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# Jahresbericht
## der Deutschen Mathematiker-Vereinigung

4 – 2005

Herausgegeben von K. Hulek
unter Mitwirkung von
U. Gather, H.-Ch. Grunau, H. Lange,
J. Rambau, A. Schied, Th. Sonar

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# Jahresbericht
der Deutschen Mathematiker-Vereinigung

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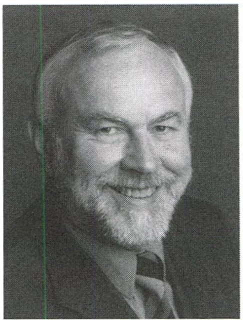
# Vorwort

Dieses Heft enthält zum ersten Mal einen Artikel in der Rubrik „Berichte aus der Forschung". M. Grötschel berichtet über die Gründung des DFG Zentrums MATHEON „Mathematics for key technologies" in Berlin und skizziert die Forschungsrichtungen und die Arbeitsweise dieses in Deutschland in dieser Art in der Mathematik einzigartigen Instituts. Die Forschungsaktivitäten von MATHEON umfassen u. a. Gebiete, die von den Life Sciences über Bereiche der Verkehrsführung und der Telekommunikation bis hin zu Finanzmathematik, Visualisierung und zu didaktischen Aktivitäten reichen.

Daneben enthält diese Ausgabe zwei Übersichtsartikel über aktuelle Gebiete der Mathematik: L.A. Imhof beschreibt in seinem Aufsatz neue Ergebnisse aus dem Bereich von Spieltheorie und Populationsentwicklungen unter stochastischen Einflüssen und J. Pach berichtet über Neues aus dem Gebiet der kombinatorischen Geometrie.

Wie stets finden Sie zudem eine Reihe von aktuellen Buchbesprechungen.

K. Hulek



# MATHEON: Introducing the DFG Research Center "Mathematics for key technologies" in Berlin

Martin Grötschel

## Abstract

- Mathematics Subject Classification: 01 A 65, 01 A 67, 01 A 74
- Keywords and Phrases: mathematics for key technologies, research center, Berlin

In October 2000, the Deutsche Forschungsgemeinschaft (DFG) created a new coordinated program called DFG research centers. This article describes the intentions and the application process of this program briefly. It particularly focusses on the establishment of the DFG Research Center MATHEON in Berlin. The MATHEON is one of six existing DFG research centers and is the only one operating in mathematics. Its complete name DFG Research Center "Mathematics for key technologies: Modelling, simulation, and optimization of real-world processes" describes the MATHEON agenda in broad terms. MATHEON is organized along application areas which currently are: life sciences, traffic and communication networks, production, electronic circuits and optical technologies, finance, visualization, and education. The mathematical work done and the special applications addressed will be outlined in the sequel.

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## Coordinated research programs

The German Research Foundation (DFG) offers quite a number of support programs, each targeted at different needs. With an allocation of roughly one third of all DFG funds (about 1,300 million Euros), the most important one is still the *Individual Grants Program* (Normalverfahren) where an individual researcher can ask for support for a particular project with a defined thematic focus and project duration.

Once it became apparent that certain research efforts require the cooperation of a larger number of scientists, coordinated research programs were created. These coordinated programs promote cooperation and structural innovation by encouraging national and international collaboration in areas of current relevance and by concentrating scientific potential at a university. Till the end of the 1990s the DFG supported *Research Units* (Forschergruppen, FOR, established in 1962), *Priority Programs* (Schwerpunktprogramme, SPP, established in 1953), *Research Training Groups* (Graduiertenkollegs, GRK, established in 1990), and *Collaborative Research Centers* (Sonderforschungsbereiche, SFB, established in 1968).

Due to the careful and unbiased refereeing processes aiming at excellence and not at uniform distribution of research funds, the existence of DFG supported projects has always been considered significant evidence of high-quality research activities at a university. Among the support programs, the Collaborative Research Centers have (probably) achieved the highest prestige. One can infer this from the fact that university presidents never fail to mention the number of SFBs when trying to indicate the excellence of research at their universities.

