even though in this sample most knot complements are fibered, as so often, 'small' examples give the wrong impression. In fact the complement of a 'generic' knot is not fibered. More generally, Joseph Maher [33] showed that a 'generic' 3-manifold is not fibered.

4.3 Gluing Handlebodies

In the previous section we constructed 3-manifolds by a gluing construction. We will now present a somewhat different gluing construction which was introduced by Poul Heegaard at the beginning of the 20th century. We start out with two handlebodies H and H' of the same genus. (A *handlebody of genus g* is the 3-manifold that is bounded by the standard genus g surface in $\mathbb{R}^3$ that is shown in Fig. 2.) We obtain a closed 3-manifold by gluing H to H' along the respective boundaries. For example, if we take H and H' to be handlebodies of genus zero, i.e. H and H' are copies of the 3-ball, then this construction gives rise to S^3.

This construction is of particular interest since one can show, surprisingly easily, that any closed 3-manifold can be obtained from two handlebodies, using a suitable gluing.

5 Special Types of 3-Manifolds

In this section we will introduce Seifert manifolds and Haken manifolds. Both classes of 3-manifolds will play a rôle in the subsequent sections. Nonetheless, if the reader is already drowning in new definitions, then this section can safely be skipped at a first reading. It suffices to know that both types of manifolds have 'enough topology' to be amenable to classical methods. In particular Seifert manifolds have been classified and Haken manifolds are relatively accessible.

5.1 Seifert Manifolds

A *Seifert manifold*, or alternatively *Seifert fibered manifold*, is defined as a 'singular circle bundle over a surface'. Some of the 3-manifolds we are already familiar with are Seifert manifolds. For example the solid torus, which can be viewed as a disjoint union of circles, is a Seifert manifold. Also, the 'Hopf fibration' shows that the 3-sphere is a Seifert manifold. With some practice one can also detect that the complement of the trefoil is a Seifert manifold. But most knot complements, e.g. the complements of the figure-8 knot and the knot 5_2, and in fact most 3-manifolds are not Seifert.

For the remainder of the paper we are not concerned with the precise definition of a Seifert manifold. What matters to us is that Seifert manifolds were completely classified by Herbert Seifert, see [49], in 1933. In particular it follows fairly quickly from the definitions that the 3-sphere is the only simply connected Seifert manifold.

5.2 Haken Manifolds

A 3-manifold M is said to be *prime* if it cannot be written as a 'connected sum' of two manifolds $M_1, M_2 \neq S^3$. A 3-manifold M is called *Haken* if it is prime and if it admits an incompressible surface, i.e. a surface F of genus ≥ 1 such that the inclusion induced map $\pi_1(F) \to \pi_1(M)$ is injective.

For example most fibered 3-manifolds are Haken. Indeed, let F be a surface of genus ≥ 1 and let $f : F \to F$ be a self-homeomorphism, then F is an incompressible surface in the corresponding fibered 3-manifold. More interestingly perhaps, basically every prime 3-manifold with boundary is Haken. In particular the complement of any non-trivial knot is Haken. On the other hand, 3-manifolds with finite fundamental groups are non-Haken. There are also many examples of 3-manifolds with infinite fundamental groups that are non-Haken, see e.g. [5] for references.

The reason Haken manifolds play such an important rôle in 3-manifold topology is the fact that they always admit a *hierarchy*. This means that given a Haken manifold M there exists a finite sequence of manifolds $M = M_1, \ldots, M_k$ such that each M_i is obtained from the previous manifold M_{i-1} by cutting along an incompressible surface, and such that the final 3-manifold M_k is a union of 3-dimensional balls. Many theorems for Haken manifolds have been proved by induction on the minimal length of such a sequence.

6 The Geometrization Theorem

In the previous sections we have seen all kinds of 3-manifolds, and at first glance 3-manifolds seem to form a rather confusing zoo. The question thus arises whether one can restore some order by finding a classification scheme or a unifying theme. Up to the mid 1970s the only 3-manifolds that were somewhat understood were Seifert manifolds and Haken manifolds. Both types of 3-manifolds are amenable to purely topological methods. But for the remaining manifolds one had no topological tools to work with, and one had absolutely no idea how to study them.

Since purely topological methods failed to deliver, it is (in hindsight!) natural to ask whether perhaps geometric methods can be brought to bear. For example, we had previously seen that every closed surface admits a geometric structure, and that for all but the two simplest closed surfaces one can exhibit hyperbolic metrics using a fairly straightforward gluing construction. Is the situation similar for 3-manifolds?

For a long time it looked like the question should be answered in the negative. In the first decades of the 20th century a few examples of hyperbolic 3-manifolds were explicitly constructed by Hugo Gieseking [20], Frank Löbell [32] and Herbert Seifert and Constantin Weber [50], but in the following 40 years no new examples of hyperbolic 3-manifolds were found. In the 1970s events suddenly took a dramatic turn. First, to everybody's surprise Robert Riley [45] showed that many knot complements, and in particular the complement of the figure-8 knot, admit a hyperbolic structure.[6]

Shortly afterwards William Thurston [55, 56] formulated the Geometrization Conjecture, which in a slightly simplified form can be formulated as follows.[7]

Geometrization Conjecture *Every* 3*-manifold admits a canonical decomposition along a* (*possibly empty*) *collection of spheres and incompressible tori*, *such that each of the resulting* 3*-manifolds is either a Seifert manifold or hyperbolic*.

To get a better understanding of the Geometrization Conjecture let us look at several examples which were already known by the time it was formulated. We have already seen several examples of Seifert manifolds and hyperbolic manifolds. In these cases one evidently does not need to decompose any further to obtain the desired result. More interestingly, if we glue the complement of the trefoil to the complement of the figure-8 knot along the boundary tori, then the resulting manifold is neither a Seifert manifold nor hyperbolic. But if we cut this manifold along the gluing torus, then the two resulting components are of course the complement of the trefoil and the complement of the figure-8 knot. Put differently, decomposing along a torus we obtain two 3-manifolds, one of which is a Seifert manifold and one of which is hyperbolic.

As we mentioned in the introduction, the Geometrization Conjecture implies the Poincaré Conjecture. It is a custom in mathematics talks to provide at least one proof.

[6]Riley [46] points out that the complement of the figure-8 knot is in fact the 2-fold cover of Gieseking's example. Hugo Gieseking was killed in France in 1915, shortly after his work on hyperbolic 3-manifold. It is conceivable that hyperbolic structures on knot complements would have been discovered much earlier if it had not been for World War I.

[7]For knots the conjecture was foreshadowed by Riley, see [46] for Riley's account.

We will do the same here, and we will quickly outline why the Geometrization Conjecture implies the Poincaré Conjecture: Let M be a simply connected closed 3-manifold and suppose the Geometrization Conjecture holds. Some basic algebraic topology quickly implies that the decomposition of M provided by the Geometrization Conjecture has to be trivial. Put differently, M is either already a Seifert manifold or a hyperbolic manifold. In Sect. 3.1 we already pointed out that the fundamental group of a hyperbolic manifold is infinite. It remains to deal with the former case. But as we already mentioned in Sect. 5.1, the classification of Seifert manifolds readily implies that M is indeed the 3-sphere.

The first major step towards a proof of the Geometrization Conjecture was Thurston's 'Monster Theorem' from the late 1970s, namely the proof of the Geometrization Theorem for Haken manifolds. As we hinted at in the previous section, the proof uses an induction argument on hierarchies. But along the way Thurston also introduced a wealth of new concepts and ideas, many of which developed into major fields of study in their own right. William Thurston was awarded the Fields medal in 1983,[8] but it took about 20 years and the efforts of many authors for all details to be written down rigorously. It is worth reading Thurston's interesting argument [57] why he did not provide the detailed proof himself.

The full proof of the Geometrization Conjecture was finally given by Perelman [39–41] in 2003 using the Ricci flow on Riemannian metrics, building on ideas pioneered by Richard Hamilton [25]. A detailed exposition of Perelman's proof is provided by John Morgan and Gang Tian [35], also an accessible outline of the ideas is given by Klaus Ecker [13] in an earlier Jahresbericht. Perelman declined the Fields medal which was awarded to him in 2006. He also declined the $1,000,000 prize offered to him by the Clay Institute for solving one of the seven Millenium Prize Problems.

It is impossible to overstate the importance of the Geometrization Theorem to 3-manifold topology. Not only does it resolve the Poincaré Conjecture, but it underpins almost every deep result on 3-manifolds. For example, it lies at the heart of the algorithm which can determine whether or not two given closed 3-manifolds are homeomorphic. (We refer to [6] for precise references.)

7 The Virtual Fibering Theorem

The Geometrization Theorem says that any 3-manifold can be canonically decomposed into Seifert manifolds and hyperbolic manifolds. In fact, in a precise sense a 'generic' 3-manifold does not need to be decomposed: it is already hyperbolic. (We refer to [5, Sect. 1] for references.[9]) Thus we see that hyperbolic 3-manifolds

[8] The ICM took place in 1983 in Warsaw. It was of course supposed to take place in 1982 but it was postponed by one year because of martial law in Poland which was in effect from December 1981 to July 1983.

[9] Interestingly the phenomenon that 'most objects are hyperbolic' also occurs in the context of group theory. Mikhail Gromov [21] showed that in a precise sense a generic finitely presented group is 'word hyperbolic'.

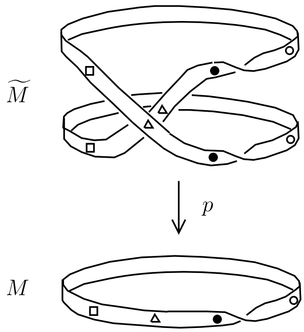

Fig. 11 The twice twisted band is a 2-fold cover of the Möbius band

lie at the heart of 3-manifold topology. The question thus arises, what can we say about the topology of hyperbolic 3-manifolds? As we mentioned before, the generic hyperbolic 3-manifold is not fibered and very many hyperbolic 3-manifolds are non-Haken. This is bad news for us 3-manifold topologists, since this means that there is little 'topology to work with' in the manifold. For example, the absence of an incompressible surface means that we cannot do our favorite trick of cutting 3-manifolds along surfaces into 'smaller' pieces.

Thurston [56] and Waldhausen [59] speculated that perhaps the picture is very different, once we are allowed to consider finite-sheeted covers. Here, loosely speaking, a manifold $\widetilde{M}$ is an n-fold sheeted cover of another manifold M if there exists a continuous map $p : \widetilde{M} \to M$ such that the preimage of each point in M consists of precisely n points in $\widetilde{M}$. In Fig. 11 we see that a twice-twisted band (which is nothing but the annulus) is a 2-fold cover of the Möbius band. The notion of a finite-sheeted cover admittedly takes a while to get used to. Suffice it to say, once one gets one's head around it, it is a very natural and central concept in topology and geometry.

In the following we say that a manifold *virtually* has a certain property, if it admits a finite-sheeted cover which has this property. For instance, as we saw above, the Möbius band is virtually an annulus. Similarly we say that a group virtually has a given property, if it admits a finite index subgroup which has that property. For example, every finitely generated abelian group is virtually torsion-free.

With this definition we can now restate Thurston's question from the introduction as follows:

Is every hyperbolic 3*-manifold virtually fibered?*

The formulation of this and closely related questions in [56] led to many decades of intense research, an overview of the results is given in [30], [5, Sect. 5.9] and [37]. But it seems fair to say that progress was limited. In fact there did not even emerge a consensus on whether one expects an affirmative or a negative answer to the above question. The first major step forward finally happened in 2007 when Ian Agol [2] proved the following theorem.

