81. Band Heft 2
ausgegeben am 30. 5. 1979

DMV

Jahresbericht der Deutschen Mathematiker-Vereinigung

Herausgegeben von
Prof. Dr. K. Jacobs, Bismarckstr. $1^1/_2$, 8520 Erlangen
unter Mitwirkung von
Prof. Dr. W. Benz, Bundesstr. 55, 2000 Hamburg 13 – Prof. Dr. P. L. Butzer, Templergraben 55, 5100 Aachen – Prof. Dr. W.-D. Geyer, Bismarckstr. $1^1/_2$, 8520 Erlangen

Verlag
B. G. Teubner, Postfach 80 1069, 7000 Stuttgart 80 (Vaihingen) – Verantwortlich für den Anzeigenteil: Walter Hirtz

Printed in Germany – GW ISSN 0012-0456
Herstellung: Schwetzinger Verlagsdruckerei GmbH, 6830 Schwetzingen

Alle für die Schriftleitung des Jahresberichts bestimmten Sendungen (Briefe, Manuskripte, Ankündigungen von neuerscheinenden Büchern) sind an Prof. Dr. K. Jacobs, Math. Institut der Universität, Bismarckstr. $1^1/_2$, 8520 Erlangen, zu richten.

Bücher, von denen eine Besprechung erfolgen soll, werden die Herausgeber bei den Verlagen anfordern.

Die Verfasser erhalten von ihren Arbeiten 75, von Buchbesprechungen 2 Sonderdrucke unentgeltlich.

Der 81. Band des Jahresberichtes umfaßt vier Hefte und wird nur geschlossen abgegeben. Der im voraus zahlbare Bezugspreis beträgt DM 74,–. Bestellungen nehmen jede Buchhandlung oder der Verlag entgegen. Für Mitglieder der DMV ist der Bezug des Jahresberichtes im Mitgliedsbeitrag enthalten. Die Bezieher werden gebeten, dem Verlag von Anschriftenänderungen unverzüglich Kenntnis zu geben.

Für den Abdruck vorgesehene Manuskripte sind in einwandfrei leserlicher und völlig druckfertiger Form (Schreibmaschinenschrift) einzureichen. Die Manuskripte müssen nach satztechnischen Richtlinien und Auszeichnungsregeln, die von der Schriftleitung des Jahresberichts zu erhalten sind, gestaltet werden.

Nachträgliche Änderungen bei der Korrektur, die 5% des Satzpreises übersteigen, müssen dem Autor angerechnet werden.

Konten der DMV: Kreissparkasse Tübingen, Girokonto 1626. – Postscheckkonto: DMV in Tübingen, Nr. 18517 – 706 beim Postscheckamt Stuttgart. – DDR: Berliner Stadtkontor, Berlin C 111, Kurstraße, Konto Nr. 20/142007.

Inhalt von Band 81, Heft 2

1. Abteilung

2. Abteilung

Jber. d. Dt. Math.-Verein.
81 (1979) 49–86
MOS classification: 41 A 02

Uniform Approximation with Constraints

B. L. Chalmers[1]), Riverside, and G. D. Taylor[2]), Fort Collins

Table of Contents

1 Introduction

The theory of uniform approximation subject to constraints has received considerable study in the last ten years. In 1973 two surveys appeared concerning this topic [94] and [157]. In this paper the survey [157] will be updated and expanded.

The usual setting of this theory is as follows. Denote by C(X) the class of all continuous real valued functions defined on X, a compact subset of [a, b], normed with the uniform (Chebyshev) norm, $\|f\| = \max\{|f(t)| : t \in X\}$. Let $W \subset C(X)$ be a class of approximants and let $K \subset W$ be a nonempty subset of W that is determined by certain constraints. In most cases W is either a Haar subspace

The organizational structure of this paper is to divide the constraints into three general classifications. Precisely, we shall classify the constraints according to the categories of equality constraints, inequality constraints and general constraints. As will be noted these classes are not necessarily clearly delineated.

2 Equality Constraints

2.1 Approximation with interpolatory constraints

The special case of Lagrange interpolatory constraints on a Haar subspace was the first problem in this classification studied and is the most complete to date. According to Lewis, Bernstein [12] treated a simple case of this problem in 1926. The first in depth study of Lagrange constraints was undertaken by Paszkowski [127], [128], [129] in 1955. This study was extended by Deutsch [31] in 1968 where a complete treatment of the best approximation problem is given. In 1969, this theory was extended by Loeb, Moursund, Schumaker and Taylor [100] to allow Hermite interpolatory constraints.

We begin with a rather complete discussion of this Hermite theory since the results obtained here are representative of those sought in the other studies. Thus, as in [100], let W be an extended n-dimensional Haar subspace of order ν defined on [a, b]. Let $T = \{t_j\}_{j=1}^k$, $a \leqq t_1 < \cdots < t_k \leqq b$, be k fixed points, $\{m_j\}_{j=1}^k$ be a given set of nonnegative integers with $\Sigma_{j=1}^k (m_j + 1) = m \leqq n$, $\max_{1 \leqslant j \leqslant k} m_j < \nu$. and $\{a_{ji}\}$ $1 \leqq j \leqq k, 0 \leqq i \leqq m_j$ be a given array of real numbers. Finally, fix $X \subset [a, b]$ a compact subset and define

$$K = \{p \in W : p^{(i)}(t_j) = a_{ji},\ 1 \leqq j \leqq k,\ 0 \leqq i \leqq m_j\}.$$

K is a subset of W satisfying the Hermite-Birkhoff constraints, $p^{(i)}(t_j) = a_{ji}$. If $m_j = 0$ for $j = 1, \ldots, k$ then this theory becomes approximation with Lagrange constraints. Except for questions concerning rates of approximation all known results of the Lagrange theory extend directly to the Hermite case. As in the case of approximating from closed subsets of a finite dimensional subspace, existence follows from a straightforward compactness argument. (Here, as elsewhere in all existence statements, we make the tacit assumption that $K \neq \emptyset$.) Thus,

Theorem 2.1.1 ([93]) Let $f \in C(X)$. *Then there exists* $p \in K$ *such that* $\| f - p \| = \inf\{ \| f - q \| : q \in K\}$.

Thus, the side conditions do not affect the standard existence result in this case. However, turning to theorems of characterization, we find that the side conditions have a very definite effect. In order to introduce these results we must first define the extremal set corresponding to a given $f \in C(X)$ and $p \in K$. The extremal set, X_p, for f with respect to p is defined to be $X_p = \{x \in X : |f(x) - p(x)| = \| f - p \|\}$. Now, in order to prove the standard characterization results, one must put restrictions on X and f. Thus, we shall assume that $X \sim T$ consists of at least $n - m + 1$ points and that $f \in \widetilde{C}(X) = \{f \in C(X); f(t_j) = a_{j0},\ 1 \leqq j \leqq k\}$. Although this restriction on f is more than is needed, it is, nevertheless, natural in some sense. Probably the restriction that would give the largest class of functions for which the following characterizations hold (and therefore uniqueness, also) is that f satisfy

$\inf\{\|f - q\| : q \in K\} > \max\{|f(t_j) - a_{j0}| : j = 1, \ldots, k\}$. With these restrictions it is possible to prove

Theorem 2.1.2 ([100]) *Let* $f \in \tilde{C}(X) \sim K$ *and* $p \in K$. *Then the following four statements are equivalent:*

(i) p *is a best approximation to* f *from* K.

(ii) $\min_{x \in X_p} \{(f(x) - p(x))(q(x) - p(x))\} \leq 0$ *for all* $q \in K$.

(iii) *The zero element* $(0, \ldots, 0)$ *is in the convex hull of the set of* $(n - m)$-*tuples* $\{\sigma(y)\hat{y} : y \in X_p\}$ *where* $\sigma(y) = \mathrm{sgn}(f(y) - p(y))$ *and* $\hat{y} = (\phi_1(y), \ldots, \phi_{n-m}(y))$ *with* $\phi_1, \ldots, \phi_{n-m}$ *a basis for* $K - p$.

(iv) *There exist* $n - m + 1$ *consecutive points* $\{x_i\}_{i=1}^{n-m+1} \subset X_p$ *such that* $\mathrm{sgn}[(f(x_i) - p(x_i))\, \Pi(x_i)] = (-1)^{i+1} \mathrm{sgn}[(f(x_1) - p(x_1))\Pi(x_1)]$ *where* $\Pi(t) = (t_1 - t)^{m_1+1} \cdots (t_k - t)^{m_k+1}$ *if* $k \neq 0$ *and* $\Pi(t) \equiv 1$ *if* $k = 0$.

Remarks. Condition (ii) is frequently referred to as the Kolmogorov criterion and can usually be shown to hold in the most general situations. Condition (iii) is usually referred to as "zero in the convex hull" and is not quite as widely applicable as (ii). The last condition (iv) is usually referred to as the alternation condition. In order to get this last characterization some sort of zero counting property must exist among the approximants and this particular characterization is used in proving the convergence of Remez-type algorithms for computation with simplified exchange procedures.

Proof. The proof follows analogously as in the classical case (no restrictions).

(i) ⇒ (ii). The proof is by contradiction. Suppose for some $q \in K$, $(f(x) - p(x)) \cdot (q(x) - p(x)) > 0$ on X_p. Thus, for positive ϵ sufficiently small, it is immediate that $p(x) - \epsilon(q(x) - p(x))$ is a better approximation to f from K at least on the critical set X_p, and in fact on all of $[a, b]$ by the usual continuity and compactness argument.

(ii) ⇒ (iii). Let $U = \{\sigma(y)\hat{y} : y \in X_p\}$. It is easily seen that U is a compact subset of R^{n-m}. Now the system of inequalities

$$\sum_{i=1}^{n-m} z_i \sigma(x) \phi_i(x) > 0, \qquad x \in X_p$$

for $z = (z_1, \ldots, z_{n-m}) \in R^{n-m}$ must be inconsistent by (ii). Thus, by the theorem on linear inequalities [22, pp. 19] we have that the origin of E^{n-m} belongs to the convex hull of $\{\sigma(y)\hat{y} : y \in X_p\}$ as desired.

$t_j \in (x_i, x_{i+1})$, is even, then $\Pi(x_i)\Pi(x_{i+1}) > 0$ so that $[p(x_i) - q(x_i)][p(x_{i+1}) - q(x_{i+1})] < 0$. Thus, $p - q$ either has a zero in $(x_i, x_{i+1}) \sim T$ since $K \subset C^{\nu}[a, b]$ or $p - q$ has a zero of order $m_j + 2$ at some $t_j \in (x_i, x_{i+1})$. Similarly, we obtain the same conclusion if the sum of the $(m_j + 1)$'s, such that $t_j \in (x_i, x_{i+1})$, is odd. Thus, counting zeros to order ν, we see that $p - q$ has n zeros, implying that $p \equiv q$, a contradiction. Thus, (iv) implies (i) and the theorem is established.

To prove uniqueness one can either give a somewhat refined argument in the (iv) implies (i) case above or give a direct proof as described below.

Theorem 2.1.3 ([100]) *Let* $f \in \tilde{C}(X)$; *then* f *has a unique best approximation from* K.

P r o o f. If $f \in K$, note that the result is immediate. Thus, assume $f \notin K$ and suppose p and q are two best approximations to f from K. Let X_p and σ be defined as in Theorem 2.1.2 corresponding to p. Then for all $x \in X_p$, $\sigma(x)[f(x) - p(x)] \geqq \sigma(x)[f(x) - q(x)]$. This implies $\sigma(x)[q(x) - p(x)] \geqq 0$ for all $x \in X_p$. Now let $y_1, \ldots, y_t$ be the zeros of $q(x) - p(x)$ which are in X_p. If $q \not\equiv p$ then we must have that $t < n - m$. Consequently there exists a $q_1 \in K - p$ such that $\sigma(y_i)q_1(y_i) = 1$, $i = 1, \ldots, t$. Then, it is a straightforward argument to conclude there exists a $\lambda > 0$ such that $\sigma(x)[q(x) - p(x) + \lambda q_1(x)] > 0$ for all $x \in X_p$. Since $q(x) + \lambda q_1(x) \in K$ this contradicts the Kolmogorov criterion (ii), showing that $q \equiv p$.

In addition, in this theory, both a strong uniqueness and a Lipschitz continuity theorem for the best approximation operator are known.

Theorem 2.1.4 ([100]) *Let* $f \in \tilde{C}(X)$ *and suppose* $p \in K$ *is the best approximation to* f. *Then there exist a constant* $\gamma = \gamma(f) > 0$ *such that for all* $q \in K$

[illegible]

Theorem 2.1.5 ([100]) *For each* $f \in \tilde{C}(X)$ *let* τf *denote the unique best approximation to* f *from* K. *Then* τ *is locally Lipschitz at each* $f \in \tilde{C}(X)$. *That is, if* $f \in \tilde{C}(X)$ *is fixed and* $g \in \tilde{C}(X)$ *arbitrary then*

$$\| \tau f - \tau g \| \leqq \frac{2}{\gamma(f)} \| f - g \|.$$

The proofs of both of these theorems are as in the standard linear theory without constraints [22], pp. 80–82. Results concerning the continuous dependence of this problem on the underlying domain of the functions have not been obtained to our knowledge. Although it appears that for this problem the results of C h e n e y [22], pp. 84–88, should apply, specific estimates as given in [22], pp. 89–94, need to be derived for the case that W consists of algebraic polynomials.

Turning to the question of computation, the convergence of the analog of the Remez single point exchange has been proved in [100]. Clearly, the multiple point exchange can also be modified to treat this problem. In the development of this Remez-like routine the following de La Vallée Poussin type of theorem was also developed.

Theorem 2.1.6 ([100]) *Let* $f \in \tilde{C}(X)$ *and let* $p \in K$ *be the best approximation to* f *from* K. *If* $q \in K$, $\{z_i\}$, $i = 1, \ldots, n - m + 1$, *is a set of* $n - m + 1$ *consecutive points in* $X \sim T$ *and* $\{\lambda_i\}$, $i = 1, \ldots, n - m + 1$, *is a set of* $n - m + 1$ *positive numbers such that*

(a) $|f(z_i) - q(z_i)| = \lambda_i$

(b) $\operatorname{sgn}\{[f(z_i) - q(z_i)]\Pi(z_i)\} = (-1)^{i+1}\operatorname{sgn}\{[f(z_1) - q(z_1)]\Pi(z_1)\}$,

then $\min_i \lambda_i \leqq \| f - p \|$, *where* $\Pi(z)$ *is defined in* Theorem 2.1.2.

As mentioned earlier, Paszkowski studied Lagrange interpolatory constraints in 1955. He developed existence, alternation, and uniqueness results, as well as results concerning the error of approximation. Deutsch extended this theory to give all the corresponding results stated above for the more general Hermite case. Before discussing results about the rate of approximation with interpolatory constraints, we want to mention a further generalization of this theory in the linear case. Specifically, this theory has been extended to include Hermite-

[134] and Kimchi and Richter-Dyn [88]. Although it is claimed in [88] that this theory can be generalized to include extended Tchebycheff systems with differentiation replaced by the suitable differentiable operator on [a, b], in both of these studies the space W is taken to be Π_{n-1}. In the first study the domain of application is a compact subset X of an interval whereas in the second it is a closed interval [a, b]. The method of description of this problem is conveniently done through the device of an incidence matrix $E^k(m) = (e_{ij})$, $1 \leq i \leq k$, $0 \leq j \leq n-1$

consecutive 1's *in one of its rows, beginning in the first column:* $e_{ij} = 1, j = 0, 1, \ldots, \mu - 1, e_{i\mu} = 0$.

Definition 2.1.2 ([88]) *Define* $\mu_i = \min\{j: e_{ij} = 0, 0 \leqq j \leqq n - 1\}, i = 1, \ldots, k$ *and* $\Pi(x) = \prod_{i=1}^{k} (x - \xi_i)^{\mu_i}$.

Then under the assumption that $E_n^k(m)$ is poised after any addition of n − m L-conditions and/or H-conditions (property P in [134]) the following alternation result follows. (To say $E_n^k(m)$ (augmented) is poised means that the interpolation problem determined by this new incidence matrix has a unique solution for any data vector.)

Theorem 2.1.7 ([88], [134]) *Let the set* K *have property* P *and* Π *be defined as above. Further assume that* $f \in \widetilde{C}(X) \sim K$ *and* $p \in K$. *Then* p *is a best approximation to* f *from* K *if and only if there exist* n − m + 1 *points* $y_1 < y_2 < \cdots < y_{n-m+1}$, *with* $y_i \in X \sim \{\xi_j: \mu_j = 0, j = 1, \ldots, k\}$ *for all* i, *such that* $|f(y_i) - p(y_i)| = \|f - p\|$ *and* $\operatorname{sgn}[(f(y_i) - p(y_i))\Pi(y_i)] = (-1)^{i+1} \operatorname{sgn}[(f(y_1) - p(y_1))\Pi(y_1)], i = 1, \ldots, n - m + 1$.

In addition, under the assumption of property P, Platte proves uniqueness, strong uniqueness and Lipschitz continuity of the best approximation operator. In [88], a study of the set of best approximations to a given $f \in \widetilde{C}(X)$ under conditions less restrictive than property P is given. A weak type alternation is developed as well as sufficient conditions for uniqueness to hold. One such sufficient condition for uniqueness is

Theorem 2.1.8 ([88]) *If* $E_n^k(m)$ *is poised after the addition of* n − m L-*conditions then there exists a unique best approximation from* K *for each* $f \in \widetilde{C}(X)$.

For details concerning the convergence of a Remez algorithm in this case, see section 4.1. In the terminology of section 4.1, K is Haar. Thus Theorem 4.1.4 assures the convergence of a Remez algorithm, while Theorem 4.1.6 provides the alternation pattern, which is the same as described above.

$$E_n(f, \underset{\sim}{\xi}) \leqq CE_n(f),$$

and for all n *sufficiently large* C *may be taken to be* 2.

Also, in [128] a Weierstrass type of theorem is given for the case that the number of points of Lagrange interpolation tends to infinity (in a controlled manner) as $n \to \infty$.

For the case of Hermite and Hermite-Birkhoff constraints, P l a t t e [134] proves that the corresponding analogue of Theorem 2.1.9 cannot hold. In fact he

$$p_n^{(i)}(x_j) = f^{(i)}(x_j),\ j = 1, 2, \ldots, m;\ i = 0, 1, \ldots, k$$

and $$\| f^{(i)} - p_n^{(i)} \| < \frac{d}{n^{k-i}}\, \omega\left(f^{(k)}; \frac{1}{n}\right).$$

Remark. Theorem 2 given in [65] has $\omega(f^{(i)}; 1/n)$ rather than $\omega(f^{(k)}; 1/n)$. However, this has been pointed out to be incorrect by Beatson [8] who independently proved the version of Theorem 2.1.12 given above (with d independent of $x_1, \ldots, x_n$).

The problem of approximating with rational functions satisfying interpolatory constraints has also been studied. In two papers, Gilormini announced results (no proofs given) corresponding to the case of Lagrange constraints [52], [53]. His announced results were that existence, Kolmogorov and alternation characterizations, uniqueness, a de La Vallée Poussin estimate, strong uniqueness and pointwise Lipschitz continuity for the best approximation operator hold in this setting, where for the last two results it is assumed that f, the function being approximated, is *normal* (that is, the best approximation for f is of full allowable degree in either its numerator or denominator). However, the claim of existence is false as shown by Loeb [101] with the following simple counter-example.

Let $W = R_1^1[0, 1]$ and $K = \{r \in R_1^1[0, 1]: r(0) = 1\}$. Define

$$f(x) = \begin{cases} 1 + 6x, & 0 \leqq x \leqq \frac{1}{3}, \\ 3 - 6\left(x - \frac{1}{3}\right), & \frac{1}{3} \leqq x \leqq \frac{2}{3}, \\ 1 + 6\left(x - \frac{2}{3}\right), & \frac{2}{3} \leqq x \leqq 1. \end{cases}$$

Note that $r^*(x) \equiv 2$ is the unique best approximation to f from $R_1^1[0, 1]$ with error 1, by the standard alternation theory. Also, the sequence $\{r_k(x)\}_{k=1}^{\infty} \subset K$ where $r_k(x) = (2x + 1/k)/(x + 1/k)$ and $\lim_{k\to\infty} \| f - r_k \| = 1$. Thus, since $r^*(x) \notin K$ there can be no best approximation to f from K.

In [54], Gilormini then stated a generalized type of existence result for approximation with rationals satisfying Lagrange interpolatory constraints and also corrected other existence claims in different rational approximation with constraints settings [55], [56]. Also, in 1969, Barrar and Loeb [6] studied uniform approximation with unisolvent families of variable degree satisfying Lagrange interpolatory constraints. A standard theory is developed and by assuming that the number of interpolating points is at least one, the possibility of a constant non-zero error curve is avoided.

(generalized rational functions) and for $f \in C^s(X)$ define $R(f)$ to be

$$R(f) = \{r \in R: r^{(j)}(y_i) = f^{(j)}(y_i), j = 0, 1, \ldots, m_i - 1, i = 1, \ldots, k\}.$$

Next, for a fixed $r \in R$, write $P + rQ$ to denote the subspace $\{p + rq: p \in P, q \in Q\}$ and define $S(r)$ to be

$$S(r) = \{h \in P + rQ: h^{(j)}(y_i) = 0, j = 0, \ldots, m_i - 1, i = 1, \ldots, k\}.$$

If $f \in C^s(X)$ and $r \in R(f)$, define $X(r)$, the *critical set* for r, to be

$$X(r) = \{x \in X: |f(x) - r(x)| = \|f - r\|\}.$$

Some of the results proved by Perrie are

Theorem 2.1.13 ([132]) *Let* $f \in C^s(X)$. r^* *is a best approximation to* f *from* $R(f)$ *if and only if* $0 \in$ *Convex Hull of* $\{\sigma(x)\hat{x}: x \in X(r^*)\}$, *where* $\sigma(x) = \operatorname{sgn}(f(x) - r^*(x))$, $\{g_1(x), \ldots, g_d(x)\}$ *is a basis for* $S(r^*)$ and $\hat{x} = (g_1(x), \ldots, g_d(x))$.

Theorem 2.1.14 ([132]) *Suppose that* $P + rQ$ *is an extended Haar subspace of order* $\nu = \max m_i + 1, r \in R$. *Suppose further that* $d = \dim S(r)$ *and* $\{g_1, \ldots, g_d\}$ *is a basis for* $S(r)$. *Then* r *is a best approximation to* $f \in C^s(X)$ *from* $R(f)$, $r \not\equiv f$, *if and only if there is a set of* $d + 1$ *points* $x_0 < x_1 < \cdots < x_d$ *in* $X(r)$ *such that* $\sigma(x_i) = (-1)^{1+k(i)}\sigma(x_{i+1}), i = 0, \ldots, d - 1$ *where* $k(i)$ *denotes the sum of the multiplicities* m_j *of the interpolating points* y_j *which lie between* x_i *and* x_{i+1}.

Definition 2.1.3 ([134]) *Assume* $\dim S(r) = d$ *for* $r \in R$ *fixed. Then* $S(r)$ *is called an* interpolating Haar subspace *if every nonzero element of* $S(r)$ *has at most* $d - 1$ *zeros distinct from the interpolating points* $y_1, \ldots, y_k$.

Theorem 2.1.15 ([134]) *If* $S(r)$ *is an interpolating Haar subspace and* r *is a best approximation to* $f \in C^s(X)$ *from* $R(f)$, *then* r *is unique.*

In addition, a strong uniqueness and a Lipschitz continuity of the best approximation operator satisfying Hermite constraints is proved for normal functions, as well as some additional results relating uniqueness and extremal signatures and some weaker convergence results. It appears that these results actually hold in the slightly more general situation where one only requires Lagrange interpolation to f and the remaining Hermite constraints are set equal to an arbitrary set of constants to determine K, the class of approximants.

To our knowledge, no computational studies of this particular problem have appeared in the literature. It is our belief that a Remez-type algorithm can be developed for this theory using ordinary rationals with the usual problems that this sort of algorithm has in the rational case. Also, the differential correction algorithm could also be generalized to treat this problem (see [81] and [93]). In a recent paper Gehner [50] studied characterization theorems for constrained approximation using optimization theory. Among the cases he considered is uniform rational approximation with Lagrange interpolatory constraints. In the conclusion of his paper he indicates that he has submitted an algorithm for computing in these problems based on the associated mathematical programming problems.

The final type of interpolatory constraints that we wish to mention in this subsection is that of approximating with rational functions having fixed numerators. Specifically, in 1972, Williams [174] considered the following problem. Denote by $D[0, b]$, $0 < b < \infty$, the set of all continuous functions f of the form $f = B \cdot g$ where $g, B \in C[0, b]$, $g(x) > 0$ for all $x \in [0, b]$ and B satisfies $B(x_\nu) = 0$

for distinct $x_\nu \in [0, b]$, $\nu = 1, \ldots, R$. f is called an oscillating decay-type function with B(x) its oscillation factor. Let V denote an n-dimensional Haar subspace of C[0, b], fix ρ a positive number, and set K equal to

$$K = \left\{ r(x) = \frac{B(x)}{(p(x))^\rho} : p \in V \text{ and } p(x) > 0 \text{ for all } x \in [0, b] \right\}.$$

He remarks that the approximation of such functions with elements of K is frequently needed in various branches of physics and chemistry. In this setting, a theory of uniform approximation of functions of D[0, b] by K is given. Existence is claimed but in a followup paper by T a y l o r and W i l l i a m s [158] a counterexample to this claim is given and sufficient assumptions including smoothness requirements. Also, it is noted that in general for $\| q \| \neq 1$, examples of nonuniqueness are readily available.

The second type of problem that we wish to describe in this subsection is the problem of simultaneous approximation and interpolation with norm preservation (SAIN Approximation). This problem was first introduced by Deutsch and Morris [32] and to date all studies have been concerned with either Weierstrass or Jackson type of results. It appears that it might be possible to combine the techniques of approximation with Lagrange interpolatory constraints [31] and that of Ross and Belford [136] to obtain characterization results for this theory. Following the paper of Lambert [89], we first state two definitions.