The statistics in Table 1, quoted from the DFG Jahresbericht 2004, shows in absolute and relative terms how much mathematics participated in the coordinated DFG programs in the year 2004.

Table 1

| | FOR | SPP | GRK | SFB | FZT |
|---|---|---|---|---|---|
| Total number | 175 | 134 | 257 | 272 | 5 |
| Total funding (M €) | 89.7 | 152.5 | 80.2 | 381.1 | 25 |
| # Math projects | 6 | 5 | 19 | 5 | 1 |
| Volume math projects | 1.2 | 7.3 | 5.2 | 5.6 | 5.0 |
| % Funding math projects | 1.3 | 4.8 | 6.5 | 1.5 | 20.0 |

The first line of Table 1 indicates the number of projects supported in each of the five coordinated research programs, the second line the total amount spent in each program in 2004 (in millions of Euros). The third, fourth, and fifth line indicate the number of mathematical projects, their total funding, and the percentage of funding spent by the DFG on mathematics in 2004.

The last column shows the numbers for the DFG research centers to be introduced below. Frank Kiefer, the DFG-Fachreferent for mathematics, pointed out that mathematicians participate considerably in programs accounted under other scientific fields. It is very difficult to estimate the funding obtained by mathematicians via this indirect way.

## DFG research centers

At the beginning of this decade a new coordinated research program came into being due to an unexpected new financial source. In August 2000, the German government sold licenses for third generation cell-phone services (UMTS) via an auction. This auction resulted in a spectacular total revenue. The federal government cashed about 50 billion Euros. Part of the auction earnings went to the Bundesministerium für Bildung und Forschung (BMBF) that, in turn, increased its support of the DFG.

The DFG decided to use some of this additional funding to start a new coordinated program: DFG research centers (in German called DFG-Forschungszentren, abbreviated FZT). The aim of the program is described in the first paragraph of the resolution of the DFG Senate of October 26, 2000:

*„Ziel des Programms ist es, in deutschen Hochschulen in begrenzter Anzahl international sichtbare und konkurrenzfähige Forschungszentren zu etablieren. Diese Zentren sollen wichtiger Bestandteil der strategischen und thematischen Planung einer Hochschule sein, ihr Profil deutlich schärfen und Prioritäten- bzw. Posterioritätensetzung verlangen. Die Konzentration von Exzellenz, Ressourcen und Kompetenz soll unter anderem durch die Anfinanzierung und Ausstattung von Professuren durch die DFG erreicht werden, deren spätere Übernahme einen wichtigen Beitrag der Grundausstattung darstellt."*

The current description (July 2005) reads:

*"Research Centers are an important strategic funding instrument to concentrate scientific research competence in particularly innovative fields and create temporary, internationally visible research priorities at research universities.*

*DFG Research Centers enable the universities to establish research priorities on the basis of existing structures. The thematic focus must incorporate a high degree of interdisciplinary cooperation. Networking with other research institutions at the university location is encouraged. DFG Research Centers are open for cooperation with partners from industry.*

*Funding may be provided for up to six professorships as well as associated independent junior research groups working within a DFG Research Center. Following the start-up funding provided by the DFG, the host university commits itself to financing the professorships from its core budget. Appropriate personnel and material resources will also be made available. Funding for each DFG Research Center averages approximately 5m per annum. Research Centers may receive funding for up to a maximum of 12 years."*

## The first round of FZT proposals

Although aiming at highest standards, international visibility, and competitiveness, there was, after the announcement of the new program, almost no time left for a deep exploration of possible options. The DFG expected the universities to submit pre-proposals (Konzepte) within eight weeks. And thus, pre-proposal writing had to start immediately. Another quote from the resolution:

*„Es wird um Konzepte (ca. 25 Seiten) für die Gründung eines konkreten Zentrums gebeten, dem bereits eine wissenschaftlich exzellente und das Profil der Hochschule prägende Struktur zugrundeliegt, welche mit den durch das Programm der DFG-Forschungszentren angebotenen Möglichkeiten ausgestaltet werden soll. Diese Konzepte, die in einer ersten, nicht thematisch definierten Auswahl-*

*runde berücksichtigt werden sollen, müssen bis zum 20. Dezember 2000 in der Geschäftsstelle der Deutschen Forschungsgemeinschaft eingetroffen sein. Der Senat wird darüber entscheiden, welchen Initiativen danach eine Antragstellung ermöglicht wird."*

By Christmas 2000, almost 90 pre-proposals had reached the DFG. They came from all parts of the country and ranged over all areas of engineering, the sciences, and the humanities.

One of the pre-proposals was written by a group of applied mathematicians working at Freie Universität (FU), Humboldt Universität (HU), or Technische Universität (TU) in Berlin or at one of the two Berlin-based mathematical research institutes, the Weierstrass Institute for Applied Analysis and Stochastics (WIAS) or the Zuse Institute Berlin (ZIB). The pre-proposal was formally submitted to the DFG by the president of TU Berlin on December 19, 2000 and finally resulted in the creation of the DFG Research Center MATHEON that will be described in this article. The main text of our December 2000 pre-proposal started as follows:

*„Ohne Mathematik tappt man doch immer im Dunkeln.*
*schrieb vor über 150 Jahren der Berliner Student Werner von Siemens an seinen Bruder Wilhelm. Schon für Siemens war die Hochtechnologie seiner Zeit ohne Mathematik nicht beherrschbar. Diese Aussage gilt heute umso mehr. Disziplinen verschmelzen miteinander; zusammen mit neuen ökonomischen Konzepten und durch staatliche Deregulierungsmechanismen werden sie zu Motoren der Weltwirtschaft: zu Zukunftstechnologien. Diese äußerst komplexen Systeme bedürfen der Mathematik, der Sprache der Wissenschaft und der Technologie. Aber die neuen Technologien benötigen nicht nur die mathematische Sprache. Ohne die Algorithmen der Mathematik ist ein effizienter, kostengünstiger und ressourcenschonender Einsatz der Technologien nicht möglich.*

*Angewandte Mathematik ist damit selbst eine Schlüsseltechnologie im globalen Wettbewerb um Ressourcen und Marktanteile; als Querschnitts- und Strukturwissenschaft besitzt sie ein besonders hohes Potential zum effektiven Einsatz. Sie wirkt jedoch im Verborgenen; ihre Beiträge zur Problemlösung sind den Endprodukten in der Regel nicht mehr anzusehen. Aber nur wer sich als fähig erweist, das innovative Potential der Angewandten Mathematik effektiv zur Entwicklung neuer Produkte und moderner gesellschaftlicher Strukturen zu nutzen, wird im Zeitalter der globalen Informationsgesellschaft auf Dauer erfolgreich konkurrieren können. Dies gilt für lokal operierende Firmen und politische Einheiten genauso wie für die Global Players."*

These sentences characterize the general view on the role of mathematics and its relations to its application areas that the members of MATHEON share. Mathematics, there is no doubt, is a field of basic research in its own right. But mathematics has also a tremendous potential for other sciences and industrial applications that both sides, mathematicians and the employers of mathematics, have not made sufficient use of. MATHEON is determined to help change this situation and to support scientific and industrial development, in particular in key technologies.