Theorem A *If M is a prime* 3*-manifold such that its fundamental group $\pi_1(M)$ is infinite and virtually 'RFRS', then M is virtually fibered.*

Here the acronym 'RFRS' stands for 'residually finite-rationally solvable', which, in all likelihood, for most readers is not particularly enlightening. In fact the precise definition of RFRS is of no concern to us. But suffice it to say that being RFRS is a very strong condition on the fundamental group of M. In fact it is so strong that at least the author of this article initially thought that the theorem would apply to a minuscule number of 3-manifolds.

It quickly turned out that this assessment was far of the mark. In 2009 Dani Wise [61–63] announced a proof, which eventually turned out to be nearly 200 pages long, that the fundamental groups of 'most' Haken hyperbolic 3-manifolds are virtually the fundamental group of a 'special cube complex'. Frédéric Haglund and Dani Wise [23] and Ian Agol [2] in turn showed that fundamental groups of special cube complexes are virtually RFRS. The proof given by Wise, once again, used a particularly intricate argument based on hierarchies. Wise's proof was a tremendous achievement, but non-Haken manifolds still seemed intractable.

At the same time, in a completely independent development, Jeremy Kahn and Vlad Markovic [27] used dynamics on hyperbolic 3-manifolds to show that fundamental groups of closed hyperbolic 3-manifolds have 'lots' of surface subgroups. By work of Nicolas Bergeron and Dani Wise [8] and Michah Sageev [48] this implies that the fundamental group of any closed hyperbolic 3-manifold is the fundamental group of a 'non-positively curved cube complex'. In light of the aforementioned result of Haglund and Wise [23] the challenge now became to promote a 'non-positively curved cube complex' to a 'special cube complex'.

This challenge could be formulated as a problem in geometric group theory, which a priori has nothing to do with 3-manifold topology. It was finally once again Agol [3], building on deep theorems of Wise [62, 63], who rose to the challenge in 2012.[10] Putting the results of Agol [3] and Wise [61–63] together gives us the following theorem.

Theorem B *The fundamental group of any hyperbolic* 3*-manifold is virtually RFRS.*

Finally, the combination of Theorems A and B gives us the desired affirmative answer to Thurston's question.

The Virtual Fibering Theorem *Every hyperbolic* 3*-manifold is virtually fibered.*

The results of Agol and Wise were rounded off by Piotr Przytycki and Dani Wise [44] who showed that Theorem B in fact holds for 'most' non-hyperbolic 3-manifolds as well.

As happens so often when an important long-standing conjecture is finally proved, the eventual proof of the Virtual Fibering Theorem delivered much more than just an answer to the initial question. In a recent book by Matthias Aschenbrenner, the author and Henry Wilton [5] it takes very dense 13 pages to just list all of the immediate

[10] It is characteristic of Ian Agol's unassuming character that the first time he publicly mentioned this result was towards the end of an introductory lecture for graduate students in Paris. Thanks to the digital camera of the author and a blog the news of Agol's theorem spread across the world of 3-manifold topologists within a few hours [60].

consequences of the work of Agol, Przytycki and Wise. Among them, the author's favorite implication is that the fundamental group of any hyperbolic 3-manifold M is linear over the integers, i.e. $\pi_1(M)$ embeds into $\mathrm{GL}(n, \mathbb{Z})$ for a suitable n. This result is totally unexpected; nobody had even dared to conjecture it before it was proved.

The results of Agol and Wise have produced a seismic shift in our understanding of 3-manifolds and related fields. For example, besides direct applications to 3-manifolds [16, 54] there have already been applications to the Cannon Conjecture [34], free-by-cyclic groups [22], and 4-manifolds with a fixed-point free circle action [10, 17, 18]. Certainly there will be many more applications in the near future, and it will surely take several years before the full impact of the work of Agol and Wise has been absorbed.

8 Thurston's Last Challenge

The field of 3-manifold topology has now undoubtedly developed a certain maturity. Nonetheless there are still many basic questions that are wide open. Already the subfield of knot theory bursts with easy-to-state but depressingly hard-to-answer questions. For example, it is still unknown whether the Jones polynomial, first introduced by Vaughan Jones [26] in 1985, detects the trivial knot. A weaker version of this question was recently answered by Peter Kronheimer and Tom Mrowka [29] in a major tour de force using Instanton Floer Homology. Many more 'elementary' knot theoretic questions and conjectures are given in a recent survey by Marc Lackenby [31].

We want to conclude this article with the formulation of the one challenge of Thurston's that is still open. In order to do so we first note that a hyperbolic metric gives us naturally a notion of a volume. A priori this volume depends of course on the choice of the hyperbolic metric. In Sect. 2.3 we had hinted at the fact that most surfaces admit many non-pairwise isometric hyperbolic metrics. Amazingly the situation is radically different in dimension 3: George Mostow [36] and Gopal Prasad [43] showed that all hyperbolic structures on a given 3-manifold are isometric. In particular the volume of a hyperbolic 3-manifold is independent of the choice of the hyperbolic structure.

Now we can finally quote Thurston's remaining challenge:

Show that volumes of hyperbolic 3*-manifolds are not all rationally related.*

Put differently, the challenge is to find two hyperbolic 3-manifolds N and M such that the ratio of the volumes is not a rational number. This challenge is related to very hard number theoretic problems, which explains why it has not been answered yet. In fact we know so little about volumes of hyperbolic 3-manifolds that it is still unknown whether there exists a hyperbolic 3-manifold such that the volume is rational (or irrational). A short discussion of this challenge can be found in the aforementioned article by Jean-Pierre Otal [37].

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CLASSICS REVISITED

The *Cours d'Analyse Infinitésimale* of Charles-Jean de La Vallée Poussin: From Innovation to Tradition

Jean Mawhin

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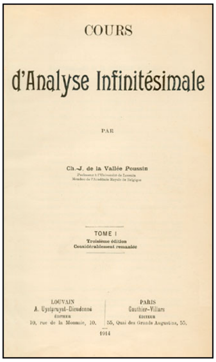
COURS

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PAR

Ch.-J. de la Vallée Poussin

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The French mathematical literature has a long tradition of extensive textbooks in mathematical analysis, which seems to have started with the publication of more or less extended versions of the lectures given at the École polytechnique, and later in Faculties of Science. Let us just mention the most famous ones published in the XIX$^{\text{th}}$ century by Lagrange, Lacroix, Cauchy, Sturm, Bertrand, Hermite, Jordan at the *École polytechnique*, and by Picard and Goursat at the Faculty of Science of Paris.

The corresponding Belgian production is less impressive, started later and is dominated by the *Cours d'analyse infinitésimale* of Charles-Jean de La Vallée Poussin (1866–1962) (in short VP), a two-volumes set, which remains famous (like its author!) for an exceptionally long life and for its international influence, both consequences of its innovative, elegant and rigorous character. This book remained until 1970 the basis for the course on differential and integral calculus of the first two years of studies leading, at the *Université Catholique de Louvain*, either to the degree of *docteur en sciences physiques et mathématiques* (later *licencié en sciences mathématiques* and *licencié en sciences physiques*) or to the degree of *ingénieur civil*. The engineer candidates formed the majority of the class population, and the content and level of the course was a compromise between the wishes of the pure and of the applied users of the material.

VP, who remains famous for his proof (independently of Jacques Hadamard (1865–1963)) of the prime number theorem, and for his original contributions to in-

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tegration, Fourier series, approximation theory, conformal maps and potential theory, graduated in engineering at the *Université Catholique de Louvain* in 1890, and in mathematics and physics in 1891. He was called by the same university in October 1891 to replace Louis-Philippe Gilbert (1832–1892), who was ill, for the course of differential and integral calculus. Gilbert died in 1892, and, from this moment, VP was in charge of those lectures until 1935 (see [24, Vol. 1] for more details).

Gilbert had already traced the way by publishing several editions of a *Cours d'analyse infinitésimale. Partie élémentaire* [26], covering his lectures on differential and integral calculus. If the 1st edition (1872) was still defining continuity through the intermediate value property, and claimed its equivalence with Cauchy's definition as well as the differentiability of any continuous function, all this was amended in the 2nd edition (1878), where Darboux's example of a nowhere differentiable continuous function was presented. The 3rd edition (1887) incorporated the recent advances made in Germany on the foundations of analysis and Gilbert, in the *Préface*, defended the opinion that rigor is not the enemy of simplicity. The posthumous 4th edition (1892), used by VP during several years as a support of his lectures, was replaced in 1898–1899 by a mimeographed *Cours d'analyse infinitésimale* in two volumes [3], that can be seen as the "zeroth edition" of VP's famous book. The order and presentation of the material remained very close to the last edition of Gilbert's treatise.

1 The First Edition (1903–1906)

Volume I of the 1st edition of VP's *Cours d'Analyse infinitésimale* [9], was published in 1903, in a format motivated in the *Preface* by its author who explains that

> this book [...] must serve together the future engineers and the students preparing the doctorate. To reach this double aim, we have adopted two different types. The text in large ones for the beginners, the one in small types providing to the preceding one all the necessary complements for advanced studies. In the text in large types, the questions are presented in their most elementary form, without never renouncing to rigor. They are reconsidered in the text in small types [...] under the most general viewpoint. [...] The text in large types of this first volume contains the material of the lectures of the first year.

The end of the *Preface* reveals that the sources of the author's inspiration are

> the *Cours d'analyse* of Mr. C. Jordan, the one of our former master Ph. Gilbert, so remarkable by its qualities of exposition, and also the *Leçons sur les applications géométriques de l'analyse* by M. Raffy.

Several of the first papers of VP had been motivated by correcting some mistakes in the 1st edition of the *Cours d'analyse* of Camille Jordan (1838–1922), who acknowledged him in the *Preface* of Volume 2 of the substantially modified 2nd edition of his *Cours*.

> We had given for the conditions under which the differentiation under the sign $\int$ is legitimate an inexact assertion. Trying to correct this error, that M. de la

> Vallée Poussin had mentioned to us, we have been led to discuss in detail the essential propositions concerning the definite integrals.

A first noticeable difference with the mimeographed edition is the larger space devoted, in the introductory part, to the theory of real numbers, based upon Dedekind's cuts, followed by a much more precise and detailed study of the continuous functions of one or several variables and their properties. There is little novelty in the chapters on differentiation. One should notice that, the author "proved" the chain rule for $f[x(t), y(t)]$ under the too weak assumptions that $f(x, y)$ has partial derivatives, and $x(t)$ and $y(t)$ are differentiable, leading to some imprecision in stating the validity conditions for Taylor's formula. An important addition is a precise study of the existence and uniqueness of implicit functions, using the intermediate value property for one unknown function followed by induction on the dimension. This is applied, in small types, to a rigorous presentation of Lagrange multipliers for constrained extrema.

In the presentation of definite simple integrals, Darboux's approach for Riemann integral is used, with, in small types, a general version of the rule of change of variables, and a thorough treatment of Darboux lower and upper integrals. Further conditions for Riemann integrability require considerations on the topology of one-dimensional sets and their Jordan's measurability. At this occasion, the *characteristic function e* of a bounded set E is introduced, and Jordan's outer and inner lengths of E are respectively defined as the upper and the lower integral of e over some $[a, b] \supset E$. As noticed by Hawkins [29],

> this connection between measure and integration could not have been stated more clearly.