Definition 2.2.1 ([89]) *The triple* (X, M, Γ) *satisfies the hypotheses of the* SAIN problem *if* X *is a normed linear space,* M *a dense subspace of* X *and* Γ *a finite dimensional space of* X^*. *The triple* (X, M, Γ) *has a* SAIN solution *at* $x \in X$ *if given* $\epsilon > 0$, *there exists* $y \in M$ *such that* $\| x - y \| < \epsilon$, $\| x \| = \| y \|$ *and* $\gamma(x) = \gamma(y)$ *for every* γ *in* Γ. *The triple* (X, M, Γ) *is said to have* property SAIN *if* (X, M, Γ) *has a* SAIN solution *for every* $x \in X$.

Definition 2.2.2 ([89]) *Let* X *be a normed linear space and* M *a dense subset of* X. *A linear functional* $x^* \in X^*$ *is said to be a* SAIN functional *if* (X, M, x^*) *has property* SAIN. *A finite sequence* $x_1^*, x_2^*, \ldots, x_n^*$ *is said to be a* SAIN sequence *in case every* $x^* \in \text{span}\{x_1^*, \ldots, x_n^*\}$ *is a* SAIN *functional.*

Now for the special case that $X = C[a, b]$ and $M = \Pi = \cup_{n \geq 0} \Pi_n$, Johnson [71] proved ([89] has a shorter alternate proof of this result)

Theorem 2.2.2 ([71]) $(C[a, b], \Pi, \Gamma)$ *has property* SAIN *if and only if* Γ *is a* SAIN *sequence.*

Since Johnson also proves that $(C[a, b], \Pi, x^*)$ has property SAIN whenever x^* has finitely atomic support, one has that if $\Gamma = \{e_{t_1}, \ldots, e_{t_k}\}$ where $e_t(f) = f(t)$ for all $f \in C[a, b]$ then $(C[a, b], \Pi, \Gamma)$ has property SAIN. That is, given $f \in C[a, b]$ and $\epsilon > 0$, there exists $p \in \Pi_n$ (some n) such that $\| f - p \| < \epsilon$, $p(t_i) = f(t_i)$, $i = 1, \ldots, k$ and $\| p \| = \| f \|$. (This special result was first proved by Deutsch and Morris.)

In addition, Johnson [72] has proved the following two theorems for

"weak SAIN" (requires $\| p \| \leqq \| f \|$ rather than $\| p \| = \| f \|$) and SAIN settings.

Theorem 2.2.3 ([72]) *Let* $f \in C(T)$ *where* T *is a compact, Hausdorff set. Suppose* $\{x_1, \ldots, x_k\} \subset T$ *and that* $|f(x_i)| < \| f \|$ *for* $i = 1, \ldots, k$. *Then there exists* C *and* N *such that for every* $n \geqq N$, *there is a* $p_n \in \Pi_n$ *for which* (1) $p_n(x_i) = f(x_i)$, $i = 1, \ldots, k$, (2) $\| p_n \| \leqq \| f \|$ *and* (3) $\| f - p_n \| \leqq C \inf\{\| f - p \|: p \in \Pi_n\}$.

Theorem 2.2.4 ([72])

These results should be compared with both the hypotheses and conclusions of Theorems 2.1.9 and 2.1.12, respectively, to see the additional cost in requiring (2) in addition to Lagrange interpolatory constraints (for the second comparison set m = n and $y_i^* = e_{x_i}$ for i = 1, . . . , n in Theorem 2.2.4).

2.3 Approximation with functional identity constraints

In 1968, Geiger [51] considered the problem of uniform approximation in $C[-1, 1]$ with

$$K = \left\{ r \in R_n^n[-1, 1]: r(x) = \frac{p(x)}{p(-x)}, p \in \Pi_n, p(x) > 0 \text{ for all } x \in [-1, 1] \right\}.$$

Note that each $r \in K$ satisfies $r(x) \cdot r(-x) \equiv 1$ for all $x \in [-1, 1]$. In this setting, existence, characterization, lower bounds for the error of approximation, normality and degree of approxi-

[illegible]

being approximated, satisfies $f(x) \cdot f(-x) = 1$ on $[-1, 1]$ then $\text{dist}(f, K) \to 0$ as $n \to \infty$ and upper and lower estimated for dist(f, K) are given in terms of $\text{dist}(f, R_n^n[-1, 1])$. For this theory the characterization theorem is

Theorem 2.3.1 ([51]) *Let* $f \in C[-1, 1] \sim K$ *and* $r \in K$. *Then* r *is a best approximation to* f *if and only if one of the following conditions holds:*

(1) 0 *is an extremal point of* $f(x) - r(x)$.

(2) *There are two extremal points* $x_1, x_2 \in [-1, 1]$ *of* $f(x) - r(x)$ *such that* $x_1 = -x_2$ *and* $f(x_1) - r(x_1) = f(x_2) - r(x_2)$.

(3) *There are* N *extremal points* $x_1, \ldots, x_N$ *such that* $0 < |x_1| < \cdots <$

the following conditions: (a) $\bar{r} \in K$; (b) If $r \in K$ and $r > 0$ on $[a, b]$, then for any $x \in [a, b]$ there is a point $y \in [a, b]$ such that $\ln(r(y)/f(y)) = -\ln(r(x)/f(x))$; (3) for any $r \in K - V^+$ there is a $q \in K \cap V^+$ such that $\|(f - g)/f\| < \|(f - r)/f\|$, where $V^+ = \{r \in V: r(x) > 0 \text{ for all } x \in [a, b]\}$. It is noted that the three following approximation problems satisfy these conditions:

(1) *Find the best approximation to* e^x on $-\alpha \leqq x \leqq \alpha$ from $R_n^m[-\alpha, \alpha]$ which satisfies the constraint $r(-x) = 1/r(x)$.

(2) For $0 < \alpha < 1$, find the best approximation to $\sqrt{x}$ on $\alpha \leqq x \leqq 1/\alpha$ *from* $R_n^m[\alpha, 1/\alpha]$ *which satisfies the constraint* $r(1/x) = 1/r(x)$.

(3) *For* $n > 0$ and $0 < \alpha < 1$, *find the best approximation to* $\sqrt{x}$ *on* $\alpha \leqq x \leqq 1/\alpha$ *from* $R_n^{n+1}[\alpha, 1/\alpha]$ *which satisfies the constraint* $xr(1/x) = r(x)$.
In this setting it is proved that

Theorem 2.3.3 ([47]) $\|(f - \bar{r})/f\| = \inf_{r \in K} \|(f - r)/r\| = e^{\bar{\lambda}} - 1$.

2.4 Approximation with linear equalities imposed on the coefficients

In 1965, Brosowski [16] studied the problem of approximating $f \in C[a, b]$ by a linear combination $\sum_{j=1}^{n} a_j h_j$ of functions in $C[a, b]$ whose coefficients satisfy certain linear equalities. Precisely, let $C = (c_{ij})$ be an $m \times n$ matrix ($m < n$) of rank m and let $\underset{\sim}{d} = (d_1, \ldots, d_m)^T \in R^m$. Then define K by

$$K = \left\{ p = \sum_{j=1}^{n} a_j h_j : \sum_{j=1}^{n} c_{ij} a_j = d_i,\ i = 1, \ldots, m \right\}$$

where it is assumed that $\operatorname{span}\langle h_1, \ldots, h_n \rangle$ is a Haar subspace of $C[a, b]$.

In this setting existence, a sufficient alternation condition and other general results are given. In 1967, Gilormini [55] announced the results of the generalization of this problem to a rational setting. Existence, characterization and other results which agree with the standard nonconstrained theory are stated. As was noted earlier, the existence claim was subsequently modified [54].

2.5 Approximation by polynomials having integer coefficients

There is now a large body of results in the area of approximation by polynomials with coefficients which are, in some sense, integers. A detailed survey by Ferguson [46], who graciously provided this subsection, is in existence and should be consulted for more details.

In the most general setting one considers a normed space of functions defined on a set X and approximates elements of the space by elements of a subspace A[X]. The simplest case of this theory is that of C[a, b], with A[a, b] the ring of polynomials with coefficients in A, the set of rational integers ($\{0, \pm 1, \pm 2, \ldots\}$). The elements of A[a, b] are called integral polynomials.

[illegible]

There are two main distinctions in this setting between approximation by integral polynomials and approximation by polynomials with arbitrary real coefficients. On the one hand, if the interval is too large then nontrivial approximation by integral polynomials may be impossible, i.e., the only approximable functions in $C[a, b]$ are those which are identically equal to an integral polynomial. This is the case if and only if $(b - a) \geqq 4$. On the other hand suppose that $f \in C[a, b]$ and $(b - a) < 4$. It is obvious that if n is an integer in $[a, b]$ and $f(n) \notin A$ then f cannot be approximated (arbitrarily closely) by integral polynomials. In this case there is a finite subset J, the "algebraic kernel" of $[a, b]$, such that an element $f \in C[a, b]$ can be approximated by integral polynomials if and only if it can be interpolated by them on J. This is proved in Hewitt and Zuckerman [64]. These results have been generalized to the "complex case" where X is a certain compact subset of the complex plane and the ring of integers A is any discrete subring of rank 2 of the complex numbers. See Fekete [40], [41] and Ferguson [43], [44]. These qualitative results have also been extended to a number theoretic setting by Cantor [17].

In order to illustrate the quantitative side of the subject we quote the following Jackson-type result of Trigub [164].

Theorem 2.5.1 ([164]) *Let* $f \in C^r[a, b]$ *and* J *be the algebraic kernel of* $[a, b]$. *Then there exists a number* C *depending on* f, r *and* $[a, b]$ *but not on* n *or* $x \in [a, b]$ *and a sequence* $\{q_n\}$ *of polynomials in* $A[a, b]$, *with degree of* $q_n \leqq n$, *such that for* $a \leqq x \leqq b$, $0 \leqq \nu \leqq r$ *and any positive integer* n *we have*

$$|f^{(\nu)}(x) - q_n^{(\nu)}(x)| \leqq C\left(\max\left\{\frac{\sqrt{(x-a)(b-x)}}{n}, \frac{1}{n^2}\right\}\right)^{r-\nu}$$

if and only if there is an element q *of* $A[a, b]$ *satisfying*

$$g^{(\nu)}(x) = f^{(\nu)}(x) \qquad 0 \leqq \nu \leqq r,\ x \in J.$$

For other results, see work by Andria [3], Ferguson [45], and Müller [115].

3 Inequality Constraints

3.1 Approximation with constraints on the ranges of the approximants

Probably the most natural approximation problem in this category is the problem of one-sided approximation. A detailed study of this problem was given by Kammerer [79] in 1959. This problem was generalized to allow more general control of the range of the approximating functions by Laurent [91], Taylor [159], [161], [162], Schumaker and Taylor [151] (restricted range approximation), and Duffin and Karlovitz [34] (non-negative approximation). In what follows we shall briefly describe the theory of restricted range approximation as developed in [161], [162]. This theory contains both the one-sided and non-negative approximation as special cases. (In the study of Laurent one-sided approximation was considered where the constraints were allowed to exist outside the domain of approximation.) Thus, let X be a compact subset of $[a, b]$ containing at least $n + 1$ points and let W be an n-dimensional Haar subspace of $C[a, b]$. Define two extended real-valued functions ℓ and u defined on X subject to the following restrictions:

(i) $-\infty \leqq \ell(x) < \infty$ *and* $-\infty < u(x) \leqq \infty$ *for all* $x \in X$,

(ii) $X_{-\infty} = \{x \in X: \ell(x) = -\infty\}$ *and* $X_{+\infty} = \{x \in X: u(x) = \infty\}$ *are open subsets of* X.

(iii) ℓ *is continuous on* $X \sim X_{-\infty}$ *and* u *is continuous on* $X \sim X_{\infty}$.

(iv) $\ell < u$ *for all* $x \in X$ *or*

(iv)′ $\ell \leqq u$ *for all* $x \in X$ *with equality holding for at most* n *points.*

In this setting one defines K by

$$K = \{p \in W: \ell(x) \leqq p(x) \leqq u(x) \text{ for all } x \in X\}.$$

In case (iv), a general theory has been developed in [159] which includes existence, characterization (zero in the convex hull and alternation – see [94] for a Kolmogorov criterion and a linear functional characterization), uniqueness, a de La Vallée Poussin estimate, strong uniqueness and a Lipschitz continuity of the best approximation operator. Except for existence and a special characterization result, all of these results are proved for the class of functions

$$\tilde{C}(X) = \{f \in C(X): \ell(x) \leqq f(x) \leqq u(x) \text{ for all } x \in X\}.$$

Here the alternation theory is dependent upon enlarging the set of critical points. Precisely, fix $f \in \tilde{C}(X)$ and $p \in K$. Define X_p and X_c by $X_p = \{x \in X: |f(x) - p(x)| = \|f - p\|\}$ and $X_c = \{x \in X: p(x) = \ell(x) \text{ or } p(x) = u(x)\}$. If $f \notin K$ and $\ell < u$ we define σ on $X_p \cup X_c$ by $\sigma(x) = \operatorname{sgn}(f(x) - q(x))$ if $x \in X_p \sim X_c$, $\sigma(x) = +1$ if $x \in X_c$ with $p(x) = \ell(x)$ and $\sigma(x) = -1$ if $x \in X_c$ with $p(x) = u(x)$. In this case σ is well defined and the following alternation theorem holds.

Theorem 3.1.1 ([159]) *Under the above assumptions* p *is a best approximation to* $f \in \tilde{C}(X) \sim K$ *if and only if there exist* n + 1 *consecutive points in* $X_p \cup X_c$, $\{x_i\}_{i=1}^{n+1}$, *at which* $\sigma(x_i) = (-1)^{i+1}\sigma(x_1)$ *for* $i = 1, \ldots, n + 1$ *holds.*

Thus by setting $\ell(x) = f(x)$ and $u(x) = +\infty$ for all $x \in X$ one has the problem of one-sided approximation (from above); by setting $\ell(x) = 0$ and $u(x) = +\infty$ one has the problem of approximating with nonnegative approximants; if one selects $x_1, \ldots, x_k$ in X, $\epsilon_i > 0$ (for all i) and sets $\ell(x_i) = f(x_i) - \epsilon_i$ and $u(x_i) = f(x_i) + \epsilon_i$ then one has the problem of finding best ϵ-interpolators [159] by the usual arguments.

For the case of (iv)′ (equality) existence and uniqueness were developed in [151] under the assumption that ℓ and u are continuous on X. In [162] a general alternation theorem was given for the case (iv)′ where it was assumed that ℓ and u have special local Taylor expansions in a neighborhood of each point where $\ell = u$ holds in addition to axioms (i)–(iii). Also, given were a zero in the convex hull theorem and it was remarked there that strong uniqueness and continuity of the best approximation operator follow by the usual arguments. More recently, Sippel [155] and Ling [98] have also considered the equality case. In [155] a slightly different alternation theory is presented with different assumptions on the behavior of the restraining curves ℓ and u at the points of equality. In [98], an alternation theory is developed for a slightly larger class of functions than those considered in the above theories. Also, in [23], [24], [25], Cimoca generalized the theory of restricted range approximation to a theory combining the work of [159] and [91], to one involving linear mappings and to one where all functions are composed with a monotone function prior to imposing the restricted range conditions, respectively. See, also, [37] for a recent paper on an aspect of one-sided approximation.

Concerning the question of computation of best restricted range approximations, Jones and Karlovitz [77] developed a Remez-type algorithm for nonnegative approximants. This was generalized to give a single point exchange for the restricted range problem ($\ell < u$) by Taylor and Winter [163]. A multiple exchange algorithm with applications to designing digital filters was developed independently by Hersey, Tufts and Lewis [63] and Gimlin, Cavin and Budge [57].

Lewis [95] gave a proof of convergence of the multiple exchange in this setting when X is a finite set and a detailed analysis of the discretization error when an interval is replaced by a finite set. In addition, he considered the digital filter design problem and discussed the numerical implementation of this algorithm. Another recent application of the restricted range multiple exchange has been by Hull and Taylor [70] where an adaptive curve fitting routine was developed.

For computation in the case of constraints of the form (iv)′, a Remez-type algorithm based on the use of generalized weight functions is described in [113]. In [98] some suggestions for the development of a Remez-type algorithm are also given. To our knowledge, there are no (running) Remez-type multiple exchange codes for this particular problem at present. Recently, Watson [171], [172] has developed algorithms for calculating best linear one-sided Chebyshev approximations and best restricted range approximations, respectively. These algorithms are based on linear programming and in the second algorithm the Haar subspace assumption is weakened.

In 1975, Johnson [73] studied the problem of whether or not K is empty. In this study necessary and sufficient conditions on the restraining curves ℓ and u are given to guarantee that K is not empty when $W = \Pi_n$ for sufficiently large n. Also given is a Weierstrass-type theorem for approximating any f satisfying $\ell(x) \leqq f(x) \leqq u(x)$ from $K = \cup_{n \geqslant 0} \{p \in \Pi_n : \ell(x) \leqq p(x) \leqq u(x) \text{ for all } x \in X\}$, X a compact subset of [a, b]. In another study, Johnson [67] derived a Weierstrass theory for a combination of one-sided approximation and a finite number of general interpolatory constraints. Precisely, a set of n bounded linear functionals, $\{\lambda_1^*, \ldots, \lambda_n^*\}$, is said to be *span indefinite* provided each nontrivial linear combination of this set is not a positive linear functional. Then, it is proved

Theorem 3.1.2 ([73]) *Suppose* $\lambda_1^*, \ldots, \lambda_n^*$ *are linearly independent bounded linear functionals on* C[a, b]. *Then, for any* $f \in C[a, b]$ *and* $\epsilon > 0$, *there is a* $p \in \Pi_k$ (*some* k) *for which* (i) $p(x) \geqq f(x)$ *for all* $x \in [a, b]$, (ii) $\lambda_i^* p = \lambda_i^* f$, $i = 1, \ldots, n$ *and* (iii) $\|f - p\| < \epsilon$ *if and only if the* $\lambda_1^*, \ldots, \lambda_n^*$ *are span indefinite on* C[a, b].

Observe that any set containing a point evaluation will not be span indefinite. Thus, one cannot have a Weierstrass theory if one wishes to combine one-sided approximations with Lagrange interpolatory constraints. However, it does seem reasonable that for smooth functions both Lagrange and Hermite constraints could be combined with one-sided approximation to give a Weierstrass type theorem. Also, a Jackson-type result in this setting is proved in [74]. More recently, Johnson has studied restricted range approximation with side conditions in [75], [76]. In these papers restricted range approximation with algebraic polynomials subject to n linear functional interpolatory constraints is considered. Both papers characterize those n-tuples of linear functionals for which a Weierstrass theory holds with the second paper obtaining results when one also assumes that the functions being approximated are continuously differentiable.

The theory of restricted range approximation has also been extended to rationals and other more general approximating families. In [102], Loeb, Moursund and Taylor developed the theory corresponding to case (iv) and for continuous ℓ and u and generalized rationals. Characterization and uniqueness results are given and for the case that the domain of approximation is an interval [a, b] and $W = R_n^m[a, b]$, an existence theorem is given. That ℓ and u are continuous is essential to the proof of this existence theorem. In [103], results concerning the continuity of the best approximation operator for restricted range approximations in a setting of generalized rational functions are given. Included is a strong uniqueness theorem and a Lipschitz continuity theorem for approximating normal functions, as well as some convergence results for the non-normal case. More general studies are the work of Taylor [164], where a general theory for the rational case with the restraining curves and approximation imposed on different sets was studied; and the work of Dunham [38], [39] and Torgna [165] where this theory was independently extended to general approximating families.

A recent problem involving constraints on the values of the approximants is that of copositive approximation which was introduced by Passow and Raymon [122]. In this theory, K is defined corresponding to $f \in C[a, b]$ by

$$K = \{p \in W: p(x)f(x) \geqq 0 \text{ for all } x \in [a, b]\}.$$

Note that $0 \in K$. Each function in K is said to be copositive with f on [a, b]. In [122], W is taken to be Π_n and existence, uniqueness and a Jackson-type theorem are proved.

Existence follows from the usual compactness argument. The uniqueness result follows via an application of restricted range theory (equality case). The Jackson theorem is developed for a restricted class of functions (proper piecewise monotone functions) giving the same order as in the nonrestricted theory ($O[\omega(f; 1/n)]$). For the same class of functions, Hill, Passow and Raymon [65] prove that there exists a sequence of polynomials $\{p_n\}$ such that $p_n \in \Pi_n$, p_n and f are simultaneously comonotone and copositive for n sufficiently large and p_n converges uniformly to f on [a, b]. Finally, in [123], Passow and Taylor develop an alternation theory for copositive approximation under the assumption that W is an extended n-dimensional Chebyshev subspace of C[a, b] of order 3.

More recently, approximation with rationals having negative poles has been studied [83], [84], [90]. In [84], existence, local characterization (alternation), local uniqueness and computational results are given. This theory was motivated by the desire to approximate e^{-x} on $[0, \infty)$ with such approximants in order to develop numerical algorithms for solving linear systems of ordinary differential equations. Many open questions remain. The degree of approximation for the special function e^{-x} from various classes of rational functions has been the focus of much work by Saff and Varga. See [150] for a recent paper of theirs.

3.2 Approximation with constraints on values of the approximants and their derivatives

In this setting the first problem studied was that of monotone approximation. The first published work was due to Shisha [153] where questions concerning the degree of approximation were studied. A second early study was that of Culbertson [26] who proved characterization and uniqueness results for a specific problem. Although historically, to some extent, results on the degree of approximation preceded the theory of best approximation for this problem, we shall first describe the theory of best approximation. Thus, fix an interval [a, b], integers $1 \leqq r_1 < \cdots < r_k$, signs $\epsilon_i = \pm 1$, $i = 1, \ldots, k$ and define $K = K(r_1, \ldots, r_k; \epsilon_1, \ldots, \epsilon_k)$ by

$$K = \{p \in \Pi_n: \epsilon_j p^{(r_j)}(x) \geqq 0, a \leqq x \leqq b, j = 0, 1, \ldots, k \text{ with } k \leqq n\}.$$

Denote $K(r; +1)$ by K_r in what follows. The first major study of this problem was given by Lorentz and Zeller [105], [106]. Existence, characterization and a partial uniqueness result were given in these studies. One of the theorems given was (set $A = \{x: |f(x) - p(x)| = \|f - p\|\}$ and $B_j = \{x: p^{(r_j)}(x) = 0\}$, $j = 1, \ldots, k$)

Theorem 3.2.1 ([106]) *A polynomial* $p \in K$ *is a polynomial of best approximation to* $f \in C[a, b] \sim K$ *if and only if there exist points* x_i $(i = 1, \ldots, \mu)$ *of* A, y_{ij} $(i = 1, \ldots, \lambda_j)$ *of* B_j $(j = 1, \ldots, k)$ *with* $\mu + \sum_{j=1}^{k} \lambda_j \leqq n + 2$ *and numbers* $a_i \neq 0$, $b_{ij} > 0$, *such that*

$$\sum_{i=1}^{\mu} a_i(f(x_i) - p(x_i))q(x_i) + \sum_{j=1}^{k} \sum_{i=1}^{\lambda_j} b_{ij} q^{(r_j)}(y_{ij}) = 0$$

for all polynomials $q \in \Pi_n$.

Roulier in [138] also gives a characterization theorem for this general problem and Culbertson [26] proved this particular characterization theorem for the special case of K_1, as well as uniqueness. Uniqueness of K_r, $r \geqq 1$, was also proved in [106]. Uniqueness for the full class K was proved by Lorentz [120]. The proof of this theorem used a sufficient condition for a certain Hermite-Birkhoff interpolation problem to have a unique solution which was proved independently in [4] and [42] (in an insignificantly weaker form in the second paper) a few years prior. Theorems corresponding to strong uniqueness and continuity of the best (monotone) approximation operator have not been investigated to our knowledge. In [96], Lewis gave an algorithm for computing the best approximation from K_r based on linear programming and discretization. In [20], Chalmers gave a general treatment of the Remez exchange which includes this theory as a special case. Finally in an excellent survey paper on monotone approximation, Lorentz [107] gave a de La Vallée Poussin type of lower estimate for this theory (among many other results). (Added in proof: Recent results have been obtained showing that a strong uniqueness theorem of order 1/2 is a best possible result for monotone approximation [177], [178]. A general study of this question is given in [179].)

In order to discuss some of the work done on questions concerning the degree of approximation in this theory, we define for $f \in C[a, b]$ and f increasing, $E_n(f) = \inf\{\|f - p\|: p \in \Pi_n\}$ and $E_n^*(f) = \inf\{\|f - p\|: p \in K\}$, where $K \subset \Pi_n$ with the precise side conditions on K stated in each instance. In 1968 studies by Roulier [139] and Lorentz and Zeller [108] extended the original work of Shisha. In the first of the papers [108], a complete analog of Jackson's theorem is obtained: (Recall $K_1 = \{p \in \Pi_n: p'(x) \geqq 0 \text{ for all } x \in [a, b]\}$)

Theorem 3.2.2 ([108], I) *For the problem* K_1, *and an increasing function* $f \in C[-1, 1]$, $E_n^*(f) \leq C\omega(f; 1/n)$ *with some absolute constant* C.

An even stronger result of this sort is given in [107], namely that $E_n^*(f) = O\left(\frac{1}{n}\,\omega\left(f'; \frac{1}{n}\right)\right)$ if $f \in C^1[a, b]$ and is monotone. Recently, DeVore [29] has extended this result to the following

Theorem 3.2.3 ([29]) *If* $f \in C^m[a, b]$ *and* f *is monotone on this interval, then for* $n > m + 1$

$$E_n^*(f) = O\left[\frac{1}{n^m}\,\omega\left(f^{(m)}; \frac{1}{n}\right)\right]$$

for the problem K_1.