Let us look, slightly more concrete, into a modern production environment. High tech companies may have made an excellent invention and may manufacture an outstanding product. But due to the very fast international information networks and global knowledge transfers they may quickly loose their competitive advantage. Understanding the invention fully, perfecting the product and its production, fabricating and distributing it at minimum cost, estimating the risks involved, avoiding environmental hazards, etc. often require very specific mathematical models and appropriate model simulations. And only mathematical models that have stood the test of simulation runs and critical reviews by experienced practitioners are utilizable for optimization. Many steps in this development process cannot be made without employing adequate mathematics tuned to the particular application. The mathematics necessary is often not available from the "bookshelf" or from software vendors. Meeting theses challenges depends on the development of mathematics in collaboration with the user and requires the design of algorithms and their implementation in a way that the user's needs are supported. MATHEON tries to participate in these processes and to make the important role of mathematics more visible.

This briefly indicated "MATHEON vision" of collaborative development of mathematics and of mathematical software has been made specific in the final application.

## Results of the first round of FZT applications

Making reasonable decisions on almost 90 very heterogeneous applications is an imposing task. The DFG decided to invite seven (quite diverse) pre-proposals to present full-fledged applications. Forty two applications (including our pre-proposal) received a positive vote and were put on hold for further consideration. Over 30 pre-proposals were rejected.

Three of the seven selected proposals finally won a research center. These centers were established in 2001:

- FZT 15 "Ocean Margins – Research Topics in Marine Geosciences for the 21st Century" in Bremen
- FZT 47 "Center for Functional Nanostructures" in Karlsruhe
- FZT 82 "Rudolf Virchow Center for Experimental Biomedicine" in Würzburg.

## The second round

While handling the seven finalists, the DFG sent out a call for new pre-proposals. This time proposals were invited that focused on two particular (though broad and transdisciplinary) areas. One of the topics selected by the DFG was:

*„Modellierung und Simulation in den Natur-, Ingenieur- und Sozialwissenschaften".*

The group of applied mathematicians from Berlin decided to try again and to submit a polished and enhanced version of the previous pre-proposal. The deadline was April 25, 2001. The DFG specified criteria for the selection; I quote:

(1) Quality of the research
(2) Extent of cooperative financial support obtained so far (from the DFG and other supporting institutions)
(3) International visibility of the site
(4) Utilizing the possibilities of the program to structure the development of the university and to focus on research areas
(5) Importance for young scientists
(6) Originality of the submitted project compared to other current supported activities.

Fourteen proposals from a wide range of groups covering, e.g., engineering, computer science, biochemistry, and medicine reached the DFG by April 25. Many applications were interdisciplinary, some involved colleagues from mathematics. Our proposal was the only one focusing on mathematics.

To be mentioned in passing: The second topic selected by the DFG was "Neurosciences: from the Molecular Foundations to Cognition". The winner of this competition was the University of Göttingen together with three Göttingen based research institutes, the Max Planck Institute for Biophysical Chemistry, the Max Planck Institute for Experimental Medicine and the German Center for Primate Research. The

- FZT 103 "Molecular Physiology of the Brain" was set up in 2002.

The opening of 2004 asked for applications in the areas "Regenerative therapies" or "Cognitive technical systems". After the initial application phase the DFG decided to drop the second area and give one center to the first. Making a selection from ten applications the DFG decided on September 2, 2005 to establish the DFG Research Center

- "Regenerative Therapy" at the Dresden University of Technology.

## The *real* application

On July 17, 2001, we learned that we were among the three finalists and that the deadline for the full application was November 15, 2001. This information arrived within the last week of the Berlin summer semester, and everybody was heading for conferences or vacation, a really bad time to meet each other and to discuss how to proceed.

At the end of August 2001, the preparation of the detailed application started. Within two and a half months about 50 mathematicians in Berlin found a joint vision for their future work, coordinated their research projects, and agreed on many details so that a streamlined application with a clear goal and focus arose. Considering that, in mathematics, it is rather unusual that such a large group works together in such a short time, this was a formidable achievement. Not only did we have to streamline our mathematical thoughts, but five institutions had to be convinced to actively support the project. Many of us had to attend lots of meetings with academic and political representatives and quite a number of committees of all kinds. The result was extremely positive. After brief explanation, unanimous support came from all academic institutions involved.