In the applications of calculus to the geometry of curves and surfaces, an orientation is given to the binormal of a curve in space, which provides a sign to the torsion. After a classical presentation of the length of a curve, Jordan's viewpoint based upon functions with bounded variation is adopted and a new proof of Jordan's theorem for simple closed curves is given, with applications to curvilinear integrals. The last chapter, devoted to series, contains Bertrand's logarithmic criteria and du Bois-Reymond's theorem on the impossibility of constructing a complete scale of convergence and divergence. Weierstrass' example of nowhere differentiable continuous function is presented, and completed in VP's paper [8].

The 1^{st} edition of Volume 2 of the *Cours d'analyse infinitésimale* [11] came out in 1906. In the *Avertissement*, it is said that one begins

> by presenting the theory of double integrals, of improper integrals, and in particular of Eulerian integrals in the simplest possible way, [...] rather different from the classical one, but equally natural for students.

This is essentially an approach, for continuous functions of two variables, by two successive simple integrations, followed by showing its equivalence to the limit of some double sums. Triple integrals are defined in a similar way. Special attention is paid to Jacobians and change of variable in multiple integrals. Three definitions are given for the area of a surface, and the reduction formulas of volume integrals into surface integrals and of surface integrals into line integrals are proved with care, but in a rather literary way. A more general study of multiple integrals using again Darboux's approach is printed in small types, following the required properties of the sets of integration and their Jordan's measurability.

For improper simple integrals, the distinction between convergence and absolute convergence is introduced. In the case of improper double integrals, special attention is paid to their reduction to simple ones, a topic to which VP contributed in 1899 [7]. His results [5] from 1892 on the uniform convergence of parametric improper integrals are also presented.

The substantial part of the volume devoted to ordinary differential equations starts with an original way of solving Cauchy's problem $y' = f(x, y)$, $y(x_0) = y_0$, when f and f'_y are continuous near (x_0, y_0). The idea consists in constructing step by step, for every $\alpha > 0$, an approximate solution by the formula

$$\varphi(x_0) = y_0, \quad \varphi(x) = y_0 + \int_{x_0}^{x} f\big(s, \varphi(s-\alpha)\big)\, ds,$$

and showing that its limit for $\alpha \to 0$ is a solution of the Cauchy problem. Rediscovered in 1925 by Leonida Tonelli (1885–1946) [44], this approach now bears his name. In the explicit integration of special differential equations, the discussion of Riccati's equation is more developed than in most textbooks. The treatment of linear differential equations and systems is rather standard, except for an unusual attention paid to Bessel's equation, and the same is true for the short introduction to first order partial and total differential equations.

A completely new chapter with respect to the mimeographed version is entitled: "Special questions: circular and Eulerian functions. Fourier series". One finds there the expression of circular and hyperbolic functions as infinite products and series of fractions, Bernoulli's numbers and polynomials, and a very detailed treatment of Euler's Beta and Gamma functions. Their use in analytic number theory may have motivated this choice, and most sections are printed in small types. Trigonometric and Fourier series are presented following Dirichlet's approach, and Cantor's uniqueness theorem of the trigonometric expansion is proved. To respect Belgian official programs, the volume ends with chapters on the calculus of variations and the calculus of differences (including Euler's summation formula). A study of the singular points of planar curves and of the curves defined on surfaces concludes the volume.

The *Avertissement* of Volume 2 ends by observing that

> this volume having taken considerable proportions, we have renounced to include the principles of the theory of functions of a complex variable. We hope to be able to publish this theory later with other questions.

This should take place in a Volume 3 announced several times but never published. The restriction to functions of real variables remained a characteristic distinction of VP's treatise, with respect to the other more or less contemporary French ones by Jordan, Émile Picard (1856–1941), Édouard Goursat (1858–1936), Georges Humbert (1859–1921), René Baire (1874–1932), and others.

2 The Second Edition (1909–1912)

Presenting the second edition of the *Cours d'analyse infinitésimale* as "considerably reworked" is not an understatement. In the *Preface* of Volume I, VP, after recalling that the structure and the use of different types was conserved,

> will not insist on the many modifications made to our first redaction and will only mention here the main one. The theory of definite integrals has been completely renewed, since our first edition, by the beautiful writings of M. Lebesgue. We have thought necessary to introduce in this course the fundamental results obtained in this new way; but we have rather considerably modified the proofs of the author, in order to eliminate the notion of transfinite, which has not yet entered our teaching methods.

This introduction in a textbook of the new integral introduced in 1902 by Henri Lebesgue (1875–1941) [33] requires a deeper study of set theory. The introductory part is completed, in small types, by a description of the cardinality of sets and of perfect sets. In the chapter on differentiation, the Dini derivatives are considered, including Scheeffer's theorem. For definite integrals, Darboux's presentation of Riemann integral is now followed by measure theory in the real line according to Émile Borel (1871–1956) and Lebesgue, starting with Borel-Lebesgue's lemma and its consequences. The exterior and interior measures are introduced, as well as Borel measurable sets. A necessary and sufficient condition for measurability is proved, followed by the properties of measure with respect to the operations on sets. Measurable real functions of a real variable are defined through the measurability of the counter-image of intervals.

The integral for bounded measurable functions is introduced following Lebesgue's division of the range of the function. Two definitions for the extension to unbounded functions are proposed, including the cut-off procedure introduced by VP in [5] for improper absolute integrals. The existence of primitives for not necessarily continuous functions follows a new way, whose interest was underlined by Lebesgue himself in his 1910 memoir on integration of functions of several variables [34], recalling that

> in my *Leçons sur l'intégration*, where I treated the case of one variable, to compare the indefinite integral of $f(x)$ to a function $F(x)$ assumed to exist and having f for derivative, I tried to evaluate $F(b) - F(a) - \int_a^b f(x)\,dx$, a and b arbitrary and fixed, by replacing the curve $F(x) = y$ by a polygonal line circumscribed for which, consequently, one can define the sides using $f(x)$. [...] M. de la Vallée-Poussin proceeds differently; he compares the function $F(x)$, x variable, to the indefinite integral $\int f(x)\,dx$ using theorems which generalize the fundamental theorem: *two functions having everywhere the same derivative only differ by a constant*, or Ludwig Scheeffer's theorem. To do this, M. de la Vallée-Poussin uses functions close to $\int f(x)\,dx$ chosen in such a way that one knows their derivative almost everywhere. This a method imitated from the one of M. de la Vallée-Poussin that I use here. I consider as an advantage of this method to require only a minor study of the functions of several variables.

The remaining of Volume I is essentially unchanged with respect to the 1^{st} edition. The section on rectifiable curves is enriched by Lebesgue's result about the differentiability almost everywhere of continuous functions having bounded variation, and the formula for the length as a Lebesgue integral. The chapter on series contains Arzelá's quasi-uniform continuity and Lebesgue's theorem on the integration of sums of series of integrable functions.

In the *Preface* to the 2^{nd} edition of Volume II, VP explained that

> the whole redaction of Volume II has seen more or less deep modifications, but the most important one follows from the introduction of the multiple integrals of M. Lebesgue. We have presented this theory following the fundamental memoirs of the author and we have been led to treat a new question which provides interesting applications of it, namely the development of functions in series of polynomials. Furthermore, the theory of trigonometric series, which owes also to M. Lebesgue its most important advances, has been completely rewritten and adapted to the level of the present knowledge. However, because of lack of space, we have sacrified the theory of Eulerian integrals contained in the first edition.

The revision starts with Chapter III, now entitled "Multiple integrals of Riemann and Lebesgue". Measure theory and Lebesgue integral follow the lines of the one-dimensional case, with the additional property that a bounded measurable function can be arbitrary closely approximated by a continuous one outside of a subset of arbitrary small measure. This result is generally attributed to Nikolai N. Lusin (1883–1950), but Lusin's note [35] is dated June 17, 1912 and VP's Volume II, May 15, 1912, so that the contributions are independent.

The study of indefinite multiple integral follows the two years old memoir of Lebesgue [34]. The *indefinite integral* of an integrable function f on E is defined on measurable subsets $e \subset E$, with measure $m(e)$, by $F(e) = \int_e f(x)\,dx$, and provides an example of countably additive and absolutely continuous set function ($F(e) \to 0$ when $m(e) \to 0$). The study of its derivative requires the introduction of Vitali's covering theorem. The derivative of F in restricted sense at x is the limit of $F(\gamma)/m(\gamma)$, where γ is a ball centered at x whose radius tends to zero. The general derivative of F at x is the limit of $F(\omega)/m(\omega)$ when ω is a measurable set containing x whose measure tends to 0 in such a way that the ratio of $m(\omega)$ with the measure of the smallest ball $\gamma \supset \omega$ remains bounded away from zero (regularity condition). The main result is Lebesgue's theorem stating that a countably additive and absolutely continuous set function has a unique finite derivative almost everywhere and is the indefinite integral of this derivative. Lebesgue-Fubini's reduction theorem for multiple integrals, which takes its most elegant and general form within Lebesgue integration theory, is followed by the Leibniz rule for differentiation under the integral sign and the Green formula in the same setting.

The approximation of functions by polynomials is treated in Chapter IV. For a continuous function f on $[a, b]$ with, without loss of generality, $0 < a < b < 1$, the n^{th} approximating polynomial P_n is defined by the integral formula

$$P_n(x) = \frac{3 \cdot 5 \cdot \ldots \cdot (2n+1)}{2(2 \cdot 4 \cdot \ldots \cdot 2n)} \int_a^b f(u)\big[1 - (u-x)^2\big]^n\,du.$$

This integral, that VP thought to be new when he introduced it in 1908 [13], had been considered a few months earlier by Edmund Landau (1877–1938) for the same purpose [32]. For a Lebesgue integrable function f, the theorem of Frederic Riesz (1880–1956) [40] that $P_n(x)$ converges to $f(x)$ at any point where $f(x)$ is the derivative of its indefinite integral (and hence almost everywhere) is proved. The approximation of functions of several variables is also considered with, for Lebesgue integrable functions, a variant of a recent theorem of Leonida Tonelli (1885–1946) [43].

Chapter IV ends with the study of trigonometric and Fourier series, completing the 1st edition, in small types, by the important advances made by Lebesgue and others using the new integral. To Dini's and Jordan's convergence criteria of the Fourier series of f, VP added the one he published in 1911 [16], on the convergence to the limit for $\alpha = 0$ of $(1/2\pi)\int_0^\alpha [f(x+\alpha) + f(x-\alpha)]\,d\alpha$, at any point x such that this function has bounded variation in α in a small interval $[0, \varepsilon]$. The summation of divergent Fourier series uses essentially Fejér's method, both for continuous and Lebesgue integrable functions. The new summation method introduced by VP in [13] is only mentioned. Follows Parseval formula for L^2-functions, du Bois-Reymond and Lebesgue singularities for continuous ones. Cantor's uniqueness result is completed by recent results of Lebesgue.

The space given to the integration of total differential equations is tripled with respect to the 1st edition and includes VP's results in [12]. The remaining of Volume II is essentially unchanged, except that Bernoulli's numbers and polynomials appear in the chapter on difference equations. When discussing the envelopes of planar curves, VP's complete treatment given in [14] is quoted.

This 2nd edition of VP's *Cours d'analyse* is, in 1912, the only textbook on analysis containing both Lebesgue integral and its application to Fourier series, and a general theory of approximation of functions by polynomials.

3 The Third Edition (1914) and Its "Ghost" Volume II

Once more, the *Avertissement* to the 3rd edition of Volume I, published in the Spring of 1914, summarizes the modifications introduced there. The first one concerns differentiability where one has

> abandoned the old definition of the total differential and adopted Stolz' one [41]. Its superiority has been emphasized by the works of MM. S. Pierpont [sic] [37], Fréchet [25] and mostly W. H. Young [46].