In the second paper of [108] it is shown however that

Theorem 3.2.4 ([108], II) *For each* $r = 1, 2, \ldots,$ *there exists* $f \in C^r[-1, 1]$ *with* $f^{(r)}(x) \geqq 0, -1 \leqq x \leqq 1,$ *for which, for the problem* K_r ($K_r = \{p \in \Pi_n: p^{(r)}(x) \geqq 0, -1 \leqq x \leqq 1\}$)

$$\limsup_{n\to\infty} E_n^*(f)/E_n(f) = +\infty.$$

Zeller [176], Roulier [140], [141], Beatson [10] and Lim [97] have continued these studies and in [140], for example, it is shown that if f has infinitely many continuous derivatives on [a, b] and $\epsilon_i f^{(r_i)}(x) > 0$ on [a, b], $i = 1, \ldots, k$ then for n sufficiently large $E_n(f) = E_n^*(f)$ using $K = K(r_1, \ldots, r_k; \epsilon_1, \ldots, \epsilon_k)$. In [141] one has

Theorem 3.2.5 ([141]) Let $f' \in C[0, 1]$ *and assume that* $f'(x) \geqq d > 0$ *on* [0, 1]. *Then, if* f *is not a polynomial,*

$$\limsup_{n\to\infty} E_n^*(f)/E_n(f) \leqq 2$$

for the problem K_1.

Theorem 3.2.6 ([141]) *Assume* $\epsilon_i f^{(r_i)}(x) > 0$ *on* $[-1, 1]$ for $i = 1, \ldots, k$ *and that for some integer* $r \geqq r_k$, $f^{(r)} \in C[-1, 1]$ *but* $f^{(r+1)} \notin C[-1. 1]$. *Then there exists* $p \in K = K(r_1, \ldots, r_k; \epsilon_1, \ldots, \epsilon_k)$ *such that*

$$|f(x) - p(x)| \leqq C_r \Delta_n(x)^r \omega(f^{(r)}; \Delta_n(x))$$

and for n *sufficiently large,* $\epsilon_i p^{(r_i)}(x) > 0$ for $-1 \leqq x \leqq 1$ and $i = 1, \ldots, k$, *where* $\Delta_n(x) = \max\{(1 - x^2)^{1/2}/n, 1/n^2\}, n > 0, \Delta_0(x) = 1$.

In [119], [120], Passow and Raymon state a theorem similar to Theorem 3.2.5 and also consider this theory when the assumption $\epsilon_i f^{(r_i)}(x) > 0$ for all $x \in [-1, 1]$ is replaced by "$\geqq$". Thus, one has (see [10] for a stronger result)

Theorem 3.2.7 ([119]) *If* f *is a function such that for all* x *the* r^{th} *difference* $\Delta^r f(x) \geqq 0$ *on* [a, b], *then for any* $\epsilon > 0$ *there is a constant* $d_{r,\epsilon} > 0$ *such that for* n *sufficiently large*

$$E_n^*(f) \leqq d_{r,\epsilon}\, \omega(f; 1/n^{1-\epsilon})$$

for the problem K_r.

In addition, Passow and Roulier [121] show that there exist functions in $C[-1, 1]$ with all r^{th} divided differences uniformly bounded away from zero for r fixed, for which infinitely many of the polynomials of best approximation to f do not have nonnegative r^{th} derivatives on $[-1, 1]$. This generalizes an earlier paper of Roulier [142]. More recently, Ubhaya [167], [168] has defined moduli of monotonicity which are one-sided analogues of the modulus of continuity and measure the extent to which a given function fails to be monotone. In this setting upper and lower bounds on the degree of approximation by monotone polynomials (K_1) for general functions are given. Also, Ubhaya [169], [170] has introduced a theory of isotone optimization. For the special case that the domain of the functions is a closed interval this theory reduces to approximating with monotone (nondecreasing) functions. See these papers for the application of duality techniques to this problem.

The theory of monotone approximation was generalized independently by space of approximants, K, is defined as follows. Fix a set of nonnegative integers, approximation. However, according to R a y m o n [135] quantitative estimates for the degree of coconvex (and higher) approximation have, for the most part, not yet been obtained.

Further generalizations of this theory included the combining of comonotone constraints with interpolatory constraints at the turning points of f. For results on the degree of approximation in this setting see papers by F o r d and R o u l i e r [48], P a s s o w and R a y m o n [123] and H i l l, P a s s o w and R a y m o n [65]. In [48], higher order derivative constraints are allowed in a Jackson theorem; whereas, the focus of [123] is to estimate the degree of a

S h i s h a [124] considered corresponding theories when W consists of "Müntz polynomials". Many gaps remain to be filled in this theory. Also, for all of the above theories other than the monotone case, a theory of best approximation remains to be done.

Two recent studies that also belong to this classification have been written by L i n g, R o u l i e r and V a r g a [99] and K e e n e r and L i n g [85]. In the first paper, three sufficient conditions for a real-valued function $f \in C[0, \infty)$ are established that ensure that for some nonnegative integer n, there is a nonnegative number r(n) so that for any $r \geqq r(n)$, the polynomial of best approximation to f on [0, r] from Π_n is increasing and nonnegative on $[r, \infty)$. The motivation for such a study was the method of proof used in the study of rational Chebyshev approximation to reciprocals of entire functions as in [111], [112], [146]. In the second study, one requires $W \subset C^1[a, b]$ and defines K by

b

$|a_k| \leqq A_k^k\}$, and $C_0 = \{f \in C[0, 1]: f(0) = 0\}$. Then, determine necessary and sufficient conditions on the sequence S so that H_S is dense in C_0. Subsequent studies have been given by Roulier [147], [148] and von Golitschek [60]. In [60], the following theorem is proved.

Theorem 3.3.1 ([60]) *Let* $\{m_q\}$ *be a sequence of positive integers for which* $0 < m_1 < m_2 < \cdots$ *and* $\sum_{q=1}^{\infty} 1/m_q = \infty$. *If* $\{w_q\}$ *is a sequence of nonnegative real numbers for which* $\lim_{q\to\infty} w_q = \infty$, *then for each* $f \in C_0$ *and each real number* $\epsilon > 0$ *there exists a polynomial* $p(x) = \sum_{q=1}^{s} c_q x^{m_q}$ *such that* $\|f - p\| < \epsilon$ *and* $|c_q| < (\epsilon w_q)^{m_q}$ *for* $q = 1, 2, \ldots, s$.

It is also remarked that this result is best possible in a certain sense. Also, this problem is studied for the interval [a, b], $0 < a < b$, where the results are quite different and, in fact, here polynomials of the form $\sum_{q=1}^{n} c_q x^{m_q}$, with $|c_q| \leqq 1$ for all q and the sequence $\{m_q\}$ as in Theorem 3.3.1, are dense in C[a, b]. More recently, Bak, von Golitschek and Leviatan [5] have considered Jackson-type questions in this setting.

In 1971, Roulier and Taylor [149] studied the problem of approximating with polynomials having bounded coefficients. Specifically, let $f \in C[0, 1]$ and $\{\alpha_\nu\}_{\nu=1}^{k}$, $\{\beta_\nu\}_{\nu=1}^{k}$ be two sets of extended real numbers such that $-\infty \leqq \alpha_\nu \leqq \beta_\nu \leqq \infty$, $-\infty < \beta_\nu$ and $\alpha_\nu < \infty$ for all ν. Let $n \geqq k - 1$ and fix $\{m_i\}_{i=1}^{k}$ a set of k integers satisfying $0 \leqq m_1 < m_2 < \cdots < m_k \leqq n$. Then K is defined by

$$K = \{p(x) = \sum_{i=0}^{n} a_i x^i : \alpha_\nu \leqq a_{m_\nu} \leqq \beta_\nu \quad \text{for} \quad \nu = 1, \ldots, k\}.$$

In this setting, existence, a necessary alternation-type characterization and a "natural" subset of functions from C[0, 1] for which uniqueness holds are established. It is easily seen that this theory is a special case of approximation with polynomials with restricted ranges on their derivatives (equality case, possibly). Subsequently, Chalmers [20] developed an elegant necessary and sufficient alternation characterization for this problem (completing that given in [149]) and a Remez-type algorithm for this problem.

Finally, in a recent paper Newman and Reddy [118] consider the problem of approximating x^n on [0, 1] by polynomials and rational functions having only nonnegative coefficients and of degree at most k ($1 \leqq k \leqq n - 1$). An explicit solution to this problem is given.

4 General Studies

In this section various results in the direction of unifying the theory described in the preceeding sections will be discussed.

4.1 Approximation with linear restrictions

We begin with a general functional analysis setting which will include all of the linear examples discussed previously in this survey. The work [20], [21] of

Chalmers will be described to do this and we believe that this approach introduces no unneeded abstractions. Thus, let E denote a compact subset of the real line and V^n an n-dimensional subspace of C(E). For some fixed index set A, let $\{L^\alpha\}_{\alpha \in A}$ be a compact set of linear functionals in the dual, $(V^n)^*$, of V^n, such that, for each p in V^n, $L^\alpha p$ is a continuous function on A, where A inherits the norm topology of $\{L^\alpha\}_{\alpha \in A}$ (i.e., dist $(\alpha_1, \alpha_2) = \| L^{\alpha_1} - L^{\alpha_2} \|_*$ where $\| \cdot \|_*$ denotes the usual induced norm on $(V^n)^*$, for each $\alpha_1, \alpha_2 \in A$). Set

$$V_0^n = \{p \in V^n; \ell(\alpha) \leq L^\alpha p \leq u(\alpha), \alpha \in A\},$$

where ℓ and u are extended real-valued functions on A with $\ell < +\infty$, $u > -\infty$, the set E_ℓ (resp. E_u) on which ℓ (resp. u) is finite is closed, ℓ (resp. u) is continuous on E_ℓ (resp. E_u), and $\ell(\alpha) \leq u(\alpha)$.

Let e_x represent point evaluation at x in E (i.e., $e_x(f) = f(x)\ \forall\ f \in C(E)$).

Fix f in $C(E) \sim V_0^n$ with the restriction that if $L^\alpha = e_x$ for some α in A and some x in E, then $\inf\{\| f - p \|; q \in K\} > \max\{\ell(\alpha) - f(x), f(x) - u(\alpha)\}$; note that this inequality is assured if for example $\ell(\alpha) \leq f(x) \leq u(\alpha)$. Call such an f *admissible*. We are concerned then with approximating such admissible f by elements of V_0^n. One can check that all of the linear examples mentioned in the previous sections of the paper are described in the above setting.

For example, ordinary monotone approximation, where $K = \{p \in \Pi_{n-1}[a, b]; p' \geqq 0\}$, fits into the scheme as follows. Set $E = [a, b]$, $V = \Pi_{n-1}$, and $A = \{(x, 1); x \in E\}$. For each $\alpha = (x, 1) \in A$, set $L^\alpha p = p'(x)$, $\ell(\alpha) \equiv 0$, $u(\alpha) \equiv +\infty$. Then $K = V_0^n = \{p \in V^n; \ell(\alpha) \leqq L^\alpha p \leqq u(\alpha), \alpha \in A\}$. Note that A, with the topology induced by the norm topology on $\{L^\alpha\}_{\alpha \in A} \subset (V^n)^*$, is homeomorphic to E with the usual topology. Thus, $L^\alpha p = p'(x)$ is a continuous function on A for each p in V^n. Moreover, any $f \in C(E) \sim V_0^n$ is vacuously admissible.

By the usual continuity and compactness argument we have the following result.

Theorem 4.1.1 (Existence) (e.g. [21]) *If* V_0^n *is not empty, then there exists a best approximation in* V_0^n *to* f.

Definition 4.1.1 ([21]) *If* $\ell(\alpha) = u(\alpha)$ *implies* α *is an isolated point of* A, *we will say* equality condition 1 (EQC1) *is satisfied.*

Most of the preceding linear examples satisfy EQC1. The remaining examples (e.g., restricted range with equality constraints and copositive approximation) fail EQC1 and they will be mentioned later in this section.

Note 4.1.1 *Throughout the remainder of this section it is to be assumed that* EQC1 *is satisfied unless otherwise stated.*

In [21] the following definitions generalizing the concept of a Haar subspace are given.

Definition 4.1.2 ([21]) *For* p *in* V_0^n *a set* $S = \{L^\alpha\}_{\alpha \in I_1} \cup \{e_x\}_{x \in I_3}$ *with* $I_1 \subset A$ *and* $I_3 \subset E$ *is called an* extremal set *for* f *and* p *provided*

(i) $L^\alpha p = u(\alpha)$ (or $\ell(\alpha)$), $\alpha \in I_1$

(ii) $|e_x(f - p)| = \| f - p \|$, $x \in I_3$

(iii) $e_x \notin \{L^\alpha\}$ if $|(f - p)(x)| = \| f - p \|$

Definition 4.1.3 ([21]) $S' = S \cup \{M^\alpha\}_{\alpha \in I_2}$ *is called an* augmented extremal set *for* f *and* p *if* S *is an extremal set for* f *and* p, $I_2 \subset I_1$ *and* $\alpha \in I_2$ *implies that* $M^\alpha p = m(\alpha)$ *where* $m(\alpha)$ *is some real number depending only on* α.

Example In the case of monotone approximation $V_0^n = \{p \in \Pi_{n-1}[a, b]; p' \geqq 0\}$, if $p \in V_0^n$ and $L^\alpha p = p'(x) = 0$ for some $x \in (a, b)$, then $M^\alpha p = p''(x) = 0$.

Notation 4.1.1 *For* f *and* p *fixed,* let S^{max} *denote the* maximal extremal *set for* f *and* p, *i.e.*, $S^{max} = \{e_x; x \in E$ *and* $|e_x(f - p)| = \|f - p\|\} \cup \{L^\alpha; \alpha \in A$ *and* $L^\alpha p = \ell(\alpha)$ (*or* $u(\alpha)$) *and if* $L^\alpha \equiv e_x$ *for some* $x \in E$ *then* $|f(x) - p(x)| < \|f - p\|\}$.

The following "0 in the convex hull" characterization theorem is found in this general setting for example in the works of Laurent [92] and Gehner [50].

Theorem 4.1.3 (Characterization) (e.g. [92]) *p is a best approximation to* f *if and only if* 0 *is in the convex hull of some* k ($\leqq$ n + 1) *members of* S^σ, *i.e.*, $0 = \sum_{i=1}^{k} \lambda_i N_i$ *where* $N_i \in S^\sigma$, $\lambda_i > 0$, $i = 1, 2, \ldots, k \leqq n + 1$.

Note that this theorem is valid in the absence of any Haar conditions. Also, as in the standard theory the presence of our Haar conditions imply k = n + 1 in Theorem 4.1.3.

Another familiar reformulation of this result holds and the proof of equivalency proceeds analogously as in Theorem 2.1.2.

Notation 4.1.2 Let $S_i^{max} = S^{max} - \{L^\alpha ; \ell(\alpha) = u(\alpha)\}$.

Theorem 4.1.3a (Kolmogorov criterion) (e.g. [92]) p *is a best approximation to* f *if and only if*

$$\max\{\max_{e_x \in S^{max}} (f(x) - p(x))q(x), \max_{L^\alpha \in S_i^{max}} - \sigma(L^\alpha) \cdot L^\alpha q\} \geqq 0$$

for all $q \in V^n \cap \{q : L^\alpha q = 0 \text{ if } \ell(\alpha) = u(\alpha)\}$.

In order to develop a Remez exchange algorithm which will work in this general setting, a new Haar-type condition which again holds for all the preceding linear examples satisfying EQC1 is formulated in [20].

Definition 4.1.6 ([20]) V^n *is* nearly Haar *on* $\Omega = \{L^\alpha\}_{\alpha \in A} \cup \{e_x\}_{x \in E}$ *provided the set of* n*-tuples* $(R_1, R_2, \ldots, R_n) \in \Omega^n$, *where the* R_i *are linearly dependent, forms a closed nowhere dense subset of* Ω^n. (Example: *montone approximation and, more generally, restricted derivatives approximation.*) V^n *is* Haar (on Ω) *if any distinct* n *elements in* Ω *are linearly independent.*

to $\|\cdot\|_{E^\nu}$ where $E^\nu = \{x_i^\nu\}_{i=k_\nu+1}^{n+1}$. Without loss of generality we may assume $E^\nu \neq \emptyset$ and by Theorem 4.1.3 the zero of $(V^n)^*$ belongs to the convex hull of S_ν^σ, where the convex combination must actually include every element of S_ν^σ if the Haar condition holds. Let $d_\nu = \|f - p_\nu\|_{E^\nu}$.

In order to construct the $(\nu + 1)^{st}$ extremal set, let N_*^ν be a functional in Ω yielding the maximum deviation of $f - p_\nu$ on Ω; i.e., N_*^ν yields

$$\max\{\max_{\substack{N=\pm e_x\\ x\in E}} N(f-p_\nu) - d_\nu,\ \max_{\substack{N=\pm L^\alpha\\ \alpha\in A}} N(f-p_\nu)\}$$

where the domain of f is extended to $\{\pm L^\alpha\}_{\alpha\in A}$ by setting $L^\alpha f = \ell(\alpha)$ and $(-L^\alpha)f = -u(\alpha)$.

Proceeding as in the standard theory, one then exchanges one of $\{N_i^\nu\}_{i=1}^{n+1}$ for N_*^ν via the Exchange Procedure (see below) to obtain a new extremal set $S_{\nu+1}^\sigma = \{N_i^{\nu+1}\}_{i=1}^{n+1}$. Continuing, one calculates $p_{\nu+1}$ (and therefore also $d_{\nu+1}$) from the n + 1 linear relations:

$$N_i^{\nu+1}p = N_i^{\nu+1}f - d_{\nu+1} \quad \text{if } N_i^{\nu+1} = \pm e_x \in S_{\nu+1}^\sigma$$
$$N_i^{\nu+1}p = N_i^{\nu+1}f \quad \text{if } N_i^{\nu+1} = \pm L^\alpha \in S_{\nu+1}^\sigma.$$

In this way one obtains the desired next extremal set $S_{\nu+1}^\sigma$ and $p_{\nu+1}$, the best approximation to f from $V_{\nu+1}^n$ with respect to $E^{\nu+1}$, and from this point the algorithm proceeds to the next iteration.

Exchange Procedure Suppose (we suppress the superscript ν)

(4.1.1) $0 = \sum_{i=1}^{n+1} \lambda_i N_i$ where $\lambda_i > 0$.

Replace some N_j by N_* so that (4.1.1) is preserved as follows: write $N_* - N_{n+1}$ as a linear combination of the $\{N_i\}_{i=1}^n$; i.e.,

(4.1.2) $N_* = N_{n+1} + \sum_{i=1}^{n} \alpha_i' N_i$.

(This step of course presumes independence of the $\{N_i\}_{i=1}^n$.) Next, multiply (4.1.2) by $\lambda > 0$ so that

(4.1.3) $\lambda N_* = \sum_{i=1}^{n+1} \alpha_i N_i$ where $\lambda_i \geqq \alpha_i$ with equality holding for some i.

Then subtracting (4.1.3) from (4.1.1) completes the exchange.

Note 4.1.3 *Computationally the above exchange involves an* n x n *matrix inversion to determine* $\{\alpha_i'\}_{i=1}^n$ *and a linear search to determine* $\lambda^{-1} = \max_i \alpha_i'/\lambda_i$.

Note 4.1.4 *The above algorithm is derived solely from the* "0 *in the convex hull" characterization of best approximation and does not require an explicit alternation theory (which can be derived in many cases as will be mentioned below).*

Theorem 4.1.4 ([20]) *If* V^n *is Haar, then the Remez algorithm converges to the best approximation.*

For V^n only nearly Haar, in [20] is introduced the notion of a non-singular discretization Ω_r of Ω with "mesh size r" on which V^n is Haar. By a modification and generalization of the work of Lewis [96] and Cheney [22], pp. 84 ff., one can establish that the Remez algorithm also works in case $V^n = \Pi_{n-1}$ is nearly

Haar (e.g., monotone, convex, and more generally restricted derivatives approximation):

Theorem 4.1.5 ([20]) *If* Π_{n-1} *is nearly Haar on* Ω *then the sequence* p_ν^r *generated by the Remez algorithm on a nonsingular* Ω_r *converges to the best approximation* p^r. *Furthermore, if* Π_{n-1} *is generalized Haar on* Ω, *then* $p^r \to p$, $r \to 0$, p *being the unique best approximation.*

In the case of restricted derivatives approximation (e.g., monotone approximation), it can in fact be shown ([96], [23], [20]) that $|\|f - p\| - \|f - p^r\|| \leq C\omega(f; r)$, where ω is the modulus of continuity function and C is some constant independent of r.

In the situation of Theorem 4.1.5, the probability that a random Ω_r be nonsingular is 1. Hence if Ω_{r_m} is a sequence of random discretizations of Ω with $r_m \to 0$, the probability that the Remez algorithm on Ω_{r_m} converges to p^{r_m} and p^{r_m} converges to p is 1.

Definition 4.1.7 ([18]) *The Remez algorithm is* iterative *if the exchange procedure can always be accomplished; i.e., if* $N_*^\nu - N_{n+1}^\nu$ *is in the span of* $\{N_i^\nu\}_{i=1}^n$ *in* (4.1.2), $\nu = 1, 2, \ldots$.

Carasso [18] shows that, in the case where A is empty (no restraints), if V^n is not necessarily Haar but if at least the functions of V^n have no common vanishing point in E, then the Remez algorithm converges to a best approximation if it is iterative. It would be of interest to know under what hypotheses this result can be extended to the problems with constraints.

As has been noted, the Remez exchange algorithm does not require an alternation theory in order to work, but depends only on the "0 in the convex hull" characterization of best approximation. From this characterization, however, can be developed in many cases an elegant alternation theory which can then be used to simplify the exchange procedure (eliminating the n x n matrix inversion and linear search). This simplification derives from the fact that in (4.1.1) of the Exchange Procedure ($0 = \sum_{i=1}^{n+1} \lambda_i N_i$ where $\lambda_i > 0$), the problem is to replace N_j by N_* so that (4.1.1) is preserved. Since each $N_i = \sigma(R_i)R_i$ for some $R_i \in S^{max}$, knowing the signs of the coefficients α_i in $0 = \sum_{i=1}^{n+1} \alpha_i R_i$, where $\lambda_i \sigma(R_i) = \alpha_i$, is equivalent to knowing how to accomplish the replacement (thus eliminating the n x n matrix inversion and linear search). An alternation theory provides this knowledge in the sense that the theory gives the desired sign pattern for each ordered n + 1 tuple from S^{max}. Thus, one adjoins $R_* = \sigma(R_*)N_*$ to $(R_1, \ldots, R_{n+1})$ and then deletes an R_i so that the resultant n + 1 tuple has the required sign pattern, i.e., (4.1.1) is preserved. The examples following the next theorem will help to illustrate how the alternation theory is used to make the exchange. (For use below define sgn R_i = sgn α_i.)

A unified alternation theory which encompasses those alternation schemes of all the preceding linear examples is found in [20]. A representative result is the following.

Notation If A and B are subsets of the real line, in the following $A \geqq B$ will indicate that for $a \in A$ either $a \geqq b$ for all $b \in B$ or $a \leqq b$ for all $b \in B$.

Theorem 4.1.6 (Alternation for pyramid-type restraints combined with poised Birkhoff interpolatory restraints) ([20]) *Let* $V_0^n = \{p \in \Pi_{n-1}[a, b] : \ell_i(x) \leqq p^{(i)}(x) \leqq u_i(x)$ *on* E_i, *where* $[a, b] \subset E_0$ *and* $E_i \geqq \cup_{j=1}^{i-1} E_j$ *for* $i = 0, \ldots, n-1\}$. *Further, let* $\{M_i^j\}_{i=1}^{m_j}$ *be poised Birkhoff interpolatory data on each* E_j, $j = 0, 1, \ldots, n-1$. *Let* $\Sigma\, m_j = m$. *Then without loss of generality (replace* f *by* $f - p_0$ *where* p_0 *satisfies the interpolatory conditions), take all the interpolatory values to be zero and consider* $V_0^{n-m} = V_0^n \cap \{p : M_i^j p = 0\}$. *Then all the extremal functionals are of the form* e_x^i *(where* $e_x^i g = g^{(i)}(x)$*) and the following alternation scheme prevails (where the* x_k *are presumed ordered):*

$$\operatorname{sgn}(e_{x_k}^t) \cdot \operatorname{sgn}(e_{x_{k+1}}^r) = \sigma,$$

where (i) *if* $r < t$, $\sigma = (-1)^{t-r+1}$;

(ii) *if* $r = t$, $\sigma = (-1)^{q+1}$, *where* q *is the number of strictly* Hermite *data in the Birkhoff data* $\{M_i^j\}$ *at all* x *where* $x_k < x < x_{k+1}$;

(iii) *if* $r > t$, $\sigma = -1$.

Note 4.1.5 *This theorem also provides, for example, the alternation scheme for Bounded Coefficients Approximation, which was not described in the preceding sections.*

Note 4.1.6 *The interpolatory conditions need not be subtracted off and the alternation statement may be made for* V_0^n *instead of* V_0^{n-m}. *But the signs of the interpolatory functionals are unimportant since such a functional can always be [illegible] as either and upper or lower restraining functional and is of course never*

Example 3 (Bounded coefficients approximation on (0, 1)) A typical example of $(N_1, \ldots, N_{n+1})$ would be

$$(-e_0^{14})(+e_0^{12})(+e_0^{11})(+e_0^{10})(-e_0^{6})(-e_0^{-5})(-e_{x_1})(+e_{x_2}) \cdots (-e_{x_{k-1}})(+e_{x_k}).$$

That is, ordinary alternation occurs among the $\{e_{x_k}\}$, but among the $\{e_{x_k}^t\}_{t>0}$, the pattern is given by (i) of Theorem 4.1.6.