In 1893, in Volume I of a remarkably modern book on differential and integral calculus [41], Otto Stolz (1842–1905), a former student of Weierstrass, had defined for the first time the modern concept of (total) differential of a real function f of n variables at a point x, equivalent to classical differentiability when $n = 1$, and lying between the existence of the partial derivatives at x and their continuity at x when $n > 1$. In VP's own terminology, the function $u(x, y)$ is differentiable at (x, y) if it is defined in a neighborhood of this point and if its variation $\Delta u = u(x + \Delta x, y + \Delta y) - u(x, y)$ can be decomposed in two parts $(A\Delta x + B\Delta y) + (|\Delta x| + |\Delta y|)\varepsilon$, with A and B independent of Δx and Δy, and $\lim_{|\Delta x|+|\Delta y|\to 0} \varepsilon = 0$. As observed by VP in his *Avertissement*, its

> superiority [...] is unquestionable: the theorems follow more directly from the principles, the theory of differentiation of explicit and implicit functions becomes sharper, and, consequently, more satisfactory.

The (justified) enthusiasm of VP led him to the following version of the implicit function theorem, attributed to William H. Young (1863–1942): given n Functions $F_1, \ldots, F_n$ of the $m+n$ variables $(x, y, \ldots, u, v, w, \ldots)$ which vanish, are totally differentiable, and have a non-zero Jacobian with respect to the $u, v, w, \ldots$ at $(a, b, \ldots, u_0, v_0, w_0, \ldots)$, there exists at least a system of functions $u, v, w, \ldots$ of $(x, y, \ldots)$ equal to $u_0, v_0, w_0, \ldots$ at $(a, b, \ldots)$, and satisfying identically the equations $F_1 = 0, \ldots, F_n = 0$ in a neighborhood of this point. This result is proved for $n = 1$ using Bolzano's intermediate value theorem, which is correct, but VP's induction argument to go from this result to arbitrary n is not. A correct proof requires more sophisticated tools, like Brouwer's fixed point theorem. Notice that, despite of VP's generous attribution, this generalized implicit function theorem can hardly be found in [46].

The other substantial modification in the 3rd edition of Volume I of the *Cours d'analyse* concerns advanced measure and integration. The main lines are again summarized in the *Avertissement*.

> We have moved to the introductory part, and simplified, measure theory, previously located in the chapter on definite integrals. We have completely reworked the theory of Lebesgue integral, but conserved the processus we had previously introduced to go from the derivative to the primitive. [...] Its use occurs in two new sections, one devoted to the problem of change of variables in a definite integral, which seems to receive here its definitive solution, the other one to the search of the primitive of a generalized second order derivative, fundamental in the theory of Fourier series.

For Lebesgue integration, much more emphasis is put on the limit process under the integral sign, with the dominated convergence and the monotone convergence theorems. For the search of primitive functions, VP's original approach of the 2nd edition is developed by expliciting the concepts of *major* (*minor*) *functions* of a Lebesgue integrable function f on $[a, b]$. They are continuous functions, infinitely close from above (below) to $\int_a^x f(t)\,dt$, having Dini derivatives greater (smaller) than $f(x)$ at any point where f is finite. They are used to express, for a function F having bounded variation on $[a, b]$, $F(b) - F(a)$ in terms of $\int_a^b F'(x)\,dx$ and the total variation of F in the set of points where F' is not determined or infinite (*VP's decomposition theorem*), and to show that the absolute continuity of F is necessary and sufficient for being the indefinite integral of its derivative. Generalizations of the major and minor functions have been the starting point of Oskar Perron (1880–1975), for introducing in 1914 his generalization of Lebesgue integral [36].

The *Encyclopédie des sciences mathématiques* (t. II, vol. 1, p. 100) having reproached to VP's original proof of Jordan's theorem for a closed simple curve "to be only indicated", details are added, "to which one can give a pure arithmetical sense". The volume ends with an "Addition to Volume II (*2nd* edition)", describing VP's recent results [17, 18] on the uniqueness of the expansion of a function in trigonometrical series.

The 3rd edition of Volume II of the *Cours d'analyse infinitésimale* never appeared. A brief explanation is given in the Introduction of VP's paper of 1915 on Lebesgue integral [20] which recalls that

> many of the obtained results [...] were already printed in August 1914 and were supposed to appear at the end of this same year in the 3rd edition of Volume II of my *Cours d'analyse*. All that has been burnt in Louvain with many other more precious things.

During World War I, the German troops invading Belgium reached Louvain ion August 19, 1914. Alleging the activity of franc-tireurs whose existence was never proved, they reacted in a very brutal way. In particular, they set fire to the old buildings of the University of Louvain during the night of August 25, destroying completely the library and its precious collections. The Uystpruyst printing house publishing VP's *Cours d'analyse*, next to the library, burnt as well, including the material related to the 3rd edition of Volume II. VP left Belgium until the end of the war, as, successively, guest professor at Harvard University, the Faculty of Science and the *Collège de France* in Paris, and the University of Geneva. In Paris, he published his famous monograph on Lebesgue integral [21], repeating in its Introduction the story of the 3rd edition of Volume II of the *Cours d'analyse*.

For a long time, the last three sections of [20] on multiple Lebesgue integrals remained the unique source for guessing the evolution of the corresponding chapter in this lost 3rd edition. The main novelty was the concept of derivative on a dyadic net, for completely additive set functions having bounded variation, which, as mentioned by VP, developed a remark made in passing by Lebesgue in [34]. This notion allowed to prove generalizations of the decomposition theorem of Volume I, and to apply it to continuous functions of two variables having bounded variation. The last section of [20] treated the change of variables in double integrals, where, again, Stolz' differentiation was used,

> showing its superiority, once more, in the present question.

Some years ago, when I was working with Paul Butzer and Pasquale Vetro on the publication of the *Collected Works* of VP [24], a great-grandson of the Belgian mathematician showed me a collection of galley proofs he was keeping in memory of his great-grandfather. He kindly allowed me to study them and copy what could be interesting. Most corresponded to published material, but a careful analysis showed me that some where galley proofs of parts of the 3rd edition of Volume II of the *Cours d'analyse*. They covered the end of chapter II on improper integrals and exact differentials, the whole Chapter III on multiple integrals and the beginning of Chapter IV on the analytical representation of functions and Fourier series. Other ones were galley proofs of the 2nd edition of the remaining part of Chapter IV, annotated by VP for the 3rd edition. With this material, that I hope to publish in the future, it is possible to reconstruct the most original part of the "ghost" edition.

With respect to the 2nd edition, Chapter II is enriched by VP's extension [19] of Goursat's technique [27] for proving Cauchy's integral theorem, to the obtention of necessary and sufficient conditions for the line integral of $P\,dx + Q\,dy$, with P and Q totally differentiable, to depend only upon its extremities. In Chapter III, the main addition is the derivative on a dyadic net of additive functions with bounded variation, similar to the one in [20]. The section on change of variables in integrals is the one given in the 2nd edition, except mentioning that the proof comes from VP's paper [15].

For Chapter IV, the modifications are essentially stylistic or consist in adding a few computational details. The reference to VP's memoir [13] is added to the theorem of the approximation of the derivatives of the function by the derivatives of polynomials. The addition to the 2nd edition of Volume II given in the 3rd edition of Volume I, mentioned above, becomes the last section of Chapter IV. A beautiful analysis of the contributions of VP to Fourier series, including the evolution of their treatment in the *Cours d'analyse*, is given by Jean-Pierre Kahane on pp. 573–586 of [24, Vol. 3].

Another collateral damage of World War I to VP's *Cours d'analyse* is that its German translation, planned just before the war, never materialized.

4 The Fourth (1921–1922) and Subsequent Editions

One could think *a priori* that the loss of the 3rd edition of Volume II would have been compensated by the publication of the 4th edition. That it is not the case is explained in the *Avertissement* to this edition of Volume I.

> The printing of Volume I, started in 1919, has met, at the beginning, serious material difficulties. To save time, I have suppressed the questions printed in small types in the old edition and, in particular, the theories related to Lebesgue integral, that I hope to include in a third volume.

The *Avertissement* to 4th edition of Volume II is more precise.

> The 3rd edition of Volume II of this *Cours d'Analyse* was burnt in Louvain in 1914 before its completion and never came to light. It contained a rather extended contribution to set theory and Lebesgue integral. Since this time, I returned to those questions and I have published [...] my book *Intégrales de Lebesgue, fonctions d'ensemble, classes de Baire* (*Paris, Gauthier-Villars, 1916*). For the reasons I have given in the *Avertissement* of Volume I, those questions have been kept apart from the present volume, but maybe will find place with other ones in a Volume III.

This planned Volume III never appeared. With the 4th edition, VP's *Cours d'analyse* more or less returned to the content of the 1st edition and remained essentially unchanged in the many subsequent editions, except for the applications of analysis to geometry, as described later, and for a more classical approach to Cauchy's existence theorem for differential equations. The elegance and rigor of the style had not been lost, but most of the XXth century material had disappeared. In Volume II, the chapter on Eulerian integrals of the 1st edition, suppressed in the 2nd one, was reintroduced, and, for Bessel equation, VP's paper of 1905 [10] on its integration in finite terms was mentioned.

Because of the absence of the 3rd edition of Volume II, the best available complete set for the *Cours d'analyse infinitésimale* is the 3rd edition of Volume I (1914) joined to the 2nd edition of Volume II (1912). They have been translated in Russian, and have been reprinted in 2003 by Gabay in Paris. At the time of their publication, their unique competitor was the first edition of *The Theory of Functions of a Real Variable and the Theory of Fourier Series* [31] published in 1907 by Ernest W. Hobson (1856–1933), giving, in a much less elegant style and a less original way, the first presentation of Lebesgue integral in book form and English language. The first treatise in German on real functions including Lebesgue's measure and integral, published in 1918, was the *Vorlesungen über reelle Funktionen* [2] of Constantin Carathéodory (1873–1950), who wrote that

> äusserst originell sind ausserdem die Darstellungen bei de la Vallée Poussin.

Carathéodory published in 1939 the first volume of another book on real functions [3]. Irony of history, the second volume never came to light, the whole edition being destroyed in the publishing house Teubner during the bombing of Leipzig by the Royal Air Force of December 4, 1943!

Volume I of the *Cours d'analyse* has seen a total of twelve editions, the last one in 1959, and Volume II nine, the last one in 1957, an exceptionally long life for a mathematical book. The difference in the number of editions for the two volumes is explained by the fact that Volume I covered the material for the first year students at the *Université Catholique de Louvain*, and Volume II the one for the second year students, whose population was substantially smaller.

In the *Preface* of the 6th edition of Volume II, VP announced that he

> revised with the greatest care the part of volume devoted to the geometrical applications. The principles of the theory of envelopes, which could look somewhat wavering, have got all the wanted precision. I have thought that the time had come to introduce in the theory of space curves the consideration of the moving frame whose use has much spread. [...] I have developed somewhat more than before the theory of the curvature of surfaces: I have made more precise the interpretation of signs, I have exposed O. Bonnet's theorem establishing the relation between the total curvature of a surface and the geodetic curvature of its contour, and finally I have extended the determination of a surface through its six parameters.

The volume gained some fifty pages, and some of the improvements came from VP's papers [22] and [23].

In 1935, VP abandoned the course on differential and integral calculus to his former student Fernand Simonart (1888–1966), a differential geometer, who contributed to the revisions of the last editions of the *Cours*. The transition was courteously announced in the *Preface* of the 8th edition.

> After having had the honor to teach myself the material of this course at the University of Louvain during forty-five years, I have been particularly happy to see this work assigned to one of my most distinguished former students, M. Fernand Simonart, to-day my colleague since already many years [...]. He has spontaneously offered me his precious help for the revision of material and the publication of this eighth edition. The book has remained essentially the same, but M. Simonart has introduced many improvements of details and judicious additions.

Those additions essentially deal with the geometrical applications of differential and integral calculus. Simonart also introduced the language of vector analysis in the theorems of Green, Stokes and Ostrogradsky.

5 Reception, Influence and Modernity

The preceding lines have, I hope, convinced the reader of the exceptional quality, both in content and in style, of the *Cours d'analyse infinitésimale*. The first editions, with their important changes, received very positive reviews. Jules Tannery (1848–1910) [42] observed, for the 1st edition of Volume I, that

> although the good books on this topic are numerous, nobody will regret the publication of this new course of analysis; the effort made by the author to found exclusively the teaching of analysis on perfectly rigorous notions is worthy of attention, especially because it is really an elementary book that he wanted to write, and that he has written.

The same reviewer noticed, concerning the same edition of Volume II,

> the very simple proof of the existence theorem for ordinary differential equations of the first order, ingeniously based upon the consideration of a function satisfying such an equation with an error smaller than any given quantity.

Robert d'Adhémar (1874–1941) [4] wrote, for the 2nd edition of Volume I, that

> the first edition of this book was excellent; the present one is a true jewel. [...] This chapter [on Lebesgue integral] where M. de la Vallée has put originality and a great force of synthesis, will be remarked by the scientists,

and, for the 2nd edition of Volume II, that

> written in this way by a very rigorous and penetrating mind, the book has the beauty of strong and classical things, and I am sure that more than one professor of analysis, before giving his lecture, [...] will read the corresponding chapter of the book of M. de la Vallée Poussin.

The same author noticed that

> Volume I, 3rd edition, and Volume II, 2nd edition, will be translated in German and we impatiently wait for a Volume III, by which the book would do a precious service to French students. [...] This sole book where one can find a didactic presentation of Lebesgue integral [...] can be compared to the one of M. Camille Jordan. There is nothing more to say.

The analysis was confirmed by Paul Mansion (1844–1919) in JFM 45.1281.03.

> Dieses Buch, dessen deutsche Übersetzung sich beim Ausbruch des Krieges in Vorbereitung befand, ist wohl—besonders wegen der Strenge der Darstellung—das beste unter allen französischen Lehrbüchern.

Reviewing the same volumes for the AMS [38], M.B. Porter observes that

> the handling throughout is clear, elegant, and concise; the various topics are illustrated by numerous carefully chosen examples selected with rare pedagogic skill to develop a real understanding of the text. [...] It is impossible to point out all the merits of these volumes, so rich in varied topics, so lucid in exposition and elegant in presentation. A unique feature of the book is that it does for Lebesgue's integral what Jordan did for Riemann's theory.

For Rolin Wavre [45], analyzing the 4th edition,

> for the real domain, the course of M. de la Vallée-Poussin is, in many points, more detailed than those of MM. Jordan, Picard or Goursat, to which one will often be tempted to compare it. A deeper analysis would be necessary to show the personal contribution of the author to the treated matters. This contribution is undoubtly very important.

For the 5th edition of Volume I, Porter [39] observes that

> even in following the conventional order of the French treatises, de la Vallée Poussin displays his usual elegance and simplicity of presentation so that the most hackneyed matters acquire a new interest. [...] The treatment of indeterminate forms is the best the reviewer knows of. [...] In conclusion, it may be said that this is one of the most valuable handbooks on modern analysis in any language and an English translation of it would be a welcome addition to our literature of the subject.

This wish was not realized, but the 8th edition was reprinted by Dover in 1946.

The influence of VP's *Cours d'analyse infinitésimale* has been deep and long lasting, as revealed by many testimonies. In the Introduction to the 8th edition of his *Course of pure Mathematics* [28], Godefrey Harold Hardy (1877–1947) expressed his indebtedness.

> I have rewritten the parts of Chs. VI and VII which deal with the elementary properties of differential coefficients. Here I have found de la Vallée-Poussin's *Cours d'analyse* the best guide, and I am sure that this part of the book is much improved.

When publishing Volume I of his monograph *Analysis* [30], Einar Hille (1894–1980) recalled in the Introduction that

> fifty years ago, in preparing for a comprehensive examination, I read Ch.J. de la Vallée Poussin *Cours d'analyse infinitésimale*, vol. 1 (1909). This treatise left a lasting impression and, when my book was planned, that of the Belgian master served as the model although the final product differs from it in many respect.

In his obituary of VP [1], John Charles Burkill (1900–1993) noticed that

> the contribution to mathematical literature for which he is most widely known is his *Cours d'Analyse* [...]. If Jordan's is the most noble of the *Cours d'Analyse* and perhaps Goursat's [...] the most widely read, it can hardly be doubted that VP's is the most elegant and lucid. After half a century it is still put before the

> more able undergraduates as a model of style, and there are parts of it which no other writer has presented with anything like the same economy and clarity.

Almost one century after its publication, VP's *Cours d'analyse infinitésimale* in its 4$^{\text{th}}$ or later edition, would not require substantial modifications to be used today as a reference text in an advanced calculus course. The XIX$^{\text{th}}$ century concept of limit of a variable should be replaced by that of limit of a sequence, with a sequence defined as a mapping from the natural integers to the real numbers. The given (correct) proofs of the fundamental properties of a continuous function on a closed interval could be simplified by replacing the rather cumbersome result about the oscillation of the function by Borel-Lebesgue lemma. Similarly for functions of several variables, where the concept of domain bounded by a curve in the plane should be replaced by the more general one of compact. All proofs about differentiation and differentiable functions are still up-to-date. The presentation of the implicit function theorem could be saved by replacing the generalized assumptions mentioned above by the standard ones. The approach to indefinite integrals needs no modification and definite (Riemann) integration in dimension one is well described using Darboux's method.

VP's approach of multiple integrals is somewhat discursive and should be made more rigorous. The presentation of numerical series and series of functions is better and more complete than in many contemporary textbooks, and the same is true for improper, curvilinear and Eulerian integrals. The theory of Fourier series, in the absence of Lebesgue integral, can hardly be improved and the chapters on differential equations do not differ essentially from the corresponding ones in a modern book of advanced calculus. Under the influence of Bourbaki's *Éléments de mathématique*, the mathematical style has been formalized in the second half of the XX$^{\text{th}}$ century, and one could be tempted, to translate VP's *Cours d'analyse infinitésimale* in the language of Jean Dieudonné's *Éléments d'analyse*. The price paid for a greatest apparent rigor would be a substantial loss in style and elegance.

Since the pioneering period where VP was including Lebesgue integration in two early editions of his *Cours d'analyse*, many other approaches have been devised to introduce Lebesgue measure and integral. The question of presenting measure before integral, or integral before measure has been warmly discussed, and is a matter of taste. However, VP's presentation remains, after one century, a very readable and valuable reference for learning Lebesgue's theory in $\mathbb{R}^n$.

6 Conclusion

The *Cours d'analyse infinitésimale* of VP, with his many editions, has been most influential all around the world in providing to beginners a clear and rigorous introduction to the foundations and applications of differential and integral calculus. When travelling abroad and telling that I was professor at the *Université Catholique de Louvain*, I was surprised to hear comments from so many interlocutors, about the fundamental role the *Cours d'analyse* had played in their mathematical training. They all praised the elegance and clarity of the style, the choice of the topics and their presentation.

On the other hand, the 2nd and 3rd editions have been the royal way used by many mathematicians of the first quarter of the XXth century to get acquainted with Lebesgue integral and its application to Fourier series. In the French literature, one had to wait for World War II to see a timid and short introduction to Lebesgue integral in the ultimate edition of Picard's *Traité d'analyse* and in Valiron's *Cours d'analyse*. No one could be compared in elegance and in comprehensiveness to the two volumes of the Belgian mathematician.

One could hardly find an introduction to analysis containing so many original contributions of his author. The *Cours d'analyse infinitésimale* is a beautiful example of the indispensable fruitful relation between research and teaching. VP's teaching influenced his research, and VP's research influenced his teaching. This is the secret of great mathematical books.

In his already quoted paper in [24, Vol. 3], Jean-Pierre Kahane insisted that

> there is something in common to the productions of youth and of maturity of de La Vallée Poussin, about Fourier analysis as well as the other domains of his activity. He is simple, elegant and precise. To read him to-day is both a good lecture of mathematics and a beautiful lecture on language and on style.

I hope this paper will contribute to motivate mathematicians to pay a first or another visit to VP's *Cours d'analyse infinitésimale*.

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BOOK REVIEW

Alexander Soifer: "The Mathematical Coloring Book. Mathematics of Coloring and the Colorful Life of Its Creators"

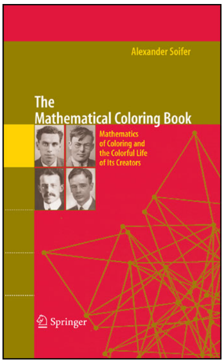
Springer-Verlag, 2009, xxx+607 pp.

Günter M. Ziegler

Published online: 18 September 2014

Keywords Coloring · Ramsey Theory · B.L. van der Waerden

1 A Private Coloring Book



Many Mathematics books, and in particular expository books and text books, tell us a lot about their authors, and thus can and should be seen as personal statements. I believe that it is valuable if they (admit that they) communicate the experiences and the views and the tastes of their authors, starting of course with the selection of topics, the selection of the material to be presented, and in telling the stories not only of the Mathematics, but also of how it was found/created, shaped, and developed.

This is what Alexander Soifer tries to deliver in his *Mathematical Coloring Book*, as he explains in the "mission statement" for his book, as part of a preface called "Greetings to the Reader" (pp. xxvii–xxviii):

> Most books in the field present mathematics as a flower, dried out between pages of an old dusty volume, so dry that the colors are faded and only theorem–proof narrative survives. Along with my previous books, *Mathematical Coloring Book* will strive to become an account of a live mathematics. I hope the book will present mathematics as a human endeavor: the reader should expect to find in it not only results, but also portraits of their creators; not only mathematical facts, but also open problems; not only new mathematical research, but

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> also new historical investigations; not only mathematical aspirations, but also moral dilemmas of the times between and during the two horrific World Wars of the twentieth century. In my view, mathematics is done by human beings, and knowing their lives and cultures enriches our understanding of mathematics as a product of human activity, rather than an abstraction which exists separately from us and comes to us exclusively as a catalog of theorems and formulas. Indeed, new facts and artifacts will be presented that are related to the history of the Chromatic Number of the Plane problem, the early history of Ramsey Theory, the lives of Issai Schur, Pierre Joseph Henry Baudet, and Bartel Leendert van der Waerden.
>
> I hope you will join me on a journey you will never forget, a journey full of passion, where mathematics and history are researched in the process of solving mysteries more exciting than fiction, precisely because those are mysteries of real affairs of human history. Can mathematics be received by all senses, like a vibrant flower, indeed, like life itself? One way to find out is to experience this book.

And in this spirit, Alexander Soifer, born 1948 in Moscow, now Professor of Mathematics, Art & Film History at the University of Colorado in Colorado Springs, and (among other functions) the editor of his own private mathematics journal *Geombinatorics*, has worked on this book for 18 years, from early 1990 until 2008, to now present us his "Mathematical Coloring Book" on xxx+607 pages.