Example 4 Consider the problem of one-sided restricted range approximation with interpolatory restraints considered in [69]. I.e., $V_0^n = \{p \in \Pi_{n-1}[a, b] : p(x) \geqq f(x)$ for all $x \in [a, b]$ and $p(y_i) = f(y_i) + \epsilon_i (\epsilon_i > 0)$ for $i = 1, 2, \ldots, m$, $a \leqq y_1 < y_2 < \cdots < y_m \leqq b$ and $m < n\}$. Then $p(y_i) = f(y_i) + \epsilon_i$, $i = 1, 2, \ldots, m$, implies that $p(x) > f(x)$ in an open neighborhood N of $\{y_i\}_{i=1}^m$. Thus the inequality conditions $p(x) \geqq f(x)$ can be ignored without loss for all x in N, and we see then that EQC1 holds. Hence in Theorem 4.1.6, take $n = 1$, $M_i^0 = e_{y_i}$, $i = 1, 2, \ldots, m = m_0$. Then $r = t = 0$ and $\sigma = (-1)^{q+1}$, where q is the number of $\{y_i\}_{i=1}^m$ between x_k and x_{k+1}. Let p be the best approximation to f and let $t_1 < t_2 < \cdots < t_s$ $(s \geqq 1)$ be the points where $f(t_i) - p(t_i) = -\|f - p\|$ $(i = 1, 2, \ldots, s)$. Further, let $u_1 < u_2 < \cdots < u_\mu$ be the points with $f(u_i) = p(u_i)$ $(i = 1, 2, \ldots, \mu)$, so that $\{z_1, \ldots, z_{n+1}\} = \{t_1, \ldots, t_s\} \cup \{u_1, \ldots, u_\mu\} \cup \{y_1, \ldots, y_m\}$, where $a \leqq z_1 < z_2 < \cdots < z_{n+1} \leqq b$. Then since $\operatorname{sgn}(e_{t_i}) = -1$ and $\operatorname{sgn}(e_{u_j}) = 1$, we can conclude that, because $\sigma = (-1)^{q+1}$,

(1) $z_i, z_{i+k} \in T, z_{i+j} \notin T$ for $j = 1, 2, \ldots, k-1 \Rightarrow k$ is even

(2) $z_i, z_{i+1} \in U, z_{i+j} \notin U$ for $j = 1, 2, \ldots, k-1 \Rightarrow k$ is even

(3) $z_i \in T, z_{i+k} \in U, z_{i+j} \notin T \cup U$ for $j = 1, 2, \ldots, k-1 \Rightarrow k$ is odd.

In Theorem 4.1.6 V^n is Haar (on Ω). If V^n is only nearly Haar (e.g., monotone approximation), then in general a complete alternation theory which is helpful in the Exchange Procedure does not exist (see [20]), although partial alternation results in this case are found in [20] and can be used to simplify the Exchange Procedure. It is interesting to note in this context the alternation theory of Kimchi and Richter-Dyn [86] which applies in particular to monotone approximation. Their alternation scheme is directly related to the partial characterization result (Lemma 4.1.1), for this scheme follows from examining the dependency among the n + 1 elements of the augmented extremal set. It is not, however, directly related to the "0 in the convex hull" characterization which is formed from the (unaugmented) extremal set, and therefore is not applicable towards simplifying the exchange procedure of the Remez algorithm. The underlying reason for this can be illustrated by the case of monotone approximation as follows. If p is a best approximation and $p'(x) = 0$ for some $x \in (a, b)$ then $p''(x) = 0$ also, so that the two functionals e_x^1 and e_x^2 occur as a pair in the augmented extremal set. It is impossible, however, in general, to treat these pairs as a unit in the Exchange Procedure as given above.

Unified results concerning strong uniqueness and continuity of the best approximation operator have been developed in this general setting by the authors [179]. Also there seems to be no general study, in this setting, of Weierstrass and Jackson-type results. The wide variance in the results of Paszkowski [128], [129] and Platte [134] alluded to in Section 2 point up the difficulty inherent in an attempt to provide unified statements along these lines.

It should be emphasized that the unified theory heretofore surveyed in this section is under the assumption that the equality condition EQC1 is satisfied. To what extent EQC1 can be relaxed in general is not clear. The results of Taylor [162], Sippel [155], Ling [98], and Schumaker and Taylor [151] constitute a relatively complete treatment in the case of restricted range approximation with equality constraints, where in order to obtain uniqueness no equality conditions are necessary [151] but for alternation and a working Remez exchange algorithm an equality condition is imposed. This condition is introduced in [162] and extended in [155], [98] and prescribes allowable behavior of the restraining curves ℓ and u at "equality points". Roughly speaking, it is shown that the order k of contact of ℓ and u at such a point is equivalent to k alternations. Another way to view the effect of the condition is that it gives rise to a problem of restricted range with Hermite interpolation (reformulated so that EQC1 holds). An interesting special case of restricted range approximation with equality constraints which cannot be reformulated so that EQC1 holds is copositive approximation where in [123] Taylor and Passow develop an alternation theory.

In [50] Gehner introduces an equality condition (see also Section 4.2) which again seems to assure that the problem can be reformulated so that EQC1 holds. It is shown that a "0 in the convex hull" theorem holds under the equality condition.

In [133], a representation theory developed by Karlin is employed by Pinkus to unify also many of the known results in the case of linear restrictions. In particular, characterization and uniqueness results are given in this paper for Hermite and Hermite-Birkhoff interpolatory, restricted range and bounded coefficient constraints.

Recently, Pence [130], [131] has developed a general theory for best global approximation in a real normed linear space by elements of a linearly constrained subset of a finite dimensional subspace with respect to certain continuous seminorms. The theory is developed via extremal linear functionals in the dual space as were the results given above and is then used to study restricted range approximation by splines.

4.2 Non-linear approximation with (non-linear) restrictions

Let $\overline{\beta} = (\beta_1, \beta_2, \ldots, \beta_n) \in \mathbf{R}^n$ and $\overline{v}(x) = (v_1(x), v_2(x), \ldots, v_n(x))$. In 4.1, $V^n = \left\{ \sum_{i=1}^{n} \beta_i v_i = \overline{\beta} \cdot \overline{v} \right\}$, where $\{v_i(\cdot)\}_{i=1}^{n}$ is a set of n fixed basis functions in C(E) and $V_0^n = \{\overline{\beta} : \ell(\alpha) \leqq L^\alpha(\overline{\beta}) = \overline{\beta} \cdot L^\alpha \overline{v} \leqq u(\alpha), \alpha \in A\}$. Then, our approximation problem can be rephrased as: minimize $\| e(\overline{\beta}) \| = \max_{x \in E} | e(x, \overline{\beta}) |$, where $e(x, \overline{\beta}) = f(x) - \phi(x, \overline{\beta})$, with $\phi(x, \overline{\beta}) = \overline{\beta} \cdot \overline{v}(x)$ and $\overline{\beta} \in V_0^n$.

In this section we relax the linearity requirements in the components of $\overline{\beta}$ on both $L^\alpha(\overline{\beta})$ and $\phi(x, \overline{\beta})$ and require only that $\partial L^\alpha / \partial \beta_i$ exists and is continuous in α and $\partial \phi / \partial \beta_i$ exists and is continuous in x, for all $\overline{\beta} \in V_0^n$, $i = 1, 2, \ldots, n$.

Several authors, in particular Hoffmann [66], [67], [68], [69], Gislason and Loeb [58], Andreasson and Watson [2], and Gehner [50], have obtained a "0 in the convex hull" – type characterization theorem for a local minimum in certain cases of this general setting. Because of the nonlinearity of ϕ in the components of $\overline{\beta}$, the theory applies only to a local (not necessarily a global) minimum. A rather comprehensive statement is given by Gehner [50] where it is shown that many approximation problems with side conditions can be viewed as optimization problems and provides a general "0 in the convex hull" – type of characterization theorem for a general mathematical optimization problem.

In [50] is introduced the *equality condition* 2 (EQC2) referred to in 4.1 which describes an admissible behavior of the inequality restraints ($\ell(\alpha) < u(\alpha)$) in the immediate neighborhood of an equality restraint ($\ell(\alpha_0) = u(\alpha_0)$). As also mentioned in 4.1, EQC2 is a condition which appears to imply that the problem can be restated so that EQC1 (which requires that there are no other restraints in the immediate neighborhood of an equality restraint) prevails. Maintaining the analogous definitions as in 4.1, we have

Theorem 4.2.1 (Characterization) ([66], [67], [68], [58], [2], [50]) *Suppose* EQC2 *holds. For* p *in* V_0^n *to be locally a best approximation to* f, *it is* *necessary that* 0 *be in the convex hull of some* k ($\leqq$ n + 1) *members of* S^σ. *If, in addition,* V_0^n *is convex, for example, the condition is also sufficient.*

Several variations of Theorem 4.2.1 may be found in the works cited. For example, the necessity holds in the absence of equality conditions if only $\phi(x, \bar{\beta})$ and $L^\alpha(\bar{\beta})$ are twice continuously differentiable in the components of $\bar{\beta}$ [2]. The theory can then be applied for example to obtain a "0 in the convex hull" theorem for rational approximation with various side conditions [50].

In [50] and [66] the authors obtain further results in this general setting in the case where there are present a finite number, k, of equality conditions $L^i(\bar{\beta}) = 0$, i = 1, 2, . . . , k, and no other conditions. In [50], a hypothesis in [66] is removed and the results apply to osculatory interpolation using such families as exponentials as well as ordinary rational functions. This setting requires the additional assumptions that E = [a, b] and that for each $\bar{\beta}$, $W(\bar{\beta}) = \left[\frac{\partial\phi}{\partial\beta_1}, \ldots, \frac{\partial\phi}{\partial\beta_n}\right]$ is a Haar subspace of dimension $d(\bar{\beta})$ with (without loss of generality), a basis $\left\{\frac{\partial\phi}{\partial\beta_1}, \ldots, \frac{\partial\phi}{\partial\beta_{d(\bar{\beta})}}\right\}$ and the k x d(A) – matrix $\left(\frac{\partial L^i(\bar{\beta})}{\partial\beta_j} : \begin{matrix} i = 1, \ldots, k \\ j = 1, \ldots, d(A) \end{matrix}\right)$ has rank k. Then corresponding to the necessity part of Theorem 4.2.1 is shown to hold, as well, a Kolmogorov condition. Furthermore, under additional assumptions, an alternation theory is developed as well as uniqueness (in fact strong uniqueness) and Lipschitz continuity of the best approximation operator.

Watson [173] develops a general method for calculating local best non-linear Chebyshev approximations on an interval, without restraints, by linear programming techniques. It would be valuable to know to what extent this algorithm can be modified for use in problems with restraints.

Gislason [59] provides an algorithm for non-linear approximation with restrictions based on a method described in Curtis [27] and Curtis and Frank [28]. Convergence is proved and rates of convergence are established under certain "normality" criteria.

Barrar and Loeb [7] obtain results on the convergence of the Remez algorithm for non-linear families. Again it would be of interest to extend this work to include restraints.

In closing, we would like to remark that we have focused on best uniform approximation by (elementary) approximants satisfying explicit constraints. Other theories exist that can be used to give analogues and extensions of the above theory. For example, generalized weight functions [114], relative error measures [62], or a purely functional analytic viewpoint [1], [19], [33], [49], [92], [152] can be employed. Finally, we wish to apologize to anyone whose work we may have inadvertently omitted and ask that you send us copies of your work so that our files may be completed. Finally, we have attempted to reference the most recent papers that we are aware of. For complete references on any one topic the references of the papers listed here should also be consulted.

References

[1] Anderson, D. M.; Osborne, M. R.: Discrete, nonlinear approximation problems in polyhedral norms. Numer. Math. **28** (1977) 143–156 and 157–170

[2] Andreassen, D. O.; Watson, G. A.: Nonlinear Chebyshev approximation subject to constraints. J. Approximation Theory **18** (1976) 241–250

[3] Andria, George D.: Approximation of continuous functions by polynomials with integral coefficients. J. Approximation Theory **4** (1971) 357–362

[4] Atkinson, K.; Sharma, A.: A partial characterization of poised Hermite-Birkhoff interpolation problems. SIAM J. Numer. Anal. **6** (1969) 230–235

[5] Bak, J.; von Golitschek, M.; Leviatan, D.: The rate of approximation by means of polynomials with restricted coefficients. Israel J. Math. **26** (1977) 265–275

[6] Barrar, R.; Loeb, H. L.: Best non-linear uniform approximation with interpolation. Arch. Rational Mech. Anal. **33** (1969) 231–237

[7] Barrar, R.; Loeb, H. L.: On the Remez algorithm for non-linear families. Numer. Math. **15** (1970) 382–391

[8] Beatson, R. K.: Degree of approximation theorems for approximation with side conditions. Ph. D. Thesis University of Canterbury, New Zealand 1978

[9] Beatson, R. K.: The asymptotic cost of Lagrange interpolatory side conditions. J. Approximation Theory **20** (1977) 288–295

[10] Beatson, R. K.: Jackson type theorems for approximation with Hermite-Birkhoff interpolatory side conditions. J. Approximation Theory **22** (1978) 95–104

[11] Beatson, R. K.: The degree of monotone approximation. Pacific J. Math. **74** (1978) 5–14

[12] Bernstein, S.: Leçons sur les propriétés extremales et la meilleure approximation des fonctions analytiques d'une variable reille. Paris: Gauthier-Villars 1926

[13] Blatt, H.-P.: Rational approximation auf $[0, \infty)$. Z. Angew. Math. Mech. **53** (1973) 182–183. See also, Numer. Math. **27** (1977) 179–190, 191–202 and J. Approximation Theory **21** (1977) 246–256

[14] Blatt, H.-P.: Some remarks on an alternation theorem of Geiger. J. Approximation Theory **11** (1974) 338–342

[15] Brink, D.; Taylor, G. D.: Chebyshev approximation by reciprocals of polynomials on $[0, \infty)$. J. Approximation Theory **16** (1976) 142–149

[16] Brosowski, B.: Über Tschebyscheffsche Approximation mit linearen Nebenbedingungen. Math. Z. **88** (1965) 105–128

[17] Cantor, D. G.: On approximation by polynomials with algebraic integer coefficients. Number Theory (Proc. Sympos. Pure Math. Vol. XII, Houston, Texas, 1967) Amer. Math. Soc., Providence, RI (1969) 1–13

[18] Carasso, C.: Convergence de l'Algorithme de Rémès. J. Approximation Theory **11** (1974) 149–158

[19] Carasso, C.: L'algorithme d'échange en optimisation convexe. Thesis Grenoble October 1973

[20] Chalmers, B. L.: The Remez exchange algorithm for approximation with linear restrictions. Trans. Amer. Math. Soc. **222** (1976) 103–131

[21] Chalmers, B. L.: A unified approach to uniform real approximation by polynomials with linear restrictions. Trans. Amer. Math. Soc. **166** (1972) 309–316

[22] Cheney, E. W.: Introduction to Approximation Theory. New York: McGraw-Hill 1966

[23] Cimoca, Gh.: On uniform approximation by functions having restricted ranges. Mathematica (Cluj) **12** (1970) 237–251

[24] Cimoca, Gh.: On a problem of restrictive uniform approximation. Mathematica (Cluj) **13** (1971) 189–197

[25] Cimoca, Gh.: On restrictive uniform approximation. Rev. Anal. Numér. Théorie Approximation **1** (1972) 37–43

[26] Culbertson, J.: On approximation by monotonic polynomials in the Chebyshev norm-characterization and uniqueness. Master's Thesis University of Maryland 1968

[27] Curtis, P. C. Jr.: n-parameter families and best approximation. Pacific J. Math. **9** (1959) 1013–1027

[28] Curtis, P. C. Jr.; Frank, W. L.: An algorithm for the determination of the polynomial of best approximation to a function defined on a finite point set. J. Approximation Theory **6** (1972) 291–300

[29] DeVore, R.: Monotone approximation by polynomials. SIAM J. Math. Anal. **8** (1977) 906–921

[29a] DeVore, R.: Monotone approximation by splines. SIAM J. Math. Anal. **8** (1977) 881–905

[30] DeVore, R.: Degree of monotone approximation. ISNM **25** (1974) 337–351

[31] Deutsch, F.: Uniform approximation with interpolatory constraints. J. Math. Anal. Appl. **24** (1968) 62–79

[32] Deutsch, F.; Morris, P. D.: On simultaneous approximation and interpolation which preserves the norm. J. Approximation Theory 2 (1969) 355–373
[33] Deutsch, F. R.; Maserick, P. H.: Applications of the Hahn-Banach theorem in approximation theory. SIAM Rev. **9** (1967) 516–530
[34] Duffin, R. J.; Karlovitz, L. A.: Formulation of linear programs in analysis I. Approximation theory. SIAM J. Appl. Math. **16** (1968) 662–675
[35] Dunham, C. B.: A difficulty in Williams' algorithm for interpolating rationals. Math. Comput. **29** (1975) 552–553
[36] Dunham, C. B.: Chebyshev approximation by interpolating rationals on [α, ∞]. Math. Comput. **29** (1975) 549–551
[37] Dunham, C. B.: Uniqueness in one-sided linear Chebyshev approximation. J. Approximation Theory **15** (1975) 275–277
[38] Dunham, C. B.: Alternating minimax approximation with unequal restraints. J. Approximation Theory **10** (1974) 199–205
[39] Dunham, C. B.: Chebyshev approximation with restricted ranges by families with the betweeness property. J. Approximation Theory **11** (1974) 254–259
[40] Fekete, M.: Approximations par polynomes avec conditions diophantiennes. C. R. Acad. Sci. Paris Sér. A–B **239** (1954) 1337–1339
[41] Fekete, M.: Approximation par des polynomes avec conditions diophantiennes II. C. R. Acad. Sci. Paris Sér. A–B **239** (1954) 1455–1457
[42] Ferguson, D.: The question of uniqueness for G. D. Birkhoff interpolation problems. J. Approximation Theory **2** (1969) 1–28
[43] Ferguson, L. B. O.: Uniform approximation by polynomials with integral coefficients I. Pacific J. Math. **27** (1968) 53–69
[44] Ferguson, L. B. O.: Uniform approximation by polynomials with integral coefficients II. Pacific J. Math. **26** (1968) 273–281
[45] Ferguson, L. B. O.: Existence and uniqueness in approximation by integral polynomials. J. Approximation Theory **10** (1974) 237–244
[46] Ferguson, L. B. O.: Approximation by polynomials with integral coefficients. Math. Surveys, Amer. Math. Soc. 1978. To appear
[47] Fike, C. T.; Sterbenz, P. H.: Minimax approximations subject to a constraint. Math. Comput. **25** (1971) 295–298
[48] Ford, W. T.; Roulier, J. A.: On interpolation and approximation by polynomials with monotone derivatives. J. Approximation Theory **10** (1974) 123–130
[49] Forst, W.: Zur linearen einseitigen Tschebyscheff-Approximation. J. Approximation Theory **16** (1976) 301–309
[50] Gehner, K. R.: Characterization theorems for constrained approximation problems via optimization theory. J. Approximation Theory **14** (1975) 51–76
[51] Geiger, C.: Über eine Klasse rationaler Tschebyscheff-Approximationen mit Nebenbedingungen. J. Approximation Theory **1** (1968) 340–354
[52] Gilormini, C.: Approximation rationnelles avec des noeuds. C. R. Acad. Sci. Paris Sér. A–B **263** (1966) A286–A287
[53] Gilormini, C.: Continuité de l'approximation rationelle de Tschebyscheff avec des noeuds. C. R. Acad. Sci. Paris Sér. A–B **264** (1967) A359–A360
[54] Gilormini, C.: Sur un contre-example de M. Henry L. Loeb. C. R. Acad. Sci. Paris Sér. A–B **266** (1968) A239–A241
[55] Gilormini, C.: Approximation par fractions rationelles généralisées dont les coefficients verifient des relations lineaires. C. R. Acad. Sci. Paris Ser. A–B. **264** (1967) A795–A798
[56] Gilormini, C.: Sur l'approximation par fractions rationelles généralisées avec constraintes sur les coefficients. C. R. Acad. Sci. Paris Ser. A–B. **265** (1967) A253–A256
[57] Gimlin, D. R.; Cavin, R. K.; Budge, M. C.: A multiple exchange algorithm for calculation of best restricted approximations. SIAM J. Numer. Anal. **11** (1974) 219–231
[58] Gislason, G.; Loeb, H. L.: Nonlinear Tchebyscheff approximation with constraints. J. Approximation Theory **6** (1972) 291–300
[59] Gislason, G. A.: An algorithm for constrained nonlinear, Tchebycheff approximations. Theory of Approximation with Applications. Alan G. Law and Badri N. Sahney, editors. Inc. New York: Academic Press 1976, 298–307
[60] von Golitschek, M.: Permissible bounds on the coefficients of generalized polynomials. In: Lorentz, G. G. (ed.): Approximation Theory. New York: Academic Press, Inc. 1973, pp. 353–357

[61] von Golitschek, M.: Generalization of the Jackson Approximation Theorems in the sense of C. H. Müntz, Bull. Amer. Math. Soc. **75** (1969) 524–528
[62] Griesel, A.: Uniform approximation with respect to a general error function. Ph. D. Thesis University of California, Riverside 1971
[63] Hersey, H. S.; Tufts, D. W.; Lewis, J. T.: Interactive minimax design of linear-phase nonrecursive digital filters subject to upper and lower constraints. IEEE Trans. Audio Electroacoust. **AU-20** (1972) 171–173
[64] Hewitt, E.; Zuckerman, H. S.: Approximation by polynomials with integral coefficients, a reformulation of the Stone-Weierstrass theorem. Duke Math. J. **26** (1959) 305–324
[65] Hill, D.; Passow, E.; Raymon, L.: Approximation with interpolatory constraints. Illinois J. Math. **20** (1976) 65–71
[65a] Hill, D.; Passow, E.; Raymon, L.: Correction to our paper: Approximation with interpolatory constraints (Illinois J. Math. **20** (1976) 65–71). Illinois J. Math. **21** (1977) 428
[66] Hoffmann, K.-H.: Zur Theorie der nichtlinearen Tschebyscheff-Approximation mit Nebenbedingungen. Numer. Math. **14** (1969) 24–41
[67] Hoffmann, K.–H.: Nichtlineare Tschebyscheff-Approximation mit Nebenbedingungen: Anwendung. Numer. Math. **14** (1970) 383–393
[68] Hoffmann, K.-H.: Tschebyscheff-Approximation unter Nebenbedingungen via Kontroll-Theorie. Z. Angew. Math. Mech. **54** (1974) 219–220
[69] Hoffmann, K.–H.: Nichtlineare Approximation unter Nebenbedingungen im normierten Raum. Z. Angew. Math. Mech. **50** (1970) 52–53
[70] Hull, J. A.; Taylor, G. D.: Restricted range adaptive curve fitting. To appear Internat. J. Numer. Methods Engrg.
[71] Johnson, D. J.: SAIN approximation in C[a, b]. J. Approximation Theory **17** (1976) 14–34
[72] Johnson, D. J.: Jackson type theorems for approximation with side conditions. J. Approximation Theory **12** (1974) 213–229
[73] Johnson, D. J.: On the nontriviality of restricted range polynomial approximation. SIAM J. Numer. Anal. **12** (1975) 183–187
[74] Johnson, D. J.: One-sided approximation with side conditions. J. Approximation Theory **16** (1976) 366–371
[75] Johnson, D. J.: Restricted range approximation with side conditions. J. Approximation Theory **19** (1977) 304–314
[76] Johnson, D. J.: RRAS approximation of differentiable functions. In: Law, A. G.; Sahney, B. N. (ed.): Theory of Approximation with Applications. New York: Academic Press, Inc. 1976, pp. 171–179
[77] Jones, R. C.; Karlovitz, L. A.: Iterative construction of constrained Chebyshev approximation of continuous functions. SIAM J. Numer. Anal. **5** (1968) 574–585
[78] Jurkat, W. B.; Lorentz, G. G.: Uniform approximation by polynomials with positive coefficients. Duke Math. J. **28** (1961) 463–474
[79] Kammerer, W.: Optimal approximations of functions; one-sided approximation and extrema preserving approximations. Ph. D. Thesis University of Wisconsin 1959
[80] Karlin, S.; Karon, J. M.: On Hermite-Birkhoff interpolation problems. J. Approximation Theory **6** (1972) 90–114
[81] Kaufman, E. H. Jr.; Taylor, G. D.: Uniform rational approximation by functions of several variables. Internat. J. Numer. Methods Engrg. **9** (1975) 297–323
[82] Kaufman, E. H. Jr.; Leeming, D. J.; Taylor, G. D.: A combined Remez-differential correction algorithm for rational approximation. Math. Comput. **32** (1978)
[83] Kaufman, E. H. Jr.; Taylor, G. D.: Best rational approximations with negative poles to e^{-x} on $[0, \infty)$. In: Saff, E. B.; Varga, R. S. (ed.): Padé and Rational Approximation Theory and Applications. New York: Academic Press, Inc. 1977, pp. 413–426
[84] Kaufman, E. H. Jr.; Taylor, G. D.: Uniform approximation with rational functions having negative poles. J. Approximation Theory **23** (1978) 364–378
[85] Keener, L.; Ling, W.: Approximation under an arc length constraint. Preprint
[86] Kimchi, E.; Richter-Dyn, N.: Properties of best approximation with interpolatory and restricted range side conditions. J. Approximation Theory **14** (1975) 101–115

[87] Kimchi, E.; Leviatan, D.: On restricted best approximation to functions with restricted derivatives. SIAM J. Numer. Anal. **13** (1976) 51–53
[88] Kimchi, E.; Richter-Dyn, N.: Best uniform approximation with Hermite-Birkhoff interpolatory side conditions. J. Approximation Theory **14** (1975) 85–100
[89] Lambert, J.: Conditions for simultaneous approximation and interpolation with norm preservation in C[a, b]. Pacific J. Math. **66** (1976) 173–180
[90] Lau, T. C.-Y.: Rational exponential approximation with real poles. Preprint
[91] Laurent, P. J.: Approximation uniforme de fonctions continues sur un compact avec constraints de type inégalite. Rev. Française Automat. Informat. Recherche Opérationnelles **1** (1967) 81–95
[92] Laurent, P. J.: Approximation et optimisation. Paris: Hermann 1972
[93] Lee, C. M.; Roberts, F. D. K.: A comparison of algorithms for rational l_∞ approximation. Math. Comput. **27** (1973) 111–121
[94] Lewis, James T.: Approximation with convex constraints. SIAM Rev. **15** (1973) 193–217
[95] Lewis, James T.: Restricted range approximation and its application to digital filter design. Math. Comput. **29** (1975) 522–539
[96] Lewis, James T.: Computation of best monotone approximations. Math. Comput. **26** (1972) 737–747
[97] Lim, K.: A note on monotone approximation. Bull. London Math. Soc. **3** (1971) 366–368
[98] Ling, W. H.: On alternation in the restricted range problem. In: Law, A. G.; Sahney, B. N. (ed.): Theory of Approximations with Applications. New York: Academic Press, Inc. 1976, pp. 64–75
[99] Ling, W. H.; Roulier, J. A.; Varga, R. S.: On approximation by polynomials increasing to the right of the interval. J. Approximation Theory **14** (1975) 285–295
[100] Loeb, H. L.; Moursund, D. G.; Schumaker, L. L.; Taylor, G. D.: Uniform generalized weight function polynomial approximation with interpolation. SIAM J. Numer. Anal. **6** (1969) 283–293
[101] Loeb, H. L.: Un contre-example à un resultat de M. Claude Gilormini. C. R. Acad. Sci. Paris Sér. A–B. **266** (1968) 237–238
[102] Loeb, H. L.; Moursund, D. G.; Taylor, G. D.: Uniform rational weighted approximations having restricted ranges. J. Approximation Theory **1** (1968) 401–411
[103] Loeb, H. L.; Moursund, D. G.: Continuity of the best approximation operator for restricted range approximation. J. Approximation Theory **1** (1968) 391–400
[104] Lorentz, G. G.: Birkhoff interpolation and the problem of free matrices. J. Approximation Theory **6** (1972) 283–290
[105] Lorentz, G. G.; Zeller, K. L.: Gleichmäßige Approximation durch monotone Polynome. Math. Z. **109** (1969) 87–91
[106] Lorentz, G. G.: Monotone approximation by algebraic polynomials. Trans. Amer. Math. Soc. **149** (1970) 1–18
[107] Lorentz, G. G.: Monotone approximation. In: Shisha, O. (ed.): Inequalities III. New York: Academic Press, Inc. 1969, pp. 201–215
[108] Lorentz, G. G.; Zeller, K. L.: Degree of approximation by monotone polynomials I and II. J. Approximation Theory **1** (1968) 501–504; **2** (1969) 265–269
[109] Lorentz, G. G.: The degree of approximation by polynomials with positive coefficients. Math. Ann. **151** (1963) 239–251
[110] Lorentz, R. A.: Uniqueness of best approximation by monotone polynomials. J. Approximation Theory **4** (1971) 401–418
[110a] McAllister, D. F.; Passow, E.; Roulier, J. A.: Algorithms for computing shape preserving spline interpolations data. Math. Comp. **31** (1977) 717–725
[111] Meinardus, G.; Varga, R. S.: Chebyshev rational approximation to certain entire functions in [0, ∞). J. Approximation Theory **3** (1970) 300–309
[112] Meinardus, G.; Reddy, A. R.; Taylor, G. D.; Varga, R. S.: Converse theorems and extensions in Chebyshev rational approximation to certain entire functions in [0, ∞). Trans. Amer. Math. Soc. **170** (1972) 171–185
[113] Moursund, D. G.: Computational aspects of Chebyshev approximation using a generalized weight function. SIAM J, Numer. Anal. **5** (1968) 126–137
[114] Moursund, D. G.: Chebyshev approximation using a generalized weight function. SIAM J. Numer. Anal. **3** (1966) 435–450

[115] M ü l l e r, Helmut: Zur Approximation durch Polynome mit gagen Koeffizienten. J. Approximation Theory **16** (1976) 43–47
[116] M y e r s, D.: Comonotone and co-convex approximation. Ph. D. Thesis Temple University 1975
[116a] M y e r s, D.: Nearly coconvex approximation of order k. J. Approximation Theory **22** (1978) 46–58
[117] N e w m a n, D. J.; P a s s o w, E.; R a y m o n, L.: Piecewise monotone polynomial approximation. Trans. Amer. Math. Soc. **172** (1972) 465–472
[118] N e w m a n, D. J.; R e d d y, A. R.: Rational approximation to x^n. Pacific J. Math. **67** (1976) 247–250
[119] P a s s o w, E.; R a y m o n, L.: Monotone and comonotone approximation. Proc. Amer. Math. Soc. **42** (1974) 390–394
[120] P a s s o w, E.; R a y m o n, L.: Restricted approximation and interpolation. Bull. Amer. Math. Soc. **80** (1974) 750–753
[121] P a s s o w, E.; R o u l i e r, J. A.: Negative theorems on generalized convex approximation. Pacific J. Math. **65** (1976) 437–447
[121a] P a s s o w, E.; R o u l i e r, J. A.: Monotone and convex spline interpolation. SIAM J. Numer. Anal. **14** (1977) 904–909
[122] P a s s o w, E.; R a y m o n, L.: Copositive polynomial approximation. J. Approximation Theory **12** (1974) 299–304
[123] P a s s o w, E.; T a y l o r, G. D.: An alternation theory for copositive approximation. J. Approximation Theory **19** (1977) 123–134
[124] P a s s o w, E.; R a y m o n, L.; S h i s h a, O.: Piecewise monotone interpolation and approximation with Muntz polynomials. Trans. Amer. Math. Soc. **218** (1976) 197–205
[125] P a s s o w, E.; R a y m o n, L.: The degree of piecewise monotone interpolation. Proc. Amer. Math. Soc. **48** (1975) 409–412
[126] P a s s o w, E.: Polynomials with positive coefficients: Uniqueness of best approximation. J. Approximation Theory **21** (1977) 352–355
[127] P a s z k o w s k i, S.: Sur l'approximation uniform avec des noeuds. Ann. Polon. Math. **2** (1955) 118–135
[128] P a s z k o w s k i, S.: On the Weierstrass approximation theorem. Colloq. Math. **4** (1956) 206–210
[129] P a s z k o w s k i, S.: On approximating with nodes. Rosprawy Mat. **14** (1957) 1–62
[130] P e n c e, D. D.: Best approximation by constrained splines. Ph. D. Thesis Purdue University 1976
[131] P e n c e, D. D.: Best uniform approximation by polynomial splines with constraints. Preprint
[132] P e r r i e, A. L.: Uniform rational approximation with osculatory interpolation. J. Comput. System Sci. **4** (1970) 509–522
[133] P i n k u s, A.: Application of representation theorems to problems of Chebyshev approximation with constraints. In: K a r l i n, S.; M i c c h e l l i, C. A.; P i n k u s, A.; S c h o e n b e r g, I. J.: Studies in Spline Functions and Approximation Theory. New York: Academic Press, Inc. 1976, pp. 83–112
[134] P l a t t e, D. M.: Approximation with Hermite-Birkhoff interpolatory constraints and related H-set theory. Ph. D. Thesis Michigan State University 1972
[135] R a y m o n, L.: Comonotone approximation and piecewise monotone interpolation. In: L a w, A. G.; S a h n e y, B. N. (ed.): Theory of Approximation with Applications. New York: Academic Press, Inc. 1976, pp. 220–237
[136] R o s s, M. D.; B e l f o r d, G. G.: Characterization of best Chebyshev approximations with prescribed norm. J. Approximation Theory **16** (1976) 315–328
[137] R i c e, J. R.: Approximation with convex constraints. J. SIAM **11** (1963) 15–32
[138] R o u l i e r, J. A.: Monotone and weighted approximation. Ph. D. Thesis Syracuse [illegible]

[143] Roulier, J. A.; Taylor, G. D.: Approximation by polynomials with restricted ranges on their derivatives. J. Approximation Theory **5** (1972) 216–227
[144] Roulier, J. A.: Best approximation to functions with restricted derivatives. J. Approximation Theory **17** (1976) 344–347
[145] Roulier, J. A.: Nearly comonotone approximation. Proc. Amer. Math. Soc. **47** (1975) 84–88
[146] Roulier, J. A.; Taylor, G. D.: Rational Chebyshev approximation on $[0, \infty)$. J. Approximation Theory **11** (1974) 208–215
[147] Roulier, J. A.: Permissible bounds on the coefficients of approximating polynomials. J. Approximation Theory **3** (1970) 117–122
[148] Roulier, J. A.: Restrictions on the coefficients of approximating polynomials. J. Approximation Theory **6** (1972) 276–282
[149] Roulier, J. A.; Taylor, G. D.: Uniform approximation by polynomials having bounded coefficients. Math. Abhandlung. **36** (1971) 126–135
[150] Saff, E. B.; Schönhage, A.; Varga, R. S.: Geometric convergence to e^{-z} by rational functions with real poles. Numer. Math. **25** (1976) 307–322
[151] Schumaker, L. L.; Taylor, G. D.: On approximating by polynomials having restricted ranges, II. SIAM J. Numer. Anal. **6** (1969) 31–36
[152] Sharma, A.: Some poised and non-poised problems of interpolation. SIAM Rev. **14** (1972) 129–151
[153] Shisha, O.: Monotone approximation. Pacific J. Math. **15** (1965) 667–671
[154] Singer, I.: Best approximation in normed linear spaces by elements of linear subspaces. New York: Springer-Verlag 1970
[155] Sippel, W.: Approximation by functions with restricted ranges. In: Lorentz, G. G. (ed.): Approximation Theory. New York: Academic Press, Inc. 1973, pp. 481–484
[156] Stafney, J. D.: A permissible restriction on the coefficients in uniform polynomial approximation to C[0, 1]. Duke Math. J. **34** (1967) 393–396
[157] Taylor, D. G.: Uniform approximation with side conditions. In: Lorentz, G. G. (ed.): Approximation Theory. New York: Academic Press, Inc. 1973, pp. 495–503
[158] Taylor, G. D.; Williams, J.: Existence questions for the problem of Chebyshev approximation by interpolating rationals. Math. Comput. **28** (1974) 1097–1103
[159] Taylor, G. D.: Approximation by functions having restricted ranges, III. J. Math. Anal. Appl. **27** (1969) 241–248
[160] Taylor, G. D.: Optimal starting approximations for Newton's method. J. Approximation Theory **3** (1970) 156–163
[161] Taylor, G. D.: On approximation by polynomials having restricted ranges, I. SIAM J. Numer. Anal. **5** (1968) 258–268
[162] Taylor, G. D.: Approximation by functions having restricted ranges: equality case. Numer. Math. **14** (1969) 71–78
[163] Taylor, G. D.; Winter, M. J.: Calculation of best restricted approximations. SIAM J. Numer. Anal. **7** (1970) 248–255
[164] Taylor, K. A.: Contributions to the theory of restricted polynomial and rational

[174] Williams, Jack: Numerical Chebyshev approximation by interpolating rationals. Math. Comput. **26** (1972) 199–206
[175] Wolibner, W.: Sur un polynome d'interpolation. Colloq. Math. **2** (1951) 136–137
[176] Zeller, K. L.: Monotone approximation. In: Lorentz, G. G. (ed.): Approximation Theory. New York: Academic Press, Inc. 1973, pp. 523–525

Added in proof:

[177] Fletcher, Y.; Roulier, J. A.: A counterexample to strong unicity in monotone approximation. prepr.
[178] [illegible]
approximation. prepr.
[179] Chalmers, B. L.; Taylor, G. D.: A unified theory of strong uniqueness in uniform approximation with constraints. prepr.
[180] Myers, D.; Raymon, L.: Exact comonotone approximation. J. Approximation Theory **24** (1978) 35–50

Prof. B. L. Chalmers
Department of Mathematics
University of California
Riverside, California 92521

Prof. G. D. Taylor
Department of Mathematics
Colorado State University
Fort Collins, Colorado 80523

(*Eingegangen: 17. 10. 1977*)

Jber. d. Dt. Math.-Verein.
81 (1979) 87–96

Global Phenomena in Bifurcations of Dynamical Systems with Simple Recurrence*)

F. Takens, Groningen

1 Introduction and definitions

We shall describe some recent results and open problems related to the topological understanding of bifurcations in generic one-parameter families of dynamical systems. The results which we describe here were obtained by S. Newhouse, J. Palis and the author during various periods of joint research at I.M.P.A. in Rio de Janeiro; they will be published as part of [6], see also [7] and [10]. As we want to give here a non-technical introduction to these aspects of bifurcation theory, we shall often restrict the generality more than absolutely necessary. First we give some definitions.

Let M be a smooth (i.e., C^∞) manifold without boundary. Here a *dynamical system* on M is just a smooth vectorfield; in particular we disregard the case of diffeomorphisms or "discrete time". The space of vectorfields on M with the C^∞ topology will be denoted by $\mathfrak{X}(M)$. If $X \in \mathfrak{X}(M)$ is a dynamical system, there is for each $t \in \mathbf{R}$ a diffeomorphism $X_t: M \to M$, the *time t map* of X; such that

$X_t(x)$ is C^∞ in (x, t);

X_0 is the identity;

$$\frac{\partial}{\partial t}(X_t(x)) = X(X_t(x)).$$

(Often the 1-parameter group $\{X_t\}$, instead of the vectorfield X, is called a dynamical system.) Sets of the form $\{X_t(x) \mid t \in \mathbf{R}\}$, for $x \in M$ fixed are called *orbits* of X. If X and Y are dynamical systems on M, then a homeomorphism $h: M \to M$ is a *topological equivalence* between X and Y if h maps orbits of X sense preserving to orbits of Y. If such a homeomorphism exists, X and Y are said to be *topologically equivalent*. X is said to be *structurally stable* if it is an interior point of its topological equivalence class.

In many cases we do not want to discuss all dynamical systems, but only the "non-exceptional ones". In order to formalize this notion, we proceed as follows: Let A be a topological space able intersection $\bigcap_{i=1}^{\infty} \mathcal{U}_i$ with each $\mathcal{U}_i$ open and dense. In case A is a Baire space we know that each residual subset of A is dense; all spaces to which we shall apply the notion of "residual subset" will be Baire spaces. Once a residual subset $B \subset A$ is choosen we can say that an element $a \in A$ is "exceptional" or "non-generic" if $a \notin B$. We shall say "for generic $a \in A$ the property Q holds" if there is a residual subset $B_Q \subset A$ such that each $a \in B_Q$ has property Q.

One should compare the notion of residual subset of a topological space with the notion of "subset of full measure", i.e., a subset the measure of whose complement is zero, in a measure space. Both are used to formalize the notion of a subset of almost all points.

In both cases the main properties which justify this formalization are that these (i.e., residual, respectively full measure) subsets are not empty, even in a certain sense dense, and that a countable intersection of them belongs again to the same class of subsets. Depending on the fact whether we have a natural topology or a natural measure we use one notion or the other; in case we have both, like on **R**, the two notions may turn out to be quite different. In our situation we deal with topological vectorspaces which have no natural measures and hence use the notions "residual" and "generic".

Next we have to discuss some notions related with recurrence. If X is a dynamical system on M, then we define its positive limit set $L^+(X)$ to be the closure of $\{x \in M \mid \exists\, \bar{x} \in M, t_i \in \mathbf{R}$ with $t_i \to +\infty$ and $X_{t_i}(\bar{x}) \to x\}$. The negative limit set $L^-(X)$ is defined the same way, except with $t_i \to -\infty$ instead of $t_i \to +\infty$. Of course neither $L^+(X)$ or $L^-(X)$ is empty (M is compact). A component of $L^+(X)$ or $L^-(X)$ may be

– a point (in such a point, X is zero, these points are called singularities, they belong to both $L^+(X)$ and $L^-(X)$);

– an embedded circle (such an embedded circle is in general a closed or periodic orbit, i.e., a non-constant orbit $\{X_t(x)\}_{t \in \mathbf{R}}$ such that for some $t_0 > 0$, $X_{t+t_0}(x) = X_t(x)$, they belong to both $L^+(X)$ and $L^-(X)$);

– or a more complicated set.

If the limit set of X consists only of a finite number of singularities and closed orbits, then we say that X has simple recurrence. The interior, in $\mathfrak{X}(M)$, of the set of all X with simple recurrence is denoted by SR(M).

The following definitions, on hyperbolicity, are needed to formulate the necessary and sufficient conditions for structural stability of dynamical systems with simple recurrence. We say that a singularity $p \in M$ of $X \in \mathfrak{X}(M)$ is hyperbolic if $(dX)_p$ has no purely imaginary eigenvalues. In this case the stable and unstable sets of p, defined by $W^s_X(p) = \{x \in M \mid X_t(x) \to p \text{ as } t \to +\infty\}$ and $W^u_X(p) = \{x \in M \mid X_t(x) \to p \text{ as } t \to -\infty\}$, are injectively immersed submanifolds: the stable and unstable manifolds of p. Their dimensions equal the number of eigenvalues of $(dX)_p$ with negative, respectively positive, real part. If γ is a closed orbit of X and Σ is a (small) co-dimension one submanifold of M which intersects γ in p and is transversal to X, then we define the Poincaré map $P: (\Sigma, p) \to (\Sigma, p)$ (at least in a small neighbourhood of p in Σ) so that P(x) is the first intersection of the positive x-orbit with Σ. We say that γ is hyperbolic if $(dP)_p$

has no eigenvalue on the unit circle. Also in this case there are stable and unstable manifolds $W^s_X(\gamma) = \{x \in M \mid X_t(x) \to \gamma \text{ as } t \to +\infty\}$ and $W^u_X(\gamma) = \{x \in M \mid X_t(x) \to \gamma \text{ as } t \to -\infty\}$. They are injectively immersed submanifolds with dimensions equal to one plus the number of eigenvalues of (dP) with norm smaller, respectively bigger, than one. For a discussion of hyperbolicity and stable and unstable manifolds of singularities and closed orbits, see [11] and [3].

The main results combining all the above notions are:

Theorem (Kupka [4], Smale [17]) *For generic* $X \in \mathfrak{X}(M)$, *all singularities and closed orbits are hyperbolic and all intersections of stable and unstable manifolds are transversal.*

Dynamical systems as in the conclusion of this theorem are called Kupka-Smale systems.

Theorem (Palis [12], Palis-Smale [13], Robinson [15]) $X \in SR(M)$ *is structurally stable if and only if it is Kupka-Smale.*

Corollary *The structurally stable elements in SR(M) are open and dense.*

Finally we come to the one-parameter families X^μ, $\mu \in [0, 1] = I$, of dynamical systems. We shall assume that our arcs, or one-parameter families, of dynamical systems X^μ are differentiable in the sense that the map which assigns to (m, μ), $m \in M$ and $\mu \in I$, the vector $X^\mu(m) \in T_m(M)$ is smooth. We give the set of these arcs of dynamical systems the C^∞ topology (of vectorfields on M x I). We say that $\bar{\mu} \in I$ is a bifurcation value of the arc X^μ if $X^{\bar{\mu}}$ is not structurally stable. So, for arcs X^μ with simple recurrence, i.e., arcs such that each $X^\mu \in SR(M)$, $\bar{\mu}$ is a bifurcation value if and only if it is not Kupka-Smale.

Since we are interested in the topological structure of these arcs near bifurcation values we define an arc X^μ, at a bifurcation value $\bar{\mu}$ to be (topologically) equivalent with an arc Y^η, at a bifurcation value $\bar{\eta}$, if there are a homeomorphism $h: (I, \bar{\mu}) \to (I, \bar{\eta})$ and a homeomorphism $H_\mu: M \to M$, depending continuously on μ, so that H_μ is a topological equivalence between X^μ and $Y^{h(\mu)}$ for μ near $\bar{\mu}$. We say that X^μ has a stable bifurcation at $\bar{\mu}$ if every arc X'^μ, sufficiently close to X^μ, has a bifurcation value $\bar{\mu}'$ such that X^μ at $\bar{\mu}$ is equivalent with X'^μ at $\bar{\mu}'$.

There are also local versions of these definitions. Let X^μ be an arc of dynamical systems with bifurcation value $\bar{\mu}$ and let $\mathcal{O}$ be an orbit of $X^{\bar{\mu}}$. Let Y^η be another such arc with bifurcation value $\bar{\eta}$ and orbit $\mathcal{B}$. Then we say that X^μ at $(\bar{\mu}, \mathcal{O})$ is equivalent with Y^η at $(\bar{\eta}, \mathcal{B})$ if there are a homeomorphism $h: (I, \bar{\mu}) \to (I, \bar{\eta})$, a homeomorphism $H_\mu: M \to M$ depending continuously on μ and a neighbourhood $\mathcal{U}$ of the closure of $\mathcal{O}$ such that $H_{\bar{\mu}}(\mathcal{O}) = \mathcal{B}$ and H_μ maps orbits of $X^\mu \mid \mathcal{U}$ to orbits of $Y^{h(\mu)} \mid H_\mu(\mathcal{U})$ for μ near $\bar{\mu}$. We say that the arc X^μ has a stable bifurcation at the bifurcation value $\bar{\mu}$ at the orbit $\mathcal{O}$ if for any nearby arc [illegible]

– for each bifurcation value $\bar{\mu}$, there is exactly one orbit $\mathcal{O}$ along which $X^{\bar{\mu}}$ is "not Kupka-Smale" in one of the following ways:
(a) *$\mathcal{O}$ is a non-hyperbolic singularity, or*
(b) *$\mathcal{O}$ is a non-hyperbolic closed orbit, or*
(c) *$\mathcal{O}$ is an orbit along which there is a non-transversal intersection of a stable and an unstable manifold;*
– the set of bifurcation values is at most countable.

If for such an arc we speak of a bifurcation which is locally stable we shall always mean locally stable at the unique orbit where Kupka-Smale is violated.

In the following sections we shall speak of generic arcs with simple recurrence. It will always be understood that these arcs have at least all the properties stated in the conclusion of the above theorem.

2 Tangencies of stable and unstable manifolds

In this and the next section we assume all arcs X^{μ} of dynamical systems to have simple recurrence. We want to determine which bifurcations of such arcs are stable.

Theorem *Let X^{μ} be an arc of dynamical systems with bifurcation value $\bar{\mu}$. If $X^{\bar{\mu}}$ has a non-transversal intersection, or tangency, of a stable and an unstable manifold, both of periodic orbits, then the bifurcation of X^{μ} at $\bar{\mu}$ is not stable; it is even not locally stable.*

This follows from a corresponding theorem on arcs of diffeomorphisms [6], I; a proof for the 2-dimensional case for diffeomorphisms can be found in [10] where it is shown that a ratio of logarithms of eigenvalues appears as a topological invariant. More detailed information about the topological invariants in dimension 2 can be found in [5].

Although the proof has not yet been completely written, we are convinced that we also have:

Theorem *Let X^{μ} be an arc of dynamical systems with bifurcation value $\bar{\mu}$. If $X^{\bar{\mu}}$ has a tangency between a stable and an unstable manifold, both of singularities at which all the eigenvalues are real and if X^{μ} satisfies some more generic conditions then X^{μ} has a stable bifurcation at $\bar{\mu}$.*

This case will be treated in detail in [6]; II. It is not hard to see that if the hypotheses of the above theorem are satisfied, the bifurcation is locally stable. The condition on the eigenvalues cannot be completely omitted because of the following.

Theorem *Let X^{μ} be an arc of dynamical systems on a 4-manifold M and let $\bar{\mu}$ be a bifurcation value. If $X^{\bar{\mu}}$ has an orbit of tangency of a 2-dimensional stable and a 2-dimensional unstable manifold, both of hyperbolic singularities with all their eigenvalues non-real, then the bifurcation of X^{μ} at $\bar{\mu}$ is not stable; it is even not locally stable.*

We sketch a proof of this theorem in section 4 of this paper. It seems likely than in n dimensions we have the following generalization:

Conjecture *Let* X^{μ} *be an arc of dynamical systems with bifurcation value* $\bar{\mu}$. *If* $X^{\bar{\mu}}$ *has a tangency of an unstable manifold* $W^u(p)$ *and a stable manifold* $W^s(q)$, p *and* q *both hyperbolic singularities, and if the weakest contracting eigenvalues at* p *and the weakest expanding eigenvalues at* q *are non-real, then the bifurcation of* X^{μ} *at* $\bar{\mu}$ *is not stable, even not locally.*

For bifurcations with for example a tangency of the stable manifold of a singularity and the unstable manifolds of a periodic orbit it is unknown whether they might be stable or unstable; even about local stability hardly anything is known in these cases.

3 Non-hyperbolic singularities and closed orbits

In the case of non-hyperbolicity of singularities and closed orbits, the local stability problem for generic bifurcations is completely solved.

Theorem *Let* X^{μ} *be an arc of dynamical systems with bifurcation value* $\bar{\mu}$. *If* X^{μ} *has a non-hyperbolic singularity or closed orbit and if the arc* X^{μ} *satisfies some generic properties, then the bifurcation of* X^{μ} *at* $\bar{\mu}$ *is locally stable. (It is not true that under these hypotheses the arc* X^{μ} *always is (globally) stable at* $\bar{\mu}$.)

For the case of singularities, this follows from [1], [14], [16] and [19]; for the case of closed orbits see [6]. We shall now consider de question of global stability in some detail on 2-manifolds.

Note that on a 2-manifold there is no problem with tangencies of stable and unstable manifolds: the only tangency (is non-transversal intersection) which is possible is that of a stable and an unstable manifold of two saddles. In this case the eigenvalues are real and hence there is no problem with the stability (since we assume that all our arcs of vectorfields have simple recurrence). The non-hyperbolic closed orbits which appear in generic arcs of vectorfields on a two-manifold are of two kinds:

(a) saddle-node closed orbits, in this case 1 is an eigenvalue of the Poincaré map; this closed orbit is attracting on one side and repelling on the other side;

(b) flip closed orbits, in this case -1 is an eigenvalue of the Poincaré map; this closed orbit is either an attractor or a repellor.