And indeed Soifer's book contains a wealth of interesting and very diverse material. But after struggling with the book for $2\frac{1}{2}$ years on the way to this review I have to say that this book has many faults, starting with the title, the dedication and the many prefaces, but more seriously with the selection of the material, the editorial choices of what to present and what not to present, which lead to fundamental problems in the "historical" sections, but also with its style, or rather: styles.

1.1 What is This Book About?

"Mathematics of Coloring" is not an established or coherent mathematical discipline or topic. Correspondingly, in this book three different themes are subsumed under this heading: The "chromatic number of the plane" (and its variants), treated in Parts II, III and V of *The Mathematical Coloring Book*, the four color problem, treated in Part IV, and a number of problems (coloring the integers, coloring the edges of a graph, etc.) that may be seen as instances of Ramsey Theory, treated in parts I, VI, VII, and VIII. One further part IX connects "coloring the plane" with a Ramsey topic, while the final Part X is called "Predicting the Future". And to a large part this book is a history book, only loosely connected to the Mathematics of Coloring.

1.1.1 The Four Color Theorem

The four color problem, which in modern language asks whether every finite planar graph has a vertex coloring with at most four colors, was first posed in 1852 by Francis Guthrie (1831–1899). This is of course a well-known problem, and treated in great detail elsewhere (see e.g. Biggs et al. [3] and Wilson [15]), including its history.

Soifer uncovered and proudly reproduces the letter from De Morgan to Hamilton that seems to be the first written record of Guthrie's problem. At the same time, my impression is that, despite saying that "Heesch's role is hard to overestimate" (p. 188), Soifer doesn't pay enough attention to Heinrich Heesch (1906–1995), who laid all the major theoretical foundations for the proof of the four color problem. (See Bigalke's biography [2].) Thus history, unfair as it can be, now presents the four color problem as a "Theorem of Appel and Haken," as Appel, Haken and Koch completed the first (massively computer-based) proof in 1976; the proof was reworked later by Robertson et al. [10] and then with a computer-checkable "formal" computer proof by Gonthier in 2005, see [6]. In retrospect one can ask: Was this a good problem? Certainly it was important, as it has driven the development of graph theory to a large extent. Nevertheless, it has made little connections to other parts of Mathematics. In 1972, when Heinrich Heesch applied to the German Science Foundation DFG to fund the computer time that he needed to carry out the computations that would complete his proof, the DFG's referee (reportedly Gerhard Ringel) said that the Four Color Theorem "wouldn't open any further perspectives. From a successful proof one could not expect consequences for more comprehensive mathematical theories." (my translation from [2, p. 222]). So Heesch couldn't complete his proof, and the fame for solving the problem instead went to Hermann Haken and his team.

1.1.2 The Chromatic Number of the Plane

The "chromatic number of the plane problem" asks how many colors you need to color the points of the plane in such a way that no two points of distance exactly 1 get the same color. This is clearly equivalent to the question about the minimal number $\chi(\mathbb{R}^2)$ of subsets A_i that cover the plane $\mathbb{R}^2$ if the subsets are required avoid the distance 1.

This problem is known as the Hadwiger–Nelson problem, but it has in various sources been attributed to Hugo Hadwiger, Edward Nelson, Paul Erdős, Martin Gardner, Frank Harary, John R. Isbell, Leo Moser, and William T. Tutte. As Soifer convincingly argues in Chapter 3 of his book, "Chromatic Number of the Plane: An Historical Essay", the problem was first posed by Edward Nelson (*1932) in 1950, who was then a student at the University of Chicago, later postdoc and since 1959 professor in Princeton. A paper by the Swiss geometer Hugo Hadwiger from 1961 presents the problem and establishes the fairly obvious bounds

$$4 \le \chi\left(\mathbb{R}^2\right) \le 7,$$

which is still all we know on the original version of the problem.

So it's originally Nelson's problem—but in particular Erdős has been instrumental in making this problem popular. Soifer's "who's done it" account of the origins of the problem is fascinating, although I find his lengthy criticism of people who give other attributions a bit tiring, and unnecessary. But clearly here he has an ax to grind, and it is not the only one in this book. If this is a natural problem (it is), it might certainly have been discovered several times. You can argue about how much credit should go to the discoverer(s) of such a problem (Soifer speaks of "authorship"), and how much praise should go to those who popularize such a problem, which in this case included

Erdős, Hadwiger, Gardner, and of course also Soifer himself, who proudly displays his part in this.

So it's a natural problem, but is it a good problem? Since Hadwiger's 1961 paper, there has been no progress on the original problem. Of course a popular method (not only in discrete or "recreational" mathematics) in such a situation is to look at variations. Thus one looks at related quantities such as the chromatic numbers $\chi(\mathbb{R}^d)$ and $\chi(\mathbb{Q}^d)$, and proves bounds for that, or at coverings by sets where no two points have distance approximately 1. One also observes that the answer may depend on whether the sets of colors A_i are required to be measurable (and in which sense), whether they are locally bounded by polygons, etc. One also speculates that the answer may depend on the axioms of set theory one is willing to use, which again is not proven, but can be seen on variations of the problem. The pertinent work by Saharon Shelah with Soifer is presented in detail in Chapter 46 of the book, which opens the last part of the book under review, Part X, "Predicting the Future". Here is the most concrete piece of evidence that is achieved:

Theorem *The minimal number of sets needed to cover the plane, if the sets are not allowed to contain two points of distance* 1 *with a rational difference vector, is* 2 *if "ZFC" is assumed* (*Zermelo–Fraenkel plus axiom of choice*), *while it is at least* 3 *and at most* 7 *if "ZFS" is assumed* (*Zermelo–Fraenkel plus countable axiom of choice and every set of real numbers is Lebesgue measurable*, *what Soifer calls the Zermelo–Fraenkel–Solovay set of axioms*).

Soifer downgrades this result (which is based on the work by Shelah and Soifer [11]) to an example, Example 46.27 (p. 550), and then presents the proof in full detail, taken from a 2007 preprint by Michael S. Payne, then an Australian undergraduate student, whose paper at the time of Soifer's writing had not been published and not even accepted for publication, and who in fact had problems getting this published. For the now published version, see [8].

Other results, for which one might really want to see the proofs, are presented without even a proof sketch. For example, Soifer presents

$$\chi(\mathbb{Q}^2) = \chi(\mathbb{Q}^3) = 2, \qquad \chi(\mathbb{Q}^4) = 4$$

as Results 11.3 and 11.4 in his book, but for the proof he refers to a "legendary unpublished manuscript" by Miro Benda and Micha Perles, whose first version was typed in Brazil in 1976, but then uses the occasion to advertise his own journal *Geombinatorics*, where he eventually published the manuscript in 2000 [1]. Micha Perles has done so beautiful and interesting mathematics, and written/published so little. Thus I feel compelled to here give the spoilers:

Proof for $\chi(\mathbb{Q}^2) = \chi(\mathbb{Q}^3) = 2$. Points in $\mathbb{Q}^2$ can be written in the reduced form $\frac{1}{e}(a,b)$, where a, b, e are integers without a common factor, and $e > 0$. It is sufficient to color the points in the additive subgroup $H_2 \subset \mathbb{Q}^2$ of points in $\mathbb{Q}^2$ that can be connected to the origin by rational vectors of length 1, i.e., that have rational difference vectors of the form $\frac{1}{e}(a,b)$ with $a^2 + b^2 = e^2$, since the plane is partitioned into translates of this set, and points in different translates are never connected by a rational vector of length 1.

Considering this modulo 4, we see that in each such vector $\frac{1}{e}(a,b) \in H_2$ the denominator e is odd, as is exactly one of a and b. Adding two such difference vectors, $\frac{1}{e}(a,b)+\frac{1}{e'}(a',b') = \frac{1}{ee'}(ae'+a'e, be'+b'e)$, we see that as e, e' are odd, the numerators modulo 2 are just added. This means that we can 2-color the set H_0 by assigning to $\frac{1}{e}(a,b)$ the value $a+b$ (mod 2).

The same type of 2-coloring can also be constructed for $\mathbb{Q}^3$. □

Proof for $\chi(\mathbb{Q}^4) = 4$. Here the same type of 2-coloring does not work, as for a reduced vector $\frac{1}{e}(a,b,c,d)$ of length 1 with $a^2+b^2+c^2+d^2 = e^2$, the value of e could be even while all the numerators a,b,c,d are odd. And indeed in this case at least four colors/sets are required, as the rational points $(0,0,0,0)$, $(1,0,0,0)$, and $(\frac{1}{2}, \pm\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ have pairwise distance 1, so they need to get distinct colors.

Now again as before we need only color the points in the additive subgroup $H_4 \subset \mathbb{Q}^4$ of points that can be reached from $(0,0,0,0)$ by rational difference vectors of length 1. These we can represent in a unique reduced way as $\frac{1}{e}(a,b,c,d)$, where now either e is odd and exactly one of a,b,c,d is odd, or e is even and all of a,b,c,d are odd. In this latter case we set $a = 2a_1+1$, $b = 2b_1+1$, etc., and from

$$4\big(a_1^2+a_1+b_1^2+b_1+c_1^2+c_1+d_1^2+d_1+1\big) = e^2,$$

which (as $a_1^2+a_1 = a_1(a_1+1)$ is even, etc.) yields that e is even, but not divisible by 4.

One now verifies that on the subset $D \subset H_0$ formed by the first type of points one can use the same 2-coloring as above, by the value $a+b+c+d$ (mod 2). This D forms an additive subgroup of $H_4 \subset \mathbb{Q}^4$ of index 2, and hence once more the same type of 2-coloring (with two new colors to be used) can be used on $H_4 \backslash D$. □

The chromatic number of the plane: Is this a good problem? Again this is a question of taste. In my view the fact that there is so little progress on the original problem in so many years, and progress only on variations, and that the answer might depend on set theory all indicate that it is not a productive, helpful problem. Also it did not much connect to theory from other areas of mathematics, except for set theory, although this may change: For example, just recently substantial tools from Fourier analysis are brought into play to estimate a quite relevant quantity, namely the maximal density $m_1(\mathbb{R}^2)$ of a (measurable!) subset $A \subset \mathbb{R}^2$ that avoids distance 1, see de Oliveira & Vallentin [5]. (Erdős has conjectured a long time ago that $m_1(\mathbb{R}^2) < \frac{1}{4}$, which would imply a result by Falconer that one needs at least five *measurable* unit-distance avoiding sets to cover the plane.)

1.1.3 Ramsey Theory

Ramsey Theory, named after Frank Plumpton Ramsey (1903–1930) for his 1930 paper [9], treats and quantifies the existence of "regular substructures" in large "arbitrary" structures. In particular, Ramsey proved that for any fixed n there is a (large) number $N(n)$ such that for *any* 2-coloring of the edges of a complete graph K_N on $N \geq N(n)$ vertices there is a monochromatic K_n-subgraph, that is, there are n vertices such that all edges that connect them have the same color. For example, it is a simple exercise to see $N(n) = 6$ will do for $n = 3$. But clearly the structural/philosophical insight of the so-called "Ramsey phenomenon" that "complete disorder is impossible," that is, that regular substructures cannot be avoided in very large structures, goes far beyond the (simple, from today's perspective) result that Ramsey proved.