Theorem *Let* X^{μ} *be a generic arc of vectorfields with simple recurrence on a* 2*-manifold and let* $\bar{\mu}$ *be a bifurcation value. If* $X^{\bar{\mu}}$ *has a tangency of a stable manifold and an unstable manifold of two saddle points or if* $X^{\bar{\mu}}$ *has a flip closed orbit, then* X^{μ} *has a stable bifurcation at* $\bar{\mu}$. *If* $X^{\bar{\mu}}$ *has a saddle-node periodic orbit with more than one separatrix (is* 1*-dimensional stable or unstable manifold) approaching it from the same side then* X^{μ} *has an unstable bifurcation at* $\bar{\mu}$. *If* X^{μ} *has a saddle node periodic orbit which is approached by at most one separatrix then* X^{μ} *has a stable bifurcation at* $\bar{\mu}$.

We conjecture that also in the case of a saddle-node periodic orbit with one separatrix approaching it at each side, the bifurcation will be stable.

This theorem is related with work of Sotomayor [19] and Guckenheimer [2] (it should be pointed out that in the last paper the case of the saddle-node periodic orbit is treated incorrectly). The proof of the above theorem is based on the following; see [6]; I.

Let X^μ be a generic arc of dynamical systems on a 2-manifold, let $\bar{\mu}$ be a bifurcation value and let $X^{\bar{\mu}}$ have a saddle-node periodic orbit γ. Let Σ be a smooth local section intersecting γ in p and let P: $(\Sigma, p) \to (\Sigma, p)$ be the Poincaré map. Then there is a unique C^∞ vectorfield Z on Σ (near p) such that the time one map Z_1 of Z equals P [21].

Let X'^μ be another such arc with bifurcation value $\bar{\mu}'$, saddle-node periodic orbit γ', section Σ' and vectorfield Z' on Σ'. If (h, H_μ) is a local equivalence between X^μ at $\bar{\mu}$ and X'^μ at $\bar{\mu}'$ (i.e. $H_{\bar{\mu}}(\gamma) = \gamma'$), then we may and do assume that $H_{\bar{\mu}}(\Sigma) = \Sigma'$ (because we may "push" H_μ along orbits of $X'^{h(\mu)}$). If $H_{\bar{\mu}}(\Sigma) = \Sigma'$ then $H_{\bar{\mu}} | \Sigma$ has to satisfy

(a) $H_{\bar{\mu}} | \Sigma$ is a conjugacy between Z and Z', i.e., for all $t \in \mathbf{R}$, $H_{\bar{\mu}} \circ Z_t = Z'_t \circ H_{\bar{\mu}}$; hence

(b) $H_{\bar{\mu}} | \Sigma \setminus p$ is differentiable.

In other words, the only freedom we have for $H_{\bar{\mu}} | \Sigma$ is to fix the image of one point on each side of p. If more than one separatrix is approaching γ from the same side, the global situation forces the choice of the image of $H_{\bar{\mu}} | \Sigma$ of more than one point on the same side of p. This is generically incompatible with the local requirements and leads to instability.

4 Spirals and tangencies

We first discuss some properties of intersecting (linear) spirals in $\mathbf{R}^3$ and then use these to indicate a proof of the last theorem of section 2. A linear spiral in $\mathbf{R}^3$ is subset L such that for some linear φ: $\mathbf{R}^3 \to \mathbf{R}^3$ and $\gamma \in (0, \infty)$,

$$\varphi(L) = \left\{ (x_1, x_2, x_3) \in \mathbf{R}^3 \mid x_1 = e^{-\frac{\gamma}{2\pi}t} \cdot \cos t, x_2 = e^{-\frac{\gamma}{2\pi}t} \cdot \sin t \text{ for some } t \in \mathbf{R} \right\}.$$

The number γ is a linear invariant of L; $e^{-\gamma} \in (0, 1)$ is called the *contraction coefficient* c(L) of L. Note that c(L) is the biggest number smaller than 1 such that $c(L) \circ L = L$ (scalar multiplication). The axis $\ell(L)$ of L is defined to be $\bar{L} \setminus L$.

Let L_1 and L_2 be two linear spirals in $\mathbf{R}^3$ with $\ell(L_1) \neq \ell(L_2)$. We want to show that $\ln(c(L_1))/\ln(c(L_2))$ is a topological invariant of the germ of the triple $(\mathbf{R}^3; L_1, L_2)$ at 0, or, more precisely:

Proposition *Let* L_1, L_2 *be linear spirals as above and let also* L'_1 *and* L'_2 *be such a pair. The contraction coefficients are denoted by* c_1, c_2, c'_1, c'_2 *and the axis by* $\ell_1, \ell_2, \ell'_1, \ell'_2$. *If there is a local homeomorphism* h: $(\mathbf{R}^3, 0) \to (\mathbf{R}^3, 0)$ *with* $h(L_i) = L'_i$ *then*

$$\frac{\ln(c_1)}{\ln(c_2)} = \frac{\ln(c'_1)}{\ln(c'_2)}.$$

Proof. Without loss of generality we may assume that $c_1 \neq c_2$ and even that $c_1 < c_2$. We first observe that for each $p \in \ell_2 \cap L_1$ there is a curve σ in $\overline{L_1 \cap L_2}$ joining p with a point in $\ell_1 \cap L_2$. To see this, we assume that ℓ_1 is the x_2-axis and

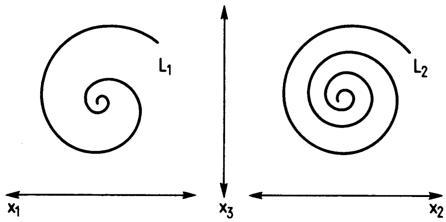

Fig. 1

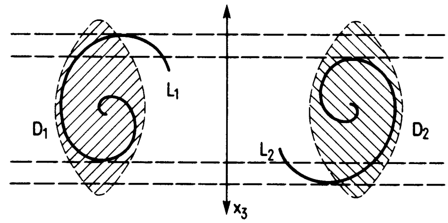

Fig. 2

ℓ_2 is the x_1-axis. The projections of L_1, L_2 on the (x_1, x_3)-, (x_2, x_3)-plane are then as indicated in Fig. 1 (where the projection of L_i is also denoted with L_i; we often make no distinction between L_i and its projection).
We observe that there is a 1-1 correspondence between curves $\sigma(t) \in \overline{L_1 \cap L_2}$, and pairs of curves $\sigma_1(t)$, $\sigma_2(t)$ with

– $\sigma_i(t)$ is in $L_i \subset (x_i, x_3)$-planes;

– the x_3-coordinates of $\sigma_1(t)$ and $\sigma_2(t)$ agree for all t.

Using this representation of curves in $\overline{L_1 \cap L_2}$ and the fact that $c_1 < c_2$ the existence (and uniqueness if L_1 is transversal with respect to L_2!) follows easily. Observe that for $c_1 > c_2$ such a curve does not exist for every $p \in \ell_2 \cap L_1$.

Next we want to construct a basis of neighbourhoods of 0 whose boundaries have no intersection with $\overline{L_1 \cap L_2}$ and such that for some constant k and any $r \in (0, 1)$, there is a neighbourhood D in this basis such that $D_{e^{-k} \cdot r}(0) \subset D \subset D_{e^{k} \cdot r}(0)$, where $D_a(0)$ is a disc of radius a around 0.

We show how to construct such a neighbourhood D with $\partial D \cap (\overline{L_1 \cap L_2}) = \emptyset$; it will then also be clear how to construct the whole basis of neighbourhoods. We use again the two projections as in figure 1, construct the projections D_1, D_2 of D and then require that D is the biggest set having these projections. For example we might have Fig. 2.
In general, to define such a neighbourhood D with $\partial D \cap (\overline{L_1 \cap L_2}) = \emptyset$ we need

(i) the projections of D_1 and D_2 on the x_3-axis to be equal;

(ii) the projections on the x_3-axis of $L_1 \cap \partial D_1$ and $L_2 \cap D_2$ to be disjoint;

(iii) the projections on the x -axis of L ∩ ∂D and L ∩ D to be disjoint

It is not hard to see that when $c_1 \neq c_2$ this construction leads to the required basis of neighbourhoods.

Now we number the points of $\ell_2 \cap L_1$ as $\ldots, p_{-1}, p_0, p_1, p_2, \ldots$ so that for each $i \in \mathbf{Z}$, 0 is between p_i and p_{i+1}, p_{i+2} is between p_i and 0, and there are no points of $\ell_2 \cap L_1$ between p_i and p_{i+2} (between refers to the order in ℓ_2). We number the points in $\ell_1 \cap L_2$ in the same way as $\ldots, q_{-1}, q_0, q_1, q_2, \ldots$. Let I be a function $\mathbf{Z} \to \mathbf{Z}$ such that there is a curve in $\overline{L_1 \cap L_2}$ from p_i to $q_{I(i)}$ for each $i \in \mathbf{Z}$.

From the previous observations we know that where $\alpha_i \sim \beta_i$ means that α_i/β_i is bounded and bounded away from zero, uniformly in i. This implies that $\lim_{i\to\infty} \frac{I(i)}{i} = \frac{\ln(c_1)}{\ln(c_2)}$; the limit is defined in terms of things which are preserved under (local) homeomorphisms; hence it is a topological invariant.

Next we define certain neighbourhoods of spirals which are related to phenomena near invariant manifolds of singularities of (linear) vectorfields with 2 contracting non-real eigenvalues and 2 expanding eigenvalues as one can see in the following example.

Let X be a linear hyperbolic vectorfield on $\mathbf{R}^4$ with unstable manifold W^u and stable manifold W^s, both of dimension 2 and suppose the contracting eigenvalues are non-real, say $\alpha \pm i\beta$ with $\beta > 0$. Let Σ be a piece of 3-dimensional affine space, containing a point $r \in W^u$ and transversal to X. Then, if we take any point $w \in W^s$ with a piece of affine plane Λ through it parallel to W^u, then $\Sigma \cap (\bigcup_{t\geq 0} X_t(\Lambda))$ is (locally) a linear spiral with contraction coefficient $e^{2\pi\alpha/\beta}$. If $\mathcal{U}$ is a small neighbourhood of w, then $\Sigma \cap (\bigcup_{t\geq 0} X_t(\mathcal{U}))$ is (locally) sort of a neighbourhood of this spiral; it is an "ϵ-δ-neighbourhood" in the sense of the following definition.

If L is a linear spiral in $\mathbf{R}^3$ and $\delta > 0$, we define $L_\delta = \bigcup e^{\delta'} \cdot L$ (scalar multiplication). For $0 < \epsilon < \delta$, we say that $\mathcal{U}$ an ϵ-δ-neighbourhood of L if $L_\epsilon \subset \mathcal{U} \subset L_\delta$. In the above example we have an ϵ-δ-neighbourhood with δ small if $\mathcal{U}$ is small.

By direct calculation one finds:

Proposition *Let* $L \subset \mathbf{R}^3$ *be a linear spiral,* $\varphi: (\mathbf{R}^3, \ell(L)) \to (\mathbf{R}^3, \ell(L))$ *a diffeomorphism and* $0 < \epsilon < \delta$ *positive constants. Then, for every* $0 < \epsilon' < \epsilon < \delta < \delta'$ *there is a neighbourhood* V *of* $0 \in \mathbf{R}^3$ *such that for every* ϵ-δ*-neighbourhood* $\mathcal{U}$ *of* L, $\varphi(\mathcal{U}) \cap V$ *is the intersection of an* ϵ'-δ'*-neighbourhood of* $d\varphi(L)$ *with* V.

In other words, "being the germ of an ϵ-δ-neighbourhood of a linear spiral" is invariant under diffeomorphisms which keep the axis straight.

We say that two linear spirals L and L′ are *separating* if $L \cap L' = \emptyset$. In this case $\ell(L) = \ell(L')$ and $c(L) = c(L')$. We say that the ϵ-δ-neighbourhoods $\mathcal{U}$ and $\mathcal{U}'$ of L, L′ are separating if $\mathcal{U} \cap \mathcal{U}' = \emptyset$. Now we come to a sort of generalization of our first proposition to a situation where spirals are replaced by ϵ-δ-neighbourhoods.

Proposition *Let* L_1, L_1', L_2, L_2' *be linear spirals in* $\mathbf{R}^3$, L_1 *separating* L_1' *and* L_2 *separating* L_2' *and* $\ell(L_1) \neq \ell(L_2)$. *Let* $\mathcal{U}_1, \mathcal{U}_1', \mathcal{U}_2, \mathcal{U}_2'$ *be* ϵ-δ*-neighbourhoods of these spirals,* ϵ, δ *small, such that* $\mathcal{U}_1$ *and* $\mathcal{U}_1'$ *are separating and* $\mathcal{U}_2$ *and* $\mathcal{U}_2'$ *are separating. Let* c_1, c_2 *be the contraction coefficients of* L_1, L_2.

Let $\tilde{L}_1, \tilde{L}_1', \tilde{L}_2$ *etc. be analogously defined. If there is a local homeomorphism* $h: (\mathbf{R}^3, 0) \to (\mathbf{R}^3, 0)$ *with* $h(\mathcal{U}_i) = \tilde{\mathcal{U}}_i$ *and* $h(\mathcal{U}_i') = \tilde{\mathcal{U}}_i'$ *then*

$$\frac{\ln c_1}{\ln c_2} = \frac{\ln \tilde{c}_1}{\ln \tilde{c}_2}.$$

P r o o f. The proof is almost the same as in the two-spiral case. Instead of $\overline{L_1 \cap L_2}$ we use $\overline{\mathcal{U}_1 \cap \mathcal{U}_2}$. A curve in $\overline{L_1 \cap L_2}$ is a curve in $\overline{\mathcal{U}_1 \cap \mathcal{U}_2}$. The construction of neighbourhoods D with $\partial D \cap (\overline{L_1 \cap L_2}) = \emptyset$ will give neighbourhoods such that even $\partial D \cap (\overline{\mathcal{U}_1 \cap \mathcal{U}_2}) = \emptyset$ if ϵ, δ are small enough. The only problem appears when we want to count points of $\ell_2 \cap \mathcal{U}_1$ (and of $\ell_1 \cap \mathcal{U}_2$). Here, and only here, we need the separating $\mathcal{U}_1'$. Define an equivalence relation on $\ell_2 \cap \mathcal{U}_1$: $p, p' \in \ell_2 \cap \mathcal{U}_1$ then $p \sim p'$ if the segment in ℓ_2 connecting p with p′ does not contain points of $\mathcal{U}_1'$. Now we number equivalence classes of $\ell_2 \cap \mathcal{U}_1$ (as we numbered the points of $\ell_2 \cap L_1$) and in the same way we number equivalence classes of $\ell_1 \cap \mathcal{U}_2$ and the proposition follows.

Note that ϵ, δ in this proposition should be so small that $L_{1\delta} \cap L_{1\delta}' = \emptyset$ and $L_{2\delta} \cap L_{2\delta}' = \emptyset$ otherwise the equivalence classes on $\ell_2 \cap \mathcal{U}_1$ or $\ell_1 \cap \mathcal{U}_2$ might not be appropriate.

Now we come finally to the main theorem of which the last theorem in section 2 is an easy consequence.

Theorem *Let* X *be a vectorfield on a* 4*-manifold with hyperbolic singularities* p *and* q. *Let* $\dim(W^u(p)) = \dim(W^s(p)) = \dim(W^u(q)) = \dim(W^u(q)) = 2$, *let the contracting eigenvalues at* p *and the expanding eigenvalues at* q *be non-real, say* $\alpha \pm i\beta$ *and* $a \pm ib$, β *and* b *positive, let* X *be linearizable at* p *and* q *and let* $W^u(p)$ *and* $W^s(q)$ *have a common orbit* γ *along which* $T(\gamma) = T(W^u(p)) \cap T(W^s(q))$.

Let $X', p', q', \alpha', \beta', a', b', \gamma'$ *be analogously defined. If there is an equivalence* h *between* X *and* X′ *mapping* γ *to* γ', *then*

$$\frac{\alpha b}{a\beta} = \frac{\alpha' b'}{a'\beta'}.$$

P r o o f. Let Σ, Σ' be smooth 3-dimensional sections intersecting γ, γ' transversally in points r, r′. We may and do assume that near r, h maps Σ to Σ'. In Σ one can choose coordinates x_1, x_2, x_3 such that $r = (0, 0, 0)$, $W^u(p) \cap \Sigma$ is the x_2-axis and $W^s(p) \cap \Sigma$ is the x_1-axis. Choose points $u_1, u_1' \in W^s(p)$, not on the same orbit and $u_2, u_2' \in W^u(q)$ also not on the same orbit. Then, if V_1, V_1', V_2, V_2' are sufficiently small neighbourhoods of u_1, u_1', u_2, u_2', then

$$\mathcal{U}_1 = (\bigcup_{t \geq 0} X_t(V_1)) \cap \Sigma, \qquad \mathcal{U}_1' = (\bigcup_{t \geq 0} X_t(V_1')) \cap \Sigma,$$

$$\mathcal{U}_2 = (\bigcup_{t \leq 0} X_t(V_2)) \cap \Sigma, \qquad \mathcal{U}_2' = (\bigcup_{t \leq 0} X_t(V_2')) \cap \Sigma$$

are ϵ-δ-neighbourhoods of linear spirals as in the assumptions of the last proposition. The contraction coefficients are $e^{2\pi\alpha/\beta}$ and $e^{-2\pi a/b}$. If we do the same construction with X′, using $\widetilde{u}_1 = h(u_1)$, $\widetilde{u}_1' = h(u_1')$ etc. and $\widetilde{V}_1 = h(V_1)$, . . . and using the fact that we may choose V_i, V_i' as small as we want we obtain ϵ-δ-neighbourhoods $\widetilde{\mathcal{U}}_1, \widetilde{\mathcal{U}}_1', \widetilde{\mathcal{U}}_2, \widetilde{\mathcal{U}}_2'$ in Σ', as in the last proposition, with contraction coefficients $e^{2\pi\alpha'/\beta'}$ and $e^{-2\pi a/b}$. $h|\Sigma$ maps $\mathcal{U}_1, \mathcal{U}_1', \widetilde{\mathcal{U}}_2, \widetilde{\mathcal{U}}_2'$ so:

$$\frac{2\pi\alpha}{\beta} \cdot \left(-\frac{2\pi a}{b}\right)^{-1} = \frac{2\pi\alpha'}{\beta'} \cdot \left(-\frac{2\pi a'}{b'}\right) \quad \text{or} \quad \frac{\alpha b}{\beta a} = \frac{\alpha' b'}{\beta' a'}.$$

Note that by [20], the assumption that X can be linearized near p and q is generically satisfied; in fact we only need a C^2 linearization.

For the proof of the last theorem in section 2, observe that the arc X^μ in that theorem admits a small deformation X'^μ which satisfies at the bifurcation value the assumptions of our present theorem. Hence X'^μ has an unstable bifurcation. But the arc X'^μ is arbitrarily close to X^μ, so also the arc X^μ has an unstable bifurcation.

References

[1] Brunovsky, P.: On one parameter families of diffeomorphisms I, II, Comment. math. univ. Carolinae, **11** (3) (1970) 559–582; **12** (4) (1971) 765–784
[2] Guckenheimer, J.: One parameter families of vectorfields on two-manifolds: another nondensity theorem. In: Peixoto, M. M. (ed.): Dynamical systems. New York-London: Academic Press 1973
[3] Hirsch, H. W.; Pugh, C. C.; Shub, M.: Invariant manifolds. Berlin – Heidelberg – New York: Springer Verlag 1977. = Lecture notes in mathematics 583
[4] Kupka, I.: Contribution à la théorie des champs génériques. In: Contributions to differential equations **2** (1963) 457–484, and **3** (1964) 411–420
[5] de Melo, W.: Moduli of stability of two-dimensional diffeomorphisms, to appear.
[6] Newhouse, S.; Palis, J.; Takens, F.: Stable arcs of dynamical systems I, II, to appear.
[7] Newhouse, S.; Palis, J.; Takens, F.: Stable arcs of diffeomorphisms. Bull. Am. Math. Assoc. **82** (1976) 499–502
[8] Newhouse, S.; Palis, J.: Bifurcation of Morse-Smale dynamical systems. In: Peixoto, M. M. (ed.): Dynamical Systems. New York – London: Academic Press 1973
[9] Newhouse, S.; Palis, J.: Cycles and bifurcation theory. Asterisque **31** (1976) 43–140
[10] Palis, J.: A differentiable invariant of topological conjugacies and moduli of stability. Astérisque **51** (1978) 335–346
[11] Palis, J.; de Melo, W.: Introducão aos sistemas dinamicos, Projeto Euclides. São Paulo: Ed. Edgard Blücher 1977
[12] Palis, J.: On Morse-Smale dynamical systems. Top. **8** (1969) 385–404
[13] Palis, J.; Smale, S.: Structural stability theorems, Global analysis. Proc. Symp. pure math. **14** (1970)
[14] Palis, J.; Takens, F.: Topological equivalence of normally hyperbolic dynamical systems. Top. **16** (1977) 335–345
[15] Robinson, R. C.: C^r structural stability implies Kupka-Smale. In: Peixoto, M. M. (ed.): Dynamical Systems. New York – London: Academic Press 1973
[16] Shoshitaishvili, A. N.: Bifurcations of topological type at singular points of parametrized vectorfields. Funct. anal. apl. **6** (1972) 169–170
[17] Smale, S.: Stable manifolds for differential equations and diffeomorphisms. Ann. Scuola Norm. Sup. Pisa **18** (1963) 97–116
[18] Sotomayor, J.: Generic bifurcations of dynamical systems. In: Peixoto, M. M. (ed.): Dynamical Systems. New York – London: Academic Press 1973
[19] Sotomayor, J.: Generic one-parameter families of vectorfields on two-dimensional manifolds. Publ. I. H. E. S. **43** (1973) 5–46
[20] Sternberg, S.: On the structure of local homeomorphisms of Euclidean space II. Amer. J. Math. **80** (1958) 623–631
[21] Takens, F.: Normal forms of certain singularities of vectorfields. Ann. Inst. Fourier **23** (2) (1973) 163–195

Prof. F. Takens
Mathematisch Instituut
Postbus 800
NL-9700 AV Groningen

(Eingegangen: 19. 9. 1978)

Buchbesprechungen

Cambridge, MA: The MIT Press 1977, 2 Vol., cloth, $ 115.–

Dies zweibändige Werk einer japanischen Autorengruppe hatte bereits drei japanische Auflagen (1954, 1960, 1968) hinter sich, als die vorliegende englische Übersetzung hergestellt wurde. Es kann sich neben dem Mathematischen Wörterbuch von J. Naas und H. L. Schmid (Berlin – Stuttgart 1961) vorteilhaft sehen lassen. Nach einer Vorbemerkung von Saunders Mac Lane, den üblichen Vorworten und Einleitungen beginnt der Hauptteil, bestehend aus 436 z. T. sehr ausführlichen Artikeln von „Abel“ bis „Zeta Functions“. Die Artikel bringen die Grundbegriffe und Hauptergebnisse der betr. Teildisziplin und schließen mit einem ausführlichen Literaturverzeichnis. Wo man hier zunächst Lücken zu entdecken meint, schließen sie sich z. T., wenn man den weit ausgebauten Anhang und das umfangreiche Namen- und Sachregister benützt. Immerhin ist es mir nicht gelungen, Hinweise auf Invarianzprinzipien der Wahrscheinlichkeitstheorie zu finden, abgesehen von den um 1950 liegenden Anfängen. Im Artikel über Distributionen fehlt ein Hinweis auf Mikusinski. Die Volterra-Lotka-Kolmogorov-Smale-Theorie der Differentialgleichungen der Ökologie fehlt. Der Artikel über „Celestial mechanics“ ist kümmerlich; man findet das berühmteste Ergebnis der Himmelsmechanik in diesem Jahrhundert, die Kolmogorov-Arnold-Moser-Theorie nicht dort erwähnt, sie wird allerdings in einem gesonderten Artikel „Three-Body Problem“ ausführlich gewürdigt. Die Namen B. de Finetti, V. Strassen, D. Ruelle, P. Huber, A. Dvoretzky, H. Bauer, P. Martin-Löf, H. O. Cordes, G.-C. Rota, R. Tutte, E. S. Andersen fehlen empfindlich im Namenregister. Mängel dieser Art wird indessen jeder Benützer eines jeden Lexikons aufdecken. Ihnen steht die außerordentliche Brauchbarkeit dieses Werks, die vor allem auf dem wohlgetroffenen Gleichgewicht zwischen Knappheit und Ausführlichkeit entwickelte Methode der finiten Elemente zur numerischen Lösung von Randwertproblemen partieller Differentialgleichungen. Schließlich befaßt sich ein Aufsatz von H. W e r n e r mit Problemkreisen der Allgemeinen Algebra (bekannter unter dem Namen „Universelle Algebra"), und R. W o l f zählt reizvolle Anwendungen von Nichtkompaktheitsmaßen in der konstruktiven Funktionalanalysis auf.

Im zweiten Teil des Buches finden sich neben einem Bericht von H. J. V o l l r a t h zum IMUK-Kongreß in Karlsruhe 1976 Beiträge von A. S c h m i t t und G. S t e i n, welche den computerorientierten Mathematikunterricht bzw. den Informatikunterricht in der Schule (mit kritischer Würdigung zweckmäßiger Programmiersprachen) zum Inhalt haben. Unter den Marginalien ist ein Artikel von D. L a u g w i t z zum Beweis von K. A p p e l und W. H a k e n des Vierfarbensatzes hervorzuheben. (Ohne daß diese großartige Leistung geschmälert werden soll, mag die verwandte Methode – mathematische Reduktion eines kombinatorischen Problems bis zu einem computerzugänglichen finiten Entscheidungsproblem – vielleicht nicht jeden Mathematiker voll zu befriedigen!)

Zusammenfassend sei festgestellt, daß das vorliegende Jahrbuch, wie schon früher, dem Nichtspezialisten eine Fülle von interessanten aktuellen Informationen aus den verschiedensten Teilgebieten der Mathematik vermittelt.

Stuttgart K. Leichtweiß

Brouwer, L. E. J., Collected Works, Vol. 1 (ed. by A. Heyting): Philosophy and Foundations of Mathematics, XVI und 628 pp., 1975. Vol. 2 (ed. by H. Freudenthal): Geometry, Analysis, Topology and Mechanics, XXVIII und 706 pp., 1976. North Holland Publishing Company: Amsterdam, Dfl. 250 per Vol.