In Part VII of his book, Soifer provides a wonderful discussion of three major results in "Ramsey Theory before Ramsey":

> **David Hilbert (1892):** *For every* $n, r > 0$ *there is some* $H(n,r)$ *such that for* $N \geq H(n,r)$ *any* r*-coloring of the integers in* $\{1, 2, \ldots, N\}$ *one color class contains an* affine n-cube*, that is, all integers of the form* $a + \sum_{i \in I} x_i$ *for fixed positive integers* $a, x_1, x_2, \ldots, x_n > 0$ *and* all *subsets* $I \subseteq \{1, 2, \ldots, n\}$.
> **Issai Schur (1916):** *For every* $n > 0$ *there is some* $S(r)$ *such that for* $N \geq S(r)$ *any* r*-coloring of the integers in* $\{1, 2, \ldots, N\}$ *one color class contains* a, b, c *such that* $a = b + c$.
> **Bartel L. van der Waerden (1927):** *For every* $n, r > 0$ *there is some* $W(n,r)$ *such that for* $N \geq W(n,r)$ *any* r*-coloring of the integers in* $\{1, 2, \ldots, N\}$ *one color class contains an arithmetic progression of length* n.

Soifer reports that Bartel Leendert van der Waerden (1903–1996), an Algebraic Geometer who got famous for his textbook *Moderne Algebra* based on courses by Emmy Noether in Göttingen and by Emil Artin in Hamburg (p. 309),

> proved this pioneering result while at Hamburg University and presented it the following year at the meeting of *D.M.V.*, *Deutsche Mathematiker Vereinigung* (German Mathematical Society) in Berlin. The result became popular in Göttingen, as the 1928 Russian visitor of Göttingen A. Y. Khinchin noticed and later reported [Khi1], but its publication [Wae2] in an obscure Dutch journal hardly helped its popularity. Only Issai Schur and his two students Alfred Brauer and Richard Rado learned about and improved upon Van der Waerden's result almost immediately[...].

This report gets a number of facts wrong. For example, the DMV meeting 1928 was held in Hamburg, and Aleksandr Khinchin writes that the result was obtained in Göttingen.

The "obscure Dutch journal" was *Nieuw Archif voor Wiskunde*. However, my main objection to Soifer's rendition is the negative and down-putting attitude and undertone: If it was popular in Göttingen, "Tagesgespräch" (talk of the town) as Khinchin writes [4], why did "only" Issai Schur and his two students learn about it, and improved it "almost immediately"? Van der Waerden's paper is called "Beweis einer Baudetschen Vermutung"—but Soifer then puts a lot of effort into proving Van der Waerden wrong and *instead* confirming his conviction that the conjecture was due to Schur. And he gets this confirmed indirectly (via Erdős), much later, from former students of Schur.

Again, if the conjecture was natural and may have been "whose time has come," why can't several people come up with it? It is quite plausible that several people came up with it independently in the 1920s, in this case Schur and Baudet. Why this urge to prove Van der Waerden wrong about the origin of the conjecture, if he apparently heard it from Baudet? And does it really make sense to talk about the “authorship of the conjecture”? The only plausible reason I can see for Soifer’s passion and persistence in his investigations and his attempts to find fault with Van der Waerden is that he badly dislikes him.

Indeed, at this point Soifer doesn’t continue on the topic of Ramsey theory, a subject that has grown and developed tremendously after Ramsey. See the “classic” account by Graham, Rothschild and Spencer [7] from 1990, but see also what Szemerédi, Gowers, Green–Tao and others have achieved afterwards, which is clearly “major mathematics”, but which is also outside the range of Soifer’s field of “coloring theory.”

1.2 Alexander Soifer vs. Bartel L. van der Waerden

Soifer instead sticks to Van der Waerden. He accuses him of “stealing” his famous algebra book from Emil Artin based on rather indirect evidence, he tries to prove that he wrongly claimed to have had a Rockefeller stipend for his time in Hamburg, he badly tries to find fault in his stay at Leipzig University during Nazi times, and so on.

At this point, I must say that I am not a historian, I have not read all materials and I have not been to the archives, so I can’t really judge this. I have no stakes in Van der Waerden, I have never met him, and I cannot (and dare not) judge him, neither his contributions to Mathematics, nor what he did or didn’t do for example as a professor in Leipzig 1931–1945. However, I object to the method. In some parts of *The Mathematical Coloring Book* Soifer tells his personal story, writes about Mathematics he has been involved in, and the people who did it and do it, people he knows and he has met. I feel he has every right to be emotional and personal in his judgement there, although of course he will write most favorable about his colleagues, friends and acquaintances. In this context, I feel he has the right to choose from which letters and emails he wants to quote (and he quotes from a lot of emails and letters). After all, in these parts it is *his* story.

When he delves into historical subjects, such as the life and times of Bartel L. van der Waerden, things change. As far as I know now, Van der Waerden was Professor of Mathematics at Leipzig University 1931–1945, and for part of that time he was the Director of the Mathematical Institute there. So Van der Waerden stayed in Nazi-Germany to the end, although he had options and offers to leave. He has courageous acts and statements on record, but also cooperated with Nazi authorities (and I certainly can’t tell whether this was “more than necessary”, whatever that could mean), and some of his actions seem to have harmed Jewish colleagues (but I don’t know and can’t judge whether any of this was intentional or even done knowingly). Certainly Van der Waerden cannot be excepted from discussion and criticism. However, it seems clear to me that it cannot be good if a historian has an ax to grind, if from the outset he *wants to prove* things about his subject of study, since this will color his judgement: He should at least *try* to be objective.

In Van der Waerden’s case, I seem to get a much more objective and trustworthy picture of his time in Leipzig from reading Reinhard Siegmund-Schultze’s account and study [12]. Indeed, Siegmund-Schultze has been in the archives himself. Siegmund-Schultze also tells me that Soifer reads neither German nor Dutch, so he can’t access the original sources, not a good basis for this. So the impression remains of a personal war. Soifer dedicates four chapters (Chapters 36–39) of the Coloring Book for his research into the Life of Van der Waerden, spanning pages 367–483, more than one hundred pages. He calls them “a report on research in progress” (p. 483). Three of these chapters have previously appeared in Soifer’s private journal *Geombinatorics* in 2004/2005. He is apparently continuing his research, and the next version of his treatment of Van der Waerden is apparently scheduled to appear with Birkhäuser in October 2014 [13].

Why this passion and scornfulness against Van der Waerden? Section 36.1 is entitled “Prologue: Why I Had to Undertake the Search for Van der Waerden” and it claims that the reason is that Van der Waerden was not treated biographically enough since he changed subjects, moved between countries, and lived to get old. And that all the accounts of Van der Waerden up to now were biased (p. 368):

> These authors apparently believed that a personal acquaintance with Professor Van der Waerden automatically made them experts on his life. Their repeating Van der Waerden’s words and explanations did contribute to mathematical folklore. However, these repetitions, mixed with “cheerleading” and lacking in archival research and critical examination of facts, hardly added up to history.[53]

With his reference to “cheerleading”, Soifer may refer to Rüdiger Thiele’s account of “Van der Waerden in Leipzig” [14], which is based on Thiele’s lecture at a festive colloquium in Leipzig, celebrating Van der Waerden’s 100th birthday in the presence of colleagues who knew and had worked with Van der Waerden. On such an occasion, of course one cannot expect a critical and balanced account, nor pure historical scholarship and objectiveness. So we get caught in the trenches of a fight between Thiele (who stays away from criticism) and Soifer (who has an ax to grind). A quote from Thiele’s lecture gets Soifer into a highly emotional and aggressive outbreak (page 368/369, if you want to look it up), which—in my opinion—proves that all that follows is not a result of historical scholarship, but rather a documentation of Soifer’s persistent personal campaign against Van der Waerden. And as it is a campaign, the reading is not agreeable, and the results and interpretations can’t be trusted. That’s not the exciting personal tour to fascinating “coloring mathematics” as it was done that I was promised in the introduction of the book!

1.3 On Style

What Alexander Soifer calls “The Coloring Book” could and should much more aptly been called “A Coloring Book” or perhaps even better “My Coloring Book”—but perhaps this title was not available due to copyright reasons, as it is the title of a famous song by Kander & Ebb. However, Soifer does not descend to pop culture, but he lets the famous poem “Pour faire le portrait d’un oiseau” by Jacques Prévert set the tone at the opening of the volume—which he presents in a translation of his own (with Maurice Stark), but leaves out the dedication that belongs to the poem, “A Elsa Henriques”. This comes after a dedication to his Father

> This coloring book is for my late father Yuri Soifer, a great painter, who introduced colors into my life.

Of course dedications are a private matter, and probably should not be commented upon, except in this case we are told twice more that Soifer's father was a great painter (pp. v, xxvii). The reader might get annoyed when reading the three laudatory Forewords (by Branko Grünbaum, Peter D. Johnson Jr., and Cecil Rousseau), and the two prefaces by the author (called "Acknowledgements" and "Greetings to the Reader"), being told again and again how great this book is. Even before the book has really started, it has been compared to Hardy's *Apology* and to Courant–Robbins' *What is Mathematics?*, Soifer has quoted Pasternak, Picasso, Kundera, and Hemingway on his behalf, *and so on*. Someone should have advised the author and made him proceed with a bit more modesty. Someone should have also forced him to cut the manuscript, at the long parts and chapters where the investigations into the colorful lives of the creators get out of hand. The "level of detail" is uneven and not sufficiently explained or justified, neither in the "historical" sections nor in the mathematical ones, where some chances to present something nice are missed (see above), but in other instances complete proofs are given with pages of messy combinatorial details and not much insight to be gained. All in all this results in a book that has interesting parts and a lot of valuable material, but as a whole fails to deliver what it promised at the outset.

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BOOK REVIEW

Camille Laurent-Gengoux, Anne Picherau, Pol Vanhaecke: "Poisson Structures"

Springer-Verlag, 2013, 461 pp.

Johannes Huebschmann

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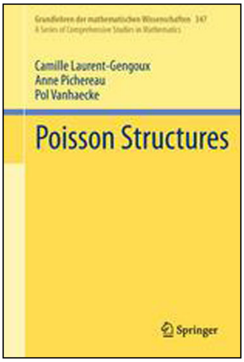

The concept of a Poisson structure is an old one: Given a commutative algebra A, for example the algebra of smooth functions on a smooth manifold, a Poisson bracket is a Lie bracket $\{\cdot, \cdot\}$ on A that turns A into a Lie algebra in such a way that, given $a \in A$, the operation $\{a, \cdot\}$ is a derivation of A relative to the multiplicative structure of A. A symplectic structure (closed non-degenerate 2-form) on a smooth manifold determines a Poisson structure on the ring of smooth functions but there are Poisson structures on smooth manifolds that do not arise from a symplectic structure.

Searching for first integrals (or constants of motion), Poisson was led to his discovery. The bracket discovered by Poisson arises from a symplectic structure. We recall that a first integral for a differential equation is a function that is constant along the solutions of the differential equation. A first integral yields a reduction of the number of degrees of freedom of the underlying differential equation.

Poisson structures have been explored by J. Liouville, S. Lie, É. Cartan, P. Dirac, and others. Looking for integrals of motion, Liouville isolated what we nowadays refer to as *Liouville integrable systems* and established a result which thereafter became fundamental for the development of analytical mechanics and prompted an entire research area, that of integrable systems, very active still today. A Liouville integrable system is one with the maximal possible number of pairwise Poisson commuting independent first integrals. The quoted result of Liouville essentially says that, under suitable additional assumptions (for example of the kind that the energy level sets be

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compact), canonical coordinates, known as action-angle variables, can be introduced, and that the underlying differential system can then be solved in an elementary manner by “quadratures”.