Brouwers Veröffentlichungen lassen sich in zwei Gruppen einteilen, zwischen denen kaum Beziehungen zu bestehen scheinen: die erste mit philosophischen und intuitionistischen Werken über die Grundlagen der Mathematik, die zweite mit nichtintuitionistischer Mathematik und hier vor allem Topologie. Dementsprechend ist sein Werk auf die beiden Bände verteilt. Der erste wurde von A. Heyting versorgt, der zweite von H. Freudenthal. Beide Bände sind sorgfältig ediert und mit zahlreichen nützlichen Anmerkungen und anderen Hinweisen versehen. Dem Temperament der beiden Herausgeber entsprechend sind die Anmerkungen von A. Heyting nüchtern und knapp (um nicht zu sagen trocken), die von H. Freudenthal ausführlicher und lebhaft, manchmal erzählerisch. Der zweite Band enthält die englische Übersetzung einer Biographie, die von den beiden Herausgebern im Jahrbuch 1966–67 der Niederl. Akad. d. Wiss. veröffentlicht wurde; außerdem eine chronologische Bibliographie aller wissenschaftlichen Werke Brouwers.

Die im ersten Band (Heyting) aufgenommenen Schriften sind chronologisch angeordnet; eine Einteilung nach Sachgebieten findet sich in der Einleitung (auf S. XIV). Drei nicht publizierte Texte, die auf Vorlesungen zurückgehen, sind in abgekürzter Form wiedergegeben, darunter die Beschreibung einer Vorlesung über intuitionistische Mathematik, die Brouwer 1946 in Cambridge, England, gehalten hat. Diese Vorlesungsausarbeitung, die anscheinend nicht vor 1954 ihre endgültige Form erhielt, konnte nicht in den vorliegenden Band aufgenommen werden. Sie ist im Besitz von B. van Rootselaar, der eine besondere Veröffentlichung dafür beabsichtigt. Das umfangreichste Werk in Band I ist die Thesis Brouwers aus dem Jahr 1907, die nach deutschen Maßstäben eher eine Habilitationsschrift als eine Dissertation ist. Sie wirkt teilweise eher philosophisch als mathematisch, stellt aber doch vor allem seine Einstellung zur Mathematik dar und bildet eine Art Programm.

Im zweiten Band (Freudenthal) sind die aufgenommenen Schriften nach Sachgebieten wie folgt geordnet: 1. Non-euclidean spaces and integral theorems. – 2. Lie groups. – 3. Toward the plane translation theorem. – 4. Vector fields on surfaces. – 5. Cantor-Schoenflies style topology. – 6. The new methods in topology. – 7. Topology of surfaces. – 8. Mechanics. Sie erstrecken sich i. w. auf die Zeit 1904–1920. Mit aufgenommen ist eine Anzahl von Briefen, insbesondere an D. Hilbert, F. Engel, O. Blumenthal sowie an H. Blaschke, E. Borel, J. Hadamard, F. Klein, P. Koebe, A. Hurwitz, die zusammen mit den (z. T. auf mündlicher Tradition beruhenden) Anmerkungen des Herausgebers ein sehr lebendiges Bild von Brouwer entstehen lassen. Für den algebraischen Topologen ist der Abschnitt 6 ein Höhepunkt; er findet dort die fundamentalen klassischen Sätze seiner Disziplin (Abbildungsgrad, Invarianz der Dimension, Invarianz des Gebiets, u. a.).

Leider scheinen zahlreiche nicht veröffentlichte Aufzeichnungen und Unterlagen Brouwers verlorengegangen zu sein, und zwar teils auf Reisen Brouwers, teils durch einen Brand in seiner Wohnung und aus ungeklärten Gründen nach seinem Tod. Dennoch stellen die beiden vorliegenden Bände eine faszinierende Dokumentation über Brouwer und zur Mathematik unseres Jahrhunderts dar, eine voll gelungene Herausgabe, auch wenn sie nicht völlig systematisch ist. Nur der Preis stört!

Heidelberg A. Dold

Die Werke von Jakob Bernoulli, hrsg. von der Naturforschenden Gesellschaft in Basel, Bd. 3, Basel: Birkhäuser 1975, X und 585 S., DM 106,–

Sechs Jahre nach der Veröffentlichung von Band 1 ist der vorliegende Band 3 als zweiter Band der Edition von Jakob Bernoullis Werken erschienen. Enthält der erste Band die Schriften zur Astronomie und Philosophia naturalis, so vereinigt Band 3 die Schriften zur Wahrscheinlichkeitsrechnung, also insbesondere die 1713 postum erschienene „Ars conjectandi“ mit den handschriftlichen und gedruckten Vorarbeiten.

Entgegen dem ursprünglichen Plan des 1966 verstorbenen Otto Spieß, des Initiators der Bernoulli-Edition, im wesentlichen diese Texte zusammen mit der von der „Ars conjectandi“ abhängigen Dissertation „De usu artis conjectandi in iure“ von Nikolaus Bernoulli zu edieren und zu kommentieren, hat der Bearbeiter van der Waerden den Umfang des Bandes beträchtlich erweitert und eine Reihe von alten und modernen Arbeiten anderer Autoren aufgenommen, darunter zwei bis dahin ungedruckte Dissertationen aus den Jahren 1967 und 1973 des Editionsmitarbeiters Karl Kohli bzw. Julian Henny und einen nachgelassenen Aufsatz von Spieß zur Vorgeschichte des Petersburger Problems.

Somit umfaßt der Band drei Abschnitte. Der erste besteht aus einer achtteiligen Historischen Einleitung van der Waerdens zur Frühgeschichte der Wahrscheinlichkeitsrechnung, die in knapper Form die wahrscheinlichkeitstheoretischen Arbeiten von Cardano, Pascal, Fermat, Huygens und Jakob Bernoulli resümiert und insbesondere die Bedeutung des vierten, unvollständigen Teils der „Ars conjectandi“ betont, der eine Anwendung der Wahrscheinlichkeitsrechnung auf „bürgerliche, sittliche und wirtschaftliche Verhältnisse“ enthalten sollte und das schwache Gesetz der großen Zahlen formuliert und beweist.

Der zweite Abschnitt, der Textteil, veröffentlicht in den Nummern 151 bis 173 (Bd. 1 endete mit Nr. 40) zum ersten Mal die Abschnitte aus den „Meditationes“ Jakob Bernoullis, seines mathematischen Tagebuches, die sich auf die „Ars conjectandi“ beziehen. Die folgenden Nummern 174–179 enthalten weitere Vorarbeiten, u. a. Zeitschriftenartikel und die Inauguralrede „De arte combinatoria“, die Bernoulli beim ersten Antritt des philosophischen Dekanates im Jahre 1692 hielt und zur Einleitung in den zweiten Teil seiner „Ars conjectandi“ umarbeitete, die „Ars conjectandi“ selbst, zusammen mit der in Form eines Briefes abgefaßten Abhandlung über das „Jeu de paume“, ohne die seinerzeit ebenfalls gleichzeitig veröffentlichte Abhandlung über unendliche Reihen (diese wurde dem Band Analysis zugewiesen), die erwähnte mathematisch-juristische Dissertation des Neffen Nikolaus sowie ein Faksimileabdruck der Schrift „Waerdye van lyf-renten naer proportie van los-renten" des holländischen „Raadspensionaris" Johan de Witt aus dem Jahre 1671. Diese Schrift wurde, wie S. 528 erwähnt wird, bei Verwendung unterschiedlicher Orthographie zweimal gedruckt. In der Tat lautet der Titel des dem Rezensenten zur Verfügung stehenden Exemplars „Waerdye van lyfrenten naer proportie van losrenten", dessen Holländisch sich nicht nur vielfach von dem des wiedergegebenen Textes unterscheidet, sondern in einigen Fällen auch einen anderen Text bietet. So heißt es S. 7 am Rand „a Door het eerste praesuppoost", während im Bernoulli-Band (S. 333) das Wort „eerste" fehlt. Der Abdruck der „Ars conjectandi" führt – ebenso wie die deutsche Übersetzung Rudolf Haussners in der Reihe Ostwaldt's Klassiker – dankenswerterweise die Seitenzahlen der Originaledition von 1713 an.

Der aus neun Nummern bestehende dritte Abschnitt – „Kommentare" genannt – enthält teils Kommentare zu Texten, die in Abschnitt 2 angeführt wurden, teils Beilagen zur Geschichte der Wahrscheinlichkeitsrechnung. So bringt K 1 deutsche Zusammenfassungen der abgedruckten 29 „Meditationes"-Artikel, die den Gedankengang verstehen lassen lassen, auch ohne daß man den lateinischen Text studiert hat. Diese Artikel werden in K 2 datiert. Anstelle eines Kommentars zur „Ars conjectandi" behandelt K 3 deren Publikationsgeschichte, wobei zahlreiche bisher unveröffentlichte Briefe aus der Korrespondenz zwischen Johann I Bernoulli, Montmort, Hermann, Nikolaus I Bernoulli u. a. herangezogen und in gut lesbarer deutscher Übersetzung zitiert werden. K 4 ist Kohlis Dissertation „Spieldauer: Von Jakob Bernoullis Lösung der 5. Aufgabe von Huygens bis zu den Arbeiten von de Moivre". Sie wurde für die Edition nicht überarbeitet, so daß die einführenden Bemerkungen auf S. 403/4 Wiederholungen der Ausführungen in der Historischen Einleitung darstellen. Während van der Waerden S. 361 auf diese Schrift vorverweist, spricht Kohli S. 405 (Stand 1967) ohne Bezugnahme auf den Band, in dem seine Sätze stehen, von dem 3. Band der Werke Bernoullis, in dem der wahrscheinlichkeitstheoretische Teil der „Meditationes" veröffentlicht werde. Ivo Schneiders Dissertation über de Moivre von 1968/9, die einen eigenen Abschnitt über rekurrente Reihen enthält, bleibt unberücksichtigt. Dies ist in der folgenden Dissertation von Henny nicht der Fall, die den wahrscheinlichkeitstheoretischen Forschungen Nikolaus und Johann Bernoullis gewidmet ist. Aber auch in ihrem Literaturverzeichnis fehlt ein Hinweis auf die vorliegende Edition von Jakob Bernoullis Schriften. K 7 gibt einen hochinteressanten Abriß der Bewertung von Leibrenten, K 8 einen Kommentar zur Dissertation von Nikolaus Bernoulli.

Die durchwegs gediegenen, mathematischen Darstellungen des Bandes kontrastieren mit der Tatsache, daß die umfangreiche Sekundärliteratur in nur sehr geringem Maße herangezogen wurde. Dafür nur wenige Beispiele. Genau dieselben Briefausschnitte, die in K 6 aus dem Briefwechsel zwischen Leibniz und Jakob Bernoulli über die Wahrscheinlichkeitsrechnung in Übersetzung zitiert werden (S. 513 muß die fehlende Bandangabe Bd. 3 lauten), sind in deutscher Übersetzung (außer zahlreichen weiteren Zitaten) bei Corrado Gini, Gedanken zum Theorem von Bernoulli, Schweiz. (!) Z. f. Volkswirtsch. Statistik 82 (1946) 401–413 nachzulesen (vgl. Gini, C.: Rileggendo Bernoulli, Metron 15 (1949) 117–132; Vassilief, A.: Le bicentenaire de la loi des grands nombres, L'enseign. math. 16,1 (1914) 92–100).

Nr. 170 sollte besser nicht „Zur Regel von Tacquet" heißen, da Tacquet in seinem 1656 erschienenen Werk (nicht 1683, s. S. 375) ausdrücklich nur Hérigones 1634 veröffentlichtes Ergebnis zitiert. Es hätte Beachtung verdient – wie es K 1 zur Nr. 154,1 auch tut –, daß bereits Leibniz dieselbe Fragestellung 1666 allgemeiner als Hérigone löst, was in der 1973 erschienenen Monographie von E. Knobloch, Die mathematischen Studien von G. W. Leibniz zur Kombinatorik, Stud. Leibn. Suppl. Bd. XI, S. 15, 30/1 genauer dargestellt wird. Dort werden auch zahlreiche Probleme unter Berücksichtigung älterer Autoren wie K. F. Hindenburg behandelt, die Henny anschneidet, z. B. Partitionen bei Leibniz (S. 35–37, 162–240; s. Henny S. 463/4, der nur den überholten, stark fehlerhaften Aufsatz von Mahnke aus den Jahren 1912/13 zitiert) oder Polynompotenzen. Die Studien K. R. Biermanns bleiben hier wie beim 'Problème des partis' (S. 487 ff.) unerwähnt.

Ergänzungsbedürftig ist die Behauptung Kohlis (S. 439), de Moivre habe die Indexschreibweise eingeführt. Bereits Mahnke hat in einem weiteren Aufsatz aus den Jahren 1912/13 gezeigt, daß diese Erfinderehre Leibniz gebührt; seit 1972 ist eine Reihe von Aufsätzen zu diesem Thema von E. Knobloch erschienen. Sehr wahrscheinlich ist de Moivre von Leibniz abhängig, spricht er doch ausdrücklich von dessen Erfindung in den 1704 veröffentlichten „Animadversiones in D. Georgii Cheynaei tractatum de fluxionum methodo inversa".

tionen anschließen als an Turingmaschinen. Dementsprechend wird von Anfang an mit Funktionalen gearbeitet. Bemerkenswert ist der vollständige Beweis des Satzes von Matijasevič, Robinson, Davis, Putnam (MRDP-Theorem) über diophantische Relationen. Das erlaubt später die Voraussetzung polynomischer Matrizen statt rekursiver unter dem Präfix und vereinfacht die Herleitung der Unentscheidbarkeitsresultate für die Arithmetik. Als weitere Sätze, die behandelt werden, seien erwähnt: der Satz von Friedberg und Mučnik über unvergleichbare rekursiv aufzählbare Turing-Grade und im Kapitel über Mengenlehre die relative Widerspruchsfreiheit von AC und GCH mit Hilfe konstruktibler Mengen.

Anstelle einer weiteren Inhaltsangabe erfolge jetzt die Angabe der Kapitelüberschriften: 0. Prerequisites. – 1. Beginning Mathematical Logic. – 2. First-Order Logic. – 3. First-Order Logic (Continued). – 4. Boolean Algebras. – 5. Model Theory. – 6. Recursion Theory. – 7. Logic-Limitative Results. – 8. Recursion Theory (Continued). – 9. Intuitionistic First-Order Logic. – 10. Axiomatic Set Theory. – 11. Nonstandard Analysis.

Das vorliegende Buch wird sicherlich ein Standardwerk über mathematische Logik werden und kann jedem ernsthaften Interessenten sehr empfohlen werden.

Kiel A. Oberschelp

Barwise, J., Admissible Sets and Structures, Berlin – Heidelberg – New York: Springer Verlag, 1975, XIV und 394 pp., cloth, DM 72,–

Die Theorie der zulässigen Mengen ist seit dem Beginn der 60er Jahre entwickelt worden. Dabei hat der Verfasser des vorliegenden Buches den Untersuchungen entscheidende Impulse gegeben. Ihm sind in erster Linie die wesentlichen Ergebnisse zu verdanken. Die Theorie der zulässigen Mengen hat sich als eine natürliche Brücke zwischen verschiedenen Gebieten der mathematischen Logik erwiesen. Z. B. wurden früher zwischen Ergebnissen zu Definierbarkeitsfragen der Modelltheorie, der Rekursionstheorie und der Mengenlehre vereinzelt Analogien bemerkt. Jedoch: „For the student of admissible sets the old boundaries between fields disappear as notions merge, techniques complement one another, and results in one field lead to results in another. This is the view of admissible sets we hope to share with the reader of the book", so schreibt der Verfasser in der Einleitung, und dieses Vorhaben ist ihm in vorbildlicher Weise gelungen. In Form der zulässigen Mengen mit Urelementen, die in diesem Buch eingeführt werden, schafft er den Rahmen zur einheitlichen Darstellung. Das Buch ist bereits jetzt das Standardwerk der Theorie der zulässigen Mengen und ist eine Pflichtlektüre für jeden Logiker. In der folgenden kurzen Inhaltsübersicht greifen wir nur einige Themenkreise heraus.

Zulässige Mengen sind die Modelle des mengentheoretischen Axiomensystems KPU (Kripke-Platek Axiome mit Urelementen). Im Teil A des Buches („The Basic Theory") werden zunächst einige Eigenschaften aus KPU hergeleitet. Mit Hilfe der konstruktiblen Hierarchie wird $HYP_{\mathfrak{M}}$ eingeführt, die kleinste zulässige Menge, die eine vorgegebene Struktur $\mathfrak{M}$ enthält. Die Bedeutung von $HYP_{\mathfrak{M}}$ (für abzählbares $\mathfrak{M}$) wird im folgenden von mehreren Seiten beleuchtet: *Mengentheoretisch* interessant ist das Ergebnis, daß für jede KPU umfassende Mengenlehre T die Mengen in $HYP_{\mathfrak{M}}$ genau diejenigen Mengen sind, welche in allen Modellen von T liegen, die $\mathfrak{M}$ als Element enthalten. Die kleinste Ordinalzahl, die nicht in $HYP_{\mathfrak{M}}$ liegt, gibt Aufschluß über die Kompliziertheit von $\mathfrak{M}$ und ist bei *modelltheoretischen* Untersuchungen von Bedeutung. Eine Teilmenge S von $\mathfrak{M}$ liegt genau dann in $HYP_{\mathfrak{M}}$, wenn sie Δ^1_1 über $\mathfrak{M}$ ist. Dies verallgemeinert das klassische *rekursionstheoretische* Ergebnis, daß die Δ^1_1 Mengen über $\mathbf{N}$ gerade die hyperarithmetischen sind. – Teil A enthält überdies die Theorie abzählbarer zulässiger Fragmente von $L_{\infty\omega}$.

Teil B des Buches („The Absolute Theory") ist der Verallgemeinerung der klassischen Rekursionstheorie auf zulässige Mengen gewidmet. Die „rekursiv aufzählbaren" Prädikate über einer zulässigen Menge sind – per definitionem – die Σ_1-Prädikate. Die Rolle der endlichen Mengen übernehmen die Elemente der zulässigen Mengen. Der Zusammenhang zwischen Σ_1-Prädikaten und induktiven Definitionen wird dargestellt.

Teil C („Towards a General Theory") führt zunächst das Studium zulässiger Fragmente von $L_{\infty\omega}$ fort, wobei jetzt die Beschränkung auf abzählbare Fragmente aufgehoben wird. – Verschiedene (rekursionstheoretische) Formulierungen des Königschen Unendlichkeitslemmas motivieren die Einführungen verschiedener Eigenschaften, den sog. König-Prinzipien. Während abzählbare zulässige Mengen diese Eigenschaften besitzen, trifft dies nicht mehr auf alle überabzählbaren zu. Barwise benutzt sie zum Studium überabzählbarer zulässiger Mengen. Er erläutert, daß Σ_1-Prädikate eine syntaktische Verallgemeinerung rekursiv aufzählbarer Prädikate sind, während strikt Π_1^1-Prädikate eine semantische Verallgemeinerung darstellen.

Das Buch ist sehr sorgfältig geschrieben. Die Ergebnisse werden gut motiviert, die Beweisideen klar herauskristallisiert. Am Ende eines jeden Paragraphen findet man historische Hinweise und Aufgaben, die weitere interessante Ergebnisse enthalten. Zum Verständnis des Buches reichen im Prinzip Grundkenntnisse in Modelltheorie, Rekursionstheorie und Mengenlehre aus.

Das vorliegende Buch ist das erste der Reihe „Perspectives in Mathematical Logic". Es ist zu wünschen, daß auch die weiteren Bücher dieser Reihe so wegweisend sind.

Freiburg J. Flum

Comfort, W. W., Negrepontis, S., The Theory of Ultrafilters, (Grundlehren der mathematischen Wissenschaften, Bd. 211), Berlin – Heidelberg – New York: Springer Verlag, 1974, X und 482 pp., cloth, DM 98,–

The book contains considerably more material than the title suggests. Even the modified title "Ultrafilters and their applications" would not give an adequate description of the amount of information contained in it. Apart from giving an up-to-date account of the theory of ultrafilters the book could very well be used as an introductory text for each of the following topics: Zermelo-Fraenkel set theory, Boolean algebras, large cardinals, combinatorial set theory and model theory of classical first order logic. The only things assumed to be known are the very basics of general topology. In later chapters some rather technical results from model theory are used without proof, but this only for very specific purposes never referred to again later.

The theory of ultrafilters grew out of two sources. In general topology it originated with the study of convergence in topological spaces without countability conditions. Strangely enough, this aspect is not mentioned in the book except by the somewhat nebulous comment in the introduction that ultrafilters are "a method of convergence to infinity". Much later ultrafilters were used as the basic notion in the construction of ultraproducts (Łos 1955). Since then ultraproducts have become one of the central and most widely used tools in model theory and general algebra. In this book the authors attempt to give a unified treatment of these developments. In the reviewer's opinion they have succeeded in their task.

There is a large amount of interplay of methods from topology, model theory and combinatorial set theory in the book. The following simplifying description of its content can therefore only give a very rough idea of the actual presentation of the material. Four of the sixteen chapters of the book (7: Basic facts about ultrafilters, 9: The Rudin-Keisler order on types of ultrafilters, 10: Good ultrafilters, 16: Ultrafilters on ω) deal with the theory of ultrafilters proper. Two chapters (11: Elementary types, 13: Saturation of ultraproducts) deal with model theory and include a very readable account of Shelah's characterization of elementary equivalence without GCH. Two chapters (14: Topology in spaces of ultrafilters, 15: Spaces homeomorphic with $(2^\alpha)\alpha$) are concerned mainly with topological aspects of the theory. The first two chapters contain basic material about Zermelo-Fraenkel set theory, Boolean algebras and certain aspects of general topology needed later in the book. The chapters 5, 6 and 7 contain Jónsson's theory of α-homogeneous universal models and its application to ordered sets and Boolean algebras. Large cardinals are treated in chapter 8 and the remaining two chapters (3: Intersection systems and families of large oscillation, 12: Families of almost disjoint sets) can probably be best described as combinatorial set theory.

In spite of the large amount of material presented the authors have managed to write a very readable book. They deserve praise for providing us with this valuable source of information.

Hamilton, Ontario G. Bruns

Knobloch, E., Die mathematischen Studien von G. W. Leibniz zur Kombinatorik, Textband im Anschluß an den gleichnamigen Abhandlungsband zum ersten Mal nach den Originalhandschriften herausgegeben (Studia Leibnitiana Supplementa, Bd. XVI), Wiesbaden: Franz Steiner Verlag 1976, 3 Falttafeln, 2 Faksimiletafeln, XII und 339 S., Ln. DM 110,–

Dieser Band stellt eine Ergänzung zum gleichnamigen Abhandlungsband des Verfassers dar, der 1973 als Bd. XI der Reihe Studia Leibnitiana Supplementa erschien (vgl. die Rezension im Jber. d. Dt. Math.-Verein. **79** (1977) 4–5). Dort wurden die Leibnizschen Handschriften zur Kombinatorik im engeren Sinn, zur Theorie der symmetrischen Funktionen und zur Theorie der Partitionen einer eingehenden Untersuchung unterzogen und dabei in chronologischer Abfolge dargestellt, wie sich die Leibnizschen Gedanken entwickelt hatten. Es zeigte sich, daß in der Geschichte der Mathematik bisher eine Reihe von Formeln mit Mathematikern des 18. und sogar des 19. Jahrhunderts in Verbindung gebracht wurden, die sich (zumindest für einfache Fälle) schon bei Leibniz finden. In dem hier anzuzeigenden Textband werden erstmals die 60 wichtigsten Handschriften herausgegeben, die die Basis jener Untersuchung bildeten. Sie wurden nach den gleichen Themenkreisen wie im Abhandlungsband geordnet bzw. da, wo mehrere Themen angesprochen werden, beim Hauptthema eingruppiert. Die wegen der reichhaltigen Leibnizschen Symbolik sehr schwierige Textwiedergabe geschah nach den Richtlinien der Akademieausgabe „Gottfried Wilhelm Leibniz: Sämtliche Schriften und Briefe", damit die Stücke später unmittelbar an passender Stelle in diese Ausgabe aufgenommen werden können. Ein Vorspann zu jedem Stück gibt Titel, Entstehungszeit, Überlieferung und Hinweise auf die im Abhandlungsband zu findenden Erläuterungen. In den Fußnoten finden sich Lesarten, von Leibniz vorgenommene Korrekturen in seinen eigenen Texten, Hinweise auf vom Herausgeber festgestellte Fehler und die Auflösungen von namentlichen Anspielungen.

Dieser Textband ermöglicht nicht nur, die Ausführungen des gleichnamigen Abhandlungsbandes des Verfassers unter Heranziehung der Originalquelle zu verfolgen, sie macht zugleich der Forschung diese Texte zum ersten Mal zugänglich und kann damit zu weiteren Studien anregen. Herr Knobloch bearbeitet inzwischen in gleicher Weise die Untersuchungen von Leibniz zur Determinantenlehre.

Hamburg C. J. Scriba

Halder, H.-R., Heise, W., Einführung in die Kombinatorik, München – Wien: Carl Hanser Verlag, 1976, XII und 304 S., geb., DM 32,–

Mit zahlreichen Beziehungen und Anwendungen beginnt das Buch mit Fakultäten, erzeugenden Funktionen, Permutationsgruppen, In- und Exklusion, Partitionen (Euler), es folgen Abzählungen von Polya, auszugsweise Graphentheorie, Existenzsätze von Turan und Ramsay. In den letzten drei Kapiteln kommen neuere Untersuchungen und Fragestellungen zu Wort: mehrfach lateinische Quadrate, sehr ausführlich endliche Geometrien, Steinersysteme und einführend Codes.

Sehr nützlich sind die überall eingestreuten platzsparenden Tabellen. Auf S. 9 wird klar, daß das Pascalsche Dreieck sowieso keines ist. Einige Zitate sollten vielleicht verbessert werden. Ausführliche Verzeichnisse über Inhalt, Literatur, Symbole, Namen, Stichworte erleichtern das Nachschlagen. Auch der Humor kommt nicht zu kurz, oft an versteckter Stelle.