For illustration, consider the Kepler problem (two-body problem). Reduction to the center of mass system reduces the original 6-dimensional configuration space to a copy of $\mathbb{R}^3$ with a point removed which for convenience we take to be the origin. The associated phase space is 6-dimensional, and conservation of angular momentum reduces the phase space to a 4-dimensional space having a punctured plane as its underlying configuration space. In standard Cartesian coordinates (q_1, q_2, p_1, p_2), the Poisson bracket $\{\cdot,\cdot\}$ is then given by $\{q_j, p_k\} = \delta_{j,k}$ $(1 \leq j, k \leq 2)$, the other Poisson brackets between coordinate functions being zero. In terms of polar coordinates (r, φ) in the punctured configuration plane, let $M = r^2\dot{\varphi}$ denote the angular momentum, let U be the central potential (a function of r), let V denote the effective potential defined by $V(r) = U(r) + \frac{M^2}{2r^2}$, and let $H = \frac{\dot{r}^2}{2} + V$ be the Hamiltonian (total energy). In the Hamiltonian formalism, the generalized momenta are given by $p_r = \dot{r}$ and $p_\varphi = r^2\dot{\varphi}$ but, to make the discussion more easily accessible to the non-expert, we stick to the more standard Lagrangian notation. The functions M and H are constants of motion; they Poisson commute, that is, $\{H, M\} = 0$ and, indeed, constitute a maximal system of independent integrals whence the Kepler problem is integrable. Indeed, since H is independent of time, the function $\dot{r}^2 = 2(H - V)$ integrates the equation of motion $\ddot{r} = -\frac{\partial V}{r}$, and an integration or “quadrature” solves the differential equation $\dot{r} = \sqrt{2(H-V)}$ for r. Likewise, an integration solves the differential equation $\dot{\varphi} = \frac{M}{r^2}$ for φ, and the resulting function

$$\varphi(r) = \int_{r_0}^{r} \frac{M/s^2 ds}{\sqrt{2(H - V(s))}}$$

describes the orbit in the (r, φ)-plane. When M is zero, this yields a line. When M is non-zero and $U = -\frac{k}{r}$ for some positive constant k (Newtonian potential) the orbits in the (r, φ)-plane are the Kepler ellipses. A variable change yields ordinary action-angle variables.

Poisson structures constitute a basic tool for Lie’s work. Indeed, given a real finite-dimensional Lie algebra $\mathfrak{g}$, the Lie bracket induces a Poisson bracket on the algebra of smooth functions on the linear dual of $\mathfrak{g}$. This is how Lie actually implemented the Lie bracket. Poisson structures thereby provided, for example, an appropriate language for the proof of Lie’s third theorem. This theorem says that any finite-dimensional Lie algebra is the Lie algebra of a Lie group. While this result is nowadays lingua franca, its upshot is the fundamental insight that a “transformation group” with finite-dimensional Lie algebra integrates to a group that is independent of the underlying manifold on which the transformation group is defined. (For infinite-dimensional “transformation groups”, referred to nowadays as “Lie pseudo groups”, no such result is available, and developing an abstract object that adequately represents an infinite-dimensional Lie pseudo group is one of the major issues in the modern theory of symmetries of differential equations.)

For many discoveries of modern symplectic geometry there are precedents in Lie’s work which could not have been spelled out without the concept of a Poisson structure. Dirac made the fundamental observation that Poisson brackets provide the right framework in which classical mechanics is seen as an approximation of quantum mechanics. Indeed, there is a story saying that Dirac's Cambridge supervisor Fowler had requested a copy of [4] from Heisenberg once it were available and that in August 1925 Heisenberg had sent a copy of the proofs to Fowler, who then turned it over to Dirac who, in turn, after a couple of weeks, made his fundamental discovery that the quantum mechanical commutator satisfies the same axioms as the Poisson bracket. Dirac also noticed the importance of Poisson brackets for classical constrained systems and developed the miraculous notion of Dirac bracket.

Since the pioneering work of Res Jost, see, e.g., [7], and A. Lichnerowicz [9], Poisson brackets have been in intense development as a research topic on its own. In a series of celebrated papers of Flato et al. [1, 2], what has come to be known as deformation quantization was developed: Given a Poisson algebra, construct a (non-commutative) formal deformation such that, suitably interpreted, the commutator in the deformed algebra recovers, up to higher order, the Poisson bracket. For intelligibility we recall that, given a commutative algebra A over a commutative ring R, a *formal deformation* of A is a non-commutative $R[\![t]\!]$-bilinear algebra structure $f\colon A[\![t]\!] \times A[\![t]\!] \to A[\![t]\!]$ on $A[\![t]\!]$ which is expressible in the form

$$f(a,b) = ab + tf_1(a,b) + t^2 f_2(a,b) + \cdots$$

where "ab" denotes the product in A, extended to $A[\![t]\!]$ in the standard manner, and where each f_j $(j \geq 1)$ is an R-bilinear map $A \times A \to A$ extended in the natural manner to an $R[\![t]\!]$-bilinear map $A[\![t]\!] \times A[\![t]\!] \to A[\![t]\!]$; here $A[\![t]\!]$ denotes the ring of formal power series in the variable t with coefficients in A. Given the algebra A, the standard tool to explore the obstructions to a recursive construction of such a deformation of A is the Hochschild complex defining Hochschild cohomology. Given a Poisson algebra A, a formal deformation quantization of A recovers the Poisson bracket through the constituent f_1. In the deformation quantization approach the requisite Hilbert space, fundamental for the interpretation of quantum mechanics, is missing, however. The formal deformation quantization program culminated in a spectacular result due to Kontsevich: Any Poisson algebra structure on a smooth manifold arises from a formal deformation. This result is a consequence of a more general one.

To explain briefly this more general result, let A be the algebra of smooth functions on a smooth manifold and consider the standard complex $\mathrm{Hoch}(A)$ defining the Hochschild cohomology of A in the Fréchet sense, endowed with the Gerstenhaber bracket; the Hochschild differential and Gerstenhaber bracket combine to a differential graded Lie algebra structure. On the other hand, the Schouten bracket (the extension of the ordinary Lie bracket of vector fields to a bracket on multi-vector fields) turns the graded vector space of multi-vector fields into a graded Lie algebra, which we view as a differential graded Lie algebra with zero differential. Kontsevich's result [8] says that an obvious map between the two differential graded Lie algebras, while plainly not a morphism of differential graded Lie algebras, extends to a morphism of what is nowadays referred to as one of L_∞ algebras (or s(trongly) h(omotopy) Lie algebras) and thereby establishes an equivalence between the two objects, viewed as L_∞ algebras. In technical terms, one refers to the situation by the phrase "the Hochschild complex, endowed with the Gerstenhaber bracket, is *formal* as a differential graded Lie algebra", and the degrees of the underlying graded objects would better be shifted by 1. In practise this means that, for the problem under discussion, the weaker property enjoyed by the morphism between the two differential graded Lie algebras is just as good as if the morphism were a true one of differential graded Lie algebras.

The book under review somewhat reflects various items of these developments and, in particular, is devoted to integrable systems and Kontsevich's formality result.

The book consists of three parts, part I (Theoretical Background), part II (Examples), part III (Applications). Part I begins with an exposition of Poisson algebras, Poisson varieties, and Poisson manifolds and offers a rather detailed discussion of a number of basic constructions in the Poisson world including the tensor product of Poisson algebras and the product of Poisson manifolds, Poisson ideals and Poisson subvarieties, holomorphic Poisson structures, field extensions, localization, etc. Next the authors treat multi-derivations, multi-vector fields, formal differentials, Schouten bracket, Lie derivative. Thereafter they introduce Poisson calculus, Poisson cohomology, and the modular class. The first part ends with an introduction to the problem of reduction in the Poisson world. In part II, the authors treat constant Poisson structures, regular and symplectic manifolds, linear Poisson structures, higher degree Poisson structures including Nambu structures, Poisson structures in dimensions two and three, r-brackets, Poisson Lie groups, that is, the classical analogues of quantum groups, and Lie bialgebras. In part III (Applications), the authors expose Liouville integrable systems including the action-angle theorem and offer an introduction into deformation quantization aimed at explaining Kontsevich's formality theorem.

Each chapter contains a series of exercises as well as a number of notes aimed at giving further hints as to how the various items in the book are interrelated and, furthermore, at placing the material in the literature. The book includes an appendix on multilinear algebra and one on real and complex differential geometry.

The book is a timely and courageous attempt to make accessible a flourishing research area to a wider audience in the form of a research monograph/textbook and as such it is very welcome. However, the authors do not reflect the collective understanding of various items treated in the book.

For example, on p. 68, the authors write "we introduce the objects which are, in a sense, dual to skew-symmetric multi-derivations of a (commutative associative) algebra $\mathcal{A}$". However, the salient feature here is that the formal differentials represent the derivations functor and the issue whether the pairing between the differentials and derivations is a duality is in general somewhat delicate. On p. 92, Lie algebra and Poisson cohomology are said to be formally very similar. Earlier in the book, the authors develop what they refer to as 'algebraic de Rham cohomology' and recall as well ordinary de Rham cohomology for manifolds. All these theories are offspring of a single theory, that of derived functors; however the book does not quote any classical standard homological algebra source such as, e.g., [3]. The observation that these above theories are offspring of a single theory, except for Poisson cohomology, goes back at least to [12] (not quoted in the book) and, for Poisson cohomology, has been worked out in [5] in the framework of Lie-Rinehart algebras. In the book under review, Poisson cohomology is defined in terms of a complex, without further explanation.

The discussion of L_∞-structures and L_∞-morphisms in Chap. 13 of the book is technically correct but lacks conceptual insight. The higher homotopies lurking behind are nowadays well understood, and the appropriate combinatorial object to handle them is the differential graded symmetric coalgebra concept. Indeed, an L_∞-structure on a graded vector space $\mathfrak{g}$ is simply a differential graded coalgebra structure on the symmetric coalgebra $S^c[s\mathfrak{g}]$ on the suspension $s\mathfrak{g}$ of $\mathfrak{g}$ (the graded vector space $\mathfrak{g}$, regraded up by 1). An L_∞-morphism is then simply a map preserving the structure. No reference is given to the existing literature on higher homotopies, see, e.g., [6] for a survey, the term “higher homotopy” does not occur, and the reader will never learn that, without Stasheff’s contributions to the idea of higher homotopies [13], Kontsevich would not even have been able to phrase the formality conjecture.

In a similar vein, on p. 399, the reader finds the phrase “a differential graded Lie algebra which admits an L_∞ quasi-isomorphism to its cohomology is called *formal*, a terminology which is borrowed from topology”. Now, in rational homotopy theory, more precisely in the framework of the Quillen model, this is what formality means; it is not just “borrowed” terminology. The terminology “formal” has been chosen to indicate that, for a formal space, the (rational) topological invariants arise as a *formal* consequence of merely the rational cohomology ring. A general space gives rise to higher homotopies that induce non-trivial higher order operations, e.g. Massey products. In the book under review, no reference is given although rational homotopy theory is nowadays well developed and standard textbooks are available. What has come to be known as the Quillen model goes back to [11]. The Quillen model involves, in a crucial manner, a celebrated theorem on the structure of Hopf algebras in characteristic zero [10]. Furthermore, in [11], Quillen discusses Lie algebra twisting cochains in detail. The defining property of a Lie algebra twisting cochain is the Maurer-Cartan equation or, equivalently, deformation equation, or master equation. The higher homotopies in Kontsevich’s formality theorem can concisely be phrased in terms of a Lie algebra twisting cochain. Elsewhere in the book, the authors describe formal deformations of commutative algebras via the Maurer-Cartan equation.

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