Ein technischer Mangel ist leider die Ähnlichkeit der verwendeten Alphabete, dadurch wird rein optisch die Lesbarkeit erschwert. Im Interesse der Verbreitung der Wissenschaft sollte man sich um die Herstellung von Kugelköpfen auch mit gotischen Buchstaben bemühen.

Insgesamt ist das vorliegende Buch als inhaltsreiche und aktuelle Einführung für Studierende der Mathematik, aber auch für Informatiker wohl zu empfehlen.

Hamburg E. Witt

Aigner, M., Kombinatorik Teil II: Matroide und Transversaltheorie (Hochschultext), Berlin – Heidelberg – New York: Springer Verlag 1976, 324 S., geh., DM 34,–

Während der erste Band des vorliegenden Werkes zum großen Teil klassische Themen der Kombinatorik enthält und das Gewicht auf der Darstellung dieser Themen unter ordnungs- und verbandstheoretischen Gesichtspunkten liegt, beschäftigt sich der zweite Band mit den wichtigsten neueren Entwicklungen der Kombinatorik, welche in der Theorie der Matroide zusammengefaßt werden können. Es handelt sich hier um die erste geschlossene Darstellung der Theorie der Matroide, so daß fast alle Abschnitte des Buches an die aktuelle Forschung heranreichen.

Ein Matroid (auch „kombinatorische Prägeometrie" genannt) ist eine Menge mit einem Abschlußoperator, welcher den Steinitzschen Austauschsatz für Basen erfüllt. Die Theorie der Matroide umfaßt weite Bereiche der Graphentheorie, der Theorie der Netzwerke, der synthetischen Geometrie und der Transversaltheorie, wobei sie diese Gebiete algebraischen Methoden zugänglich macht. Es ist nun ein „Hauptziel dieses Buches, dem Leser ... Zusammenhänge zwischen der algebraischen und der rein kombinatorischen Interpretation diskreter Probleme klarzumachen. Interessante Beispiele hierfür sind der Zusammenhang von Einbettung und Färbung von Graphen zu Matroidbegriffen auf der Kantenmenge des Graphen, oder von Maximum-Minimum-Sätzen aus der Transversaltheorie zu korrespondierenden Rangformeln. Gleichzeitig ist damit auch der Standort dieses Buches im Vergleich zu anderen Lehrbüchern über diese Themen festgelegt, insofern als fast alle Standardsätze wie auch neuere Resultate aus diesen Gebieten von Obersätzen aus der Matroidtheorie abgeleitet werden".

Das Buch ist als Textbuch zu einer Vorlesung konzipiert und entstanden, worauf auch die Vielzahl von Übungsaufgaben von unterschiedlichem Schwierigkeitsgrad hinweist.

Wuppertal H. Scheid

Cameron, P. J. (ed.), Combinatorial Surveys: Proceedings of the Sixth British Combinatorial Conference, New York – London: Academic Press 1977, 274 pp., $ 13.65/£ 7.00

The British Combinatorial Conference ist entgegen allem, was ihr Name sagt, zu einer bedeutenden internationalen Angelegenheit geworden. Dies zeigt auch der vorliegende Band der Proceedings of the Sixth BCC, in dem sieben Übersichtsvorträge, die von eingeladenen Rednern auf der Tagung gehalten wurden, publiziert sind. Er zeigt ferner die Vielseitigkeit dieser Disziplin der Mathematik und die lebhafte Entwicklung, in der sie sich befindet. Allen, die sich mit Kombinatorik beschäftigen oder die auch nur wissen möchten, was derzeit in diesem schönen und hochinteressanten Zweig der Mathematik geschieht, kann eine Lektüre dieses Bandes nur wärmstens empfohlen werden.

Inhaltsverzeichnis: Buekenhout, F., What is a Subspace? Cameron, P. J., Extensions of Designs: Variations on a Theme. Lovasz, L., Flats in Matroids and Geometric Graphs. Ray-Chaudhuri, D. K., Combinatorial Characterization Theorems for Geometric Incidence Structures. Sloane, N. J. A., Binary Codes, Lattices and Sphere Packings. White, A. T., Graphs of Groups on Surfaces. Woodall, D. R., Zeros of Chromatic Polynomials.

Kaiserslautern H. Lüneburg

Hammer, J., Unsolved problems concerning lattice points (Research Notes in Mathematics, Series No. 15), London – San Francisco – Melbourne: Pitman 1977, 128 pp., £ 5.00/$ 9.50

Der Verfasser stellt in den beiden ersten Kapiteln des Buches eine große Anzahl von Resultaten und ungelösten Fragen unterschiedlichen Schwierigkeitsgrades mit zugehörigen Literaturangaben zusammen. Dabei werden die Geometrie der Zahlen, die Diskrete Geometrie und Teile der Diophantischen Approximationen und der Kombinatorischen Geometrie erfaßt; ferner werden Graphentheorie und Integralgeometrie berührt. Die Betonung liegt auf zweidimensionalen Problemen und entspricht den persönlichen Interessen des Autors. Daß der Schwierigkeitsgrad mancher Probleme, z. B. des Mordellschen Umkehrproblems für den Minkowskischen Linear-

Diese große Entwicklung im ganzen nachzuzeichnen, eine Analyse der vielen verschiedenen Beweise und Ideen (bereits 1963 erschien im Amer. Math. Monthly **70**, S. 397, der angeblich 152. Beweis des quadratischen Reziprozitätsgesetzes) zu geben, überschreitet Umfang und Anlage des vorliegenden Buches. Es ist geschrieben für Leser, die kaum Vorkenntnisse in Algebra oder Zahlentheorie vorweisen können (in 2 Anhängen werden die benutzten, ganz elementaren Eigenschaften von Primzahlen, Kongruenzen etc. entwickelt bzw. referiert), die aber die Mühe nicht scheuen, durch das zunächst undurchdringlich erscheinende Dickicht elementarer Einzelheiten hindurchzufinden und verschiedene Pfade zu dem gleichen, so merkwürdig verborgenen Ziel mitzuverfolgen.

Der Autor beschränkt sich daher im wesentlichen auf eine ausführliche, Wiederholungen nicht scheuende Darstellung der elementaren Teile des Beitrages, den Gauß geleistet hat, in verschiedenen Varianten. Nach einem 20 Seiten umfassenden, sachlichen und historischen Überblick über das Thema wird die Theorie der quadratischen Reste und der erste Gaußsche Beweis entwickelt unter weitgehender Anlehnung an die entsprechenden Artikel in den Disquisitiones arithmeticae (in der deutschen Übersetzung von E. Netto, Ostwalds Klassiker Nr. 122). Es folgen 14 Variationen, in denen unter breiter Entwicklung der jeweils benötigten Hilfsmittel weitere Beweise vorgestellt werden, die im Kern fast immer auf Gauß zurückgehen. Es ist hier nicht der Ort, diese verschiedenen Variationen zu skizzieren; es sei aber bemerkt, daß sich (was bei dem sonst sehr sorgfältig geschriebenen Text auffällt) in die Schilderung der 4. Variation (des von Kronecker 1872 in der Berliner Akademie vorgetragenen Beweises des Herrn Zeller, Bezirksschulinspektor und Pfarrer zu Weiler bei Schorndorf in Württemberg) eine kleine Unkorrektheit eingeschlichen hat. Die Beziehungen und Verwandtschaften zwischen verschiedenen Variationen festzustellen bleibt der Entdeckerfreude des Lesers vorbehalten. Unter diesem Leser kann ich mir Oberschüler, Studenten, Lehrer und andere Liebhaber der Mathematik vorstellen, denen das vorgeführte Ringen um einen Meilenstein der Mathematikgeschichte eine lebendige Einführung in die Gedankenwelt der Zahlentheorie bietet. Gerade für diesen Personenkreis wäre es aber vielleicht gut gewesen, den Gaußschen Ausführungen ein Kapitel mit Eulers Darstellung der quadratischen Reste und der Reziprozität voranzustellen, die m. E. den Vorzug der größeren Unmittelbarkeit genießt.

Den beiden Anhängen folgt ein (nicht ganz zuverlässiges) Literaturverzeichnis, das sich auf das Notwendigste beschränkt. Das anschließende Mathematikverzeichnis ist als Erstinformation anregend und nützlich; daß sich unter den vielen Zahlentheoretikern auch Henri Lebesgue eingefunden hat, beruht vermutlich auf einer Personenverwechslung mit V. A. Lebesgue, der mehrere Noten zum quadratischen Reziprozitätsgesetz verfaßte, i. ü. aber nicht in die illustre Reihe prominenter Namen paßt.

Dem reichhaltigen Büchlein sind eifrige Leser zu wünschen, die von hier aus einen Einstieg in die Originalliteratur finden können. Auch für Proseminare ist es gut geeignet.

Erlangen W.-D. Geyer

Koblitz, N., p-adic Numbers, p-adic Analysis and Zeta-Function (Graduate Texts in Mathematics), Berlin – Heidelberg – New York: Springer Verlag 1977, 122 pp., cloth, DM 27,90

In dem ersten Kapitel des Buches wird an die Konstruktion der p-adischen Zahlen und deren einfachste arithmetischen Eigenschaften erinnert, im dritten Kapitel wird der algebraische Abschluß von $\mathbf{Q}_p$ und dessen Komplettierung Ω beschrieben. Im vierten Kapitel werden dann einige Tatsachen über Potenzreihen über Ω (z. B. über Konvergenzradien, elementare Funktionen und Newton-Polygone) hergeleitet. Angewendet werden diese Kapitel auf die Zetafunktion in verschiedenen Versionen: Im zweiten Kapitel wird die p-adische Interpolation der Riemannschen Zetafunktion mit Hilfe der p-adischen Maße nach Mazur behandelt, und im fünften Kapitel wird

Dworks Satz über die Rationalität der Zetafunktion einer Hyperfläche über einem endlichen Körper bewiesen.

Der Autor des Buches ist bemüht, möglichst elementar und suggestiv in ein sehr interessantes Gebiet der Mathematik einzuführen, was mit einigen Einschränkungen auch gut gelingt. Es scheint manchmal nicht ganz eindeutig entschieden zu sein, welches Schwierigkeitsniveau eingehalten werden soll. In den zahlreichen Übungsaufgaben sind bunt gemischt einfache Rechenaufgaben neben für das Weitere sehr wichtige (und auch unbedenklich verwendete) Resultate gestellt; sie sind unbedingt bei einem Durcharbeiten des Buches sorgfältig anzuschauen. Besonders nachteilig macht sich der Darstellungsstil m. E. im dritten Kapitel bemerkbar, in dem man sich auch wundert, daß die Fortsetzungen von Bewertungen nur für lokal kompakte Körper (und nicht für komplette Körper) beschrieben werden und daß nicht etwas genauer auf die Erweiterungen von $\mathbb{Q}_p$ von gegebenem Grad eingegangen wird (obwohl Krasners Lemma bewiesen wird).

Ein weiterer Nachteil der bemüht elementaren Darstellung ist, daß der gemeinsame Hintergrund der verschiedenen Zetafunktionen, insbesondere der geometrischen und der arithmetischen Zetafunktionen, nicht genügend klar wird; aus den Definitionen kann man das ohne weiteres nicht erkennen. Auch sollte man im zweiten Kapitel die kurze Zusammenfassung der Theorie der analytischen Zetafunktion (§ 7), die die Integraldarstellung und die Funktionalgleichung bringt, vielleicht schon vor der Behandlung der p-adischen Zetafunktion durchlesen: Die Definition der p-adischen Zetafunktion und die „Integraldarstellung" durch Maße wird dann motivierter.

Zusammenfassend kann man sagen, daß das vorliegende Buch vielleicht nicht als Einführung in die p-adischen Zahlen (und schon gar nicht in die p-adische Analysis, von der nur die elementarsten Bereiche gestreift werden) benutzt werden sollte, aber gut geeignet ist, Studenten mit einigen Kenntnissen in Algebra und Zahlentheorie einen interessanten und lebendigen Einblick in die Welt der Zetafunktionen zu geben, der zu tieferen Studien anregen sollte.

Saarbrücken G. Frey

Knopfmacher, J., Abstract Analytic Number Theory (North Holland Math. Library, Vol. 12), Amsterdam-New York: North Holland Publishing Company 1975, IX + 322 S., geb., Dfl 75,–

Um den Inhalt dieses Buches zu beschreiben, scheint uns ein kurzer geschichtlicher Rückblick zweckmäßig. Hilbert hat 1900 in seinem Problem 8 unter anderem die Aufgabe gestellt, die für die rationalen Primzahlen gewonnenen Ergebnisse auf die Ideale in einem gegebenen Zahlkörper k zu übertragen; es folgt dort ein Hinweis auf die Dedekindsche Zeta-Funktion von k. Schon 1903 hat daraufhin Landau den Primidealsatz bewiesen. Es folgt 1917 der Beweis der Funktionalgleichung für die Dedekindsche Zeta-Funktion durch Hecke. All das findet sich schon in dem Buch von Landau, *Einführung in die elementare und analytische Theorie der algebraischen Zahlen und der Ideale* (1917), Nachdruck Chelsea, New York 1949. Von ähnlichem Charakter, aber mit vielen neueren Literaturhinweisen ist auch die Vorlesungsausarbeitung von I. S. Gal, *Lectures on Algebraic und Analytic Number theory,* Minneapolis 1961. Dann ging es weiter mit Heckes Zeta-Funktion mit Größencharakteren 1920 und den L-Funktionen von Artin 1924.

Das vorliegende Buch verfolgt eine andere Richtung, welche im wesentlichen von Beurling 1937 ihren Ausgang nahm. Die Beobachtung, daß die Menge $\mathbf{N}$ der natürlichen Zahlen durch einen Zählvorgang entsteht und daß für sie dann die eindeutige Primfaktorzerlegung gilt, läßt man außer acht und stellt die Primfaktorzerlegung an den Anfang. $\mathbf{N}$ wird dann nur noch aufgefaßt als Halbgruppe mit den abzählbar vielen Primzahlen als Erzeugenden. Ähnlich verfährt man mit den Idealen von k. Dieser abstrakte oder axiomatische Standpunkt ist von großer Allgemeinheit; er hat den Vorteil, daß sehr viele mathematische Objekte darunterfallen und daß vieles vereinheitlicht wird; er hat den Nachteil, daß oft nur die oberflächlichen Ergebnisse über diese

Objekte so gewonnen werden; er hat den Vorteil, daß neuartige Fragen der Aufzählung und der Mittelwertbildung in den Blickpunkt rücken. Eine Halbgruppe mit Norm heißt arithmetisch. Insbesondere lassen sich zahlentheoretische Funktionen für arithmetische Halbgruppen studieren, und man hat Möbius-Umkehrung, Ramanujan-Summen und ähnliches. Axiom A des Buches verallgemeinert die Beobachtung $\sum_{0<n\leqslant x} 1 = [x] = x + O(1)$. Daran schließt sich an die abstrakte Primzahltheorie. Verschiedenartige erzeugende Funktionen spielen eine wichtige Rolle. Für die Einzelheiten verweisen wir auf das Buch selbst, welches mit großer Sachkenntnis und sehr sorgfältig geschrieben ist; es zeigt, daß sich der Verfasser in vielen Gebieten der Mathematik umgesehen hat. Die Vielfalt der Ergebnisse und Beispiele (etwa aus Zahlentheorie, Algebra, Topologie)

Schoeneberg, B., Elliptic modular functions – an introduction, Translated from the German by J. R. Smart and E. A. Schwandt (Grundlehren der mathematischen Wissenschaften, Bd. 203), Berlin – Heidelberg – New York: Springer Verlag 1974, VIII und 233 pp., cloth, DM 68,–

Klein and Fricke's monumental treatise on elliptic modular functions appeared in the 1890s and it is only in the last fifteen years that its dominance as a textbook has faded with the appearance of several new books on the subject that expound more recent developments. Professor Schoeneberg's book provides an excellent introduction to some of these and to the basic theory.

In his first three chapters he confines his attention to the full modular group and derives the standard properties of this group and of the modular functions and forms belonging to it. Throughout the book the dimension (weight) is restricted to be an integer and the corresponding multiplier systems are taken to be constant. Klein's invariant J is obtained by use of the Riemann mapping theorem, although it is shown later, after Eisenstein series have been introduced, that an appeal to this theorem can be avoided.

Subgroups of the modular group, together with the corresponding functions and forms, appear in Chapters IV and V. Fundamental regions, quotient spaces, the associated Riemann surfaces and their genera are briefly but clearly introduced and various examples are given. The Riemann-Roch theorem is stated and there is a good expository account of divisors and differentials leading to formulae for the dimensions of various subspaces of modular forms. Transformations of order n and modular equations are discussed in Chapter VI.

In Chapter VII Eisenstein series of higher level N and dimension $-k$ ($k > 2$) are introduced. These series are sums of terms of the form $(m_1\tau + m_2)^{-k}$, where the integers m_1, m_2 run through various congruence classes modulo N. An alternative definition, not mentioned, imposes the further restriction $(m_1, m_2) = 1$ (so that m_1 and m_2 form the second row of a unimodular matrix) and is more convenient for some purposes; thus, in Theorem 2 on p. 158 a single series of this type can replace the linear combination there mentioned. For $k = 1$ or 2 Hecke's method is used to extend the results.

The final two chapters are concerned with the more specialised subjects of integrals of $\mathscr{P}$-division values and general theta series. There is a rather inadequate index, which does not, for example, list $\mathscr{P}$-division values (defined on p. 157), although it contains d i v i s i o n v a l u e with a reference to p. 209.

Modular function theory is now so vast a subject that a book of reasonable size cannot include every aspect and so necessarily reflects the particular interests of the author. Thus the present work does not contain any mention of Poincaré series representing cusp forms, and there is no discussion of the theory of Hecke operators.

The book is beautifully produced and attractively set out. It is a welcome addition to the literature of the subject.

Glasgow R. A. Rankin

Lang, S., Introduction to Modular Forms (Grundlehren der mathematischen Wissenschaften, Bd. 222), Berlin – Heidelberg – New York: Springer Verlag 1976, IX und 261 pp., cloth DM 54,–

Wie zu erwarten war, hat das neue Interesse für die Theorie der Modulformen dazu geführt, daß S. Lang auch für dieses Gebiet eine Einführung geschrieben hat. Diese Einführung stellt nicht nur die klassische Theorie in übersichtlicher Form dar, sondern verschafft dem Leser auch Anschluß an neuere Entwicklungen. Das Buch enthält folgende Teile: I. Klassische Theorie (Hecke-Operatoren, Petersson-Produkt) mit einem Appendix von D. Zagier, in dem ein elementarer Beweis der Eichler-Selbergschen Spurformel für $SL_2\mathbf{Z}$ gegeben wird. II. Perioden von

Spitzenformen (Modulsymbole, Eichler-Shimura-Isomorphismus für $SL_2\mathbf{Z}$). III. Modulformen auf Kongruenzuntergruppen (Atkin-Lehner Theorie für $\Gamma_1(N)$, Formalismus für die Dedekindsche η-Funtion). IV. Kongruenzen und Galoisdarstellungen (Reduktion modp, Zusammenhang mit Darstellungen von Galoisgruppen in $GL_2(F_l)$ mit einem Appendix von W. Feit über exzeptionelle Untergruppen von GL_2. V. p-adische Distributionen (Werte von p-adischen L-Funktionen).

Wie der Verfasser im Vorwort erklärt, behandelt er vorwiegend solche Teile der Theorie, für die es noch keine systematische Einführung gibt. Dies und das Interesse des Autors hat zur Folge, daß der Schwerpunkt dieses Buches bei der arithmetischen Theorie liegt. Das Buch bietet deshalb nicht einen einführenden Überblick über die ganze Theorie der Modulformen. Für andere wichtige Teile der Theorie (wie z. B. Dirichletsche Reihen, Poincaresche Reihen, Quadratische Formen) sei der Leser auf die Bücher „Lectures on Modular Forms" von R. Gunning und „Modular Forms and Dirichlet Series" von A. Ogg verwiesen. Trotz dieser Beschränkung kann ich dieses Buch jedem, der sich für die Theorie der Modulformen interessiert, empfehlen. Interessant sind vor allem die Teile II und IV. Das Kapitel über Modulsymbole enthält den Beweis des Manin-Drinfeldschen Satzes, der besagt, daß Divisoren, die vom Grade Null sind und aus Spitzen bestehen, von endlicher Ordnung in der Divisorklassengruppe sind. Es ist bemerkenswert, daß der Verfasser bei der Behandlung von dem Eichler-Shimura-Isomorphismus (über Perioden von Modulformen, d. h. Integrale der Gestalt $\int_0^{i\infty} f(z)\, z^j dz$) Kohomologie nicht benutzt. Teil IV ist den Ergebnissen von Serre und Swinnerton-Dyer (vgl. Lecture Notes in Mathematics Bd. 350) über Kongruenzen für die Koeffizienten von Modulformen und die Erklärung davon mit Hilfe von Darstellungen von Galoisgruppen gewidmet. Für die Existenz solcher Darstellungen werden ohne Beweis Sätze von Deligne benutzt.

Das Buch ist gut lesbar, obwohl man es dem Buch ansehen kann, daß es schnell geschrieben wurde (Druckfehler, falsche Literaturhinweise). Es enthält eine ausführliche Bibliographie, die hauptsächlich auf rezente Veröffentlichungen hinweist. Der Leser sei verwiesen auf Modular Functions of One Variable VI (Lecture Notes in Mathematics Bd. 627) für ein Erratum bzgl. Appendix I. Meiner Meinung nach ist das Buch sehr gut geeignet als Einführung in die arithmetische Theorie der Modulformen.

Amsterdam G. van der Geer

Pilz, G., Near-rings (North-Holland Mathematics Studies, vol. 23), Amsterdam – New York – Oxford: North Holland Publishing Company 1977, XIV und 393 pp., Dfl 62.50

Zunächst einige Definitionen: Ein (Rechts)- F a s t r i n g (N, +, .) ist durch drei Axiome charakterisiert:
(a) (N, +) ist eine (nicht notwendig kommutative) Gruppe; (b) (N, .) ist eine Halbgruppe;
(c) $(n_1 + n_2)n_3 = n_1n_3 + n_2n_3$ für alle $n_i \in N$.
Im folgenden bezeichnet N einen Fastring. Aus (a) und (c) folgt $0n = 0$ für alle $n \in N$. Gilt dazuhin noch $n0 = 0$, also $0n = n0 = 0$ für alle $n \in N$, so heißt der Fastring N n u l l s y m m e t r i s c h. Ein F a s t k ö r p e r ist ein Fastring N mit der Eigenschaft, daß $(N \setminus \{0\}, .)$ eine Gruppe ist. Mit einer trivialen Ausnahme sind alle Fastkörper nullsymmetrisch.

Während Fastkörper schon 1905 in einer Arbeit von L. E. Dickson auftreten (Trans. Amer. Math. Soc. **6** (1905) 198–204), kommen (spezielle) Fastringe zum ersten Mal in Arbeiten von O. Ore, P. Furtwängler – O. Taussky und O. Taussky aus den dreißiger Jahren vor. H. Wielandt entwickelte um 1937 eine Strukturtheorie von Fastringen; er veröffentlichte jedoch nur eine kurze Mitteilung in Dt. Math. **3** (1938) 9–10. Mit D. W. Blacketts Arbeit über „Simple and semisimple near-rings" (Proc. Amer. Math. Soc. **4** (1953) 772–785; Kurzfassung einer unter Anleitung von E. Artin verfaßten Dissertation) begann eine rasche Entwicklung der Fastringtheorie. Bis 1976 erschienen rund 550 Arbeiten über Fastringe und Fastkörper.

Die vorliegende Monographie von Günter Pilz ist die erste – und sehr gelungene – Abhandlung über das Gebiet in Buchform. Der Verfasser stellt die gesamte Fastringtheorie systematisch dar. Dabei wird auf die Voraussetzung der Nullsymmetrie soweit möglich verzichtet. Fastkörper werden als spezielle Klasse von Fastringen aufgefaßt und verhältnismäßig kurz abgehandelt. Dies ist sinnvoll, weil gute Ergebnisberichte über Fastkörper vorliegen, zum Beispiel die Arbeit von H. Wähling im Jber. d. Dt. Math.-Verein. **76** (1975) 41–103. Eine Reihe von Resultaten in dem zu besprechenden Buch sind neu oder erscheinen in einer – gegenüber der Originalarbeit – deutlich verbesserten Form. Es ist zu erwarten, daß das Buch neues Interesse an Fastringen weckt und die Forschung auf diesem Gebiet fördert.

Der Verfasser stellt diejenigen Begriffe und Resultate deutlich heraus, die über Verallgemeinerungen der klassischen Ringtheorie weit hinausgehen und die tiefere Einsichten in die Struktur der Fastringe vermitteln. Anwendungen, zum Beispiel die Konstruktion von taktischen Konfigurationen bzw. Blockplänen mit Hilfe planarer Fastringe, werden ebenfalls gebührend betont.

Der Stil des Buches ist knapp, aber klar. Der Stoff ist übersichtlich gegliedert. Einzelne einfache oder an anderer Stelle leicht zugängliche Beweise wurden ausgelassen. Dies gilt besonders für den Abschnitt über Fastkörper. Der Verfasser verwendet eine wohl überlegte, konsequente Bezeichnungsweise und Terminologie. Es ist zu hoffen, daß damit der terminologische Wirrwarr in der Fastringtheorie wirksam eingedämmt wird. Die Ökonomie der Darstellung beruht auch auf dem geschickten Einsatz einfacher Hilfsmittel aus der Universellen Algebra.

Besonders hervorzuheben sind:

1. eine Liste von Fastringen kleiner Ordnung, die man J. R. Clay verdankt,
2. eine Zusammenstellung von interessanten offenen Fragen und Problemen[1]),
3. die ungemein reichhaltige, praktisch vollständige Bibliographie, deren Wert durch ein [illegible] Klassifikationsschema noch beträchtlich erhöht wird. Eine nachahmenswerte Spezia-