The Deutsche Forschungsgemeinschaft also expected financial support from the State of Berlin. This was (sort of) granted initially, but the promise never materialized. To fill the financial gap the five academic institutions jointly made a substantial effort to match what the DFG expected from the host state of the research center. All in all, this process showed that the academic institutions in Berlin are able to act, to define priorities, and to move forward in a joint and coordinated effort – despite the almost hopeless financial situation in Berlin.

On November 14, 2001 we were ready and shipped a 412-page application book to DFG via express mail signed by the presidents/directors of FU, HU, TU, WIAS, and ZIB. A really hectic period of time was over.

## The final selection

The submission of the application was not the final word. On January 21 and 22, 2002 each of the three finalists (groups from Berlin, Darmstadt, and Heidelberg) had half a day to present the application orally and visually. Eighteen persons, including representatives from all participating institutions and the State of Berlin, travelled to Bonn, gave thirteen short presentations, and passed the detailed interrogation by the international group of reviewers with distinction.

The DFG Hauptausschuss finished the competition on May 8, 2002, and declared our application the winner. The new center took up operation on June 1, 2002 under the name "Mathematics for key technologies: Modelling, simulation, and optimization of real-world processes" with the DFG abbreviation FZT 86 (reflecting the fact that our proposal was the 86th application for a Forschungszentrum reaching the DFG).

As much as we were happy to receive the grant, celebrated in a spontaneous party, we could feel the disappointment of our competitors. They worked as hard as we did but did not gain any financial support. "The winner takes it all" competitions result, unfortunately, in many frustrated participants.

Figure 1: Opening ceremony, TU Audimax

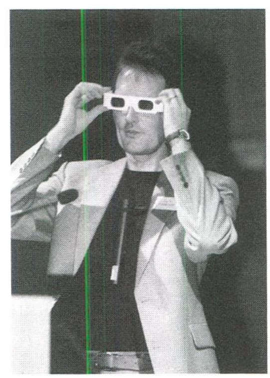


Figure 2: 3-d show

A formal celebration took place on November 11, 2002. The „feierliche Eröffnung" in the TU Audimax was attended by more than 1,000 persons, see Figure 1, listening to words of welcome, an overview of the MATHEON research activities, and seven brief talks by young MATHEON members about their projects. The "3-d show" introduced by Christof Schütte, see Figure 2, was the most spectacular part of the presentation.

## Naming the center

The full name mentioned above is a "monster", and it turned out that nobody really used it. Whenever an article about our center was published, the name of our center was altered in an arbitrary way or dropped at all. The abbreviation FZT 86 sounds more like a name for a database or a bookkeeper. So we started looking for a "better" name for our DFG Research Center. We asked center members, colleagues, and many other persons for suggestions and ended up with creations such as CAMB, MathKeyTech, MathKT, Keymathics, or KeyMath Center. Finally, we called Töchter+Söhne, a student run spin-off company of the Universität der Künste in Berlin, for creative input. They came up with the name MATHEON. One can find various interpretations (you can view it as mathe-on, i.e., switching on mathematics, or one can employ the similarity to pantheon), but it is just an artificial name that is apparently new and sounds good.

Before coming to a final naming decision in 2004 we used the name MATHEON at various public occasions. Most people found that this name made them curious (journalists, persons from other sciences). That is what we wanted to achieve. We also consulted Greek colleagues who, for instance, commented: *"It is for sure that* MATHEON *is not a Greek word. However, it sounds perfectly like a Greek word. I could give you two possible explanations, not of the meaning (which is evident) but of the structure of this word: –* MATH *(from mathematics) and* THEON *(means God) → Ma[th]eon. – Adding* EON *(which is a usual extension in Ancient Greek, for instance* PANTHEON = *the ensemble or house of Gods) to* MATH. *You understand that, in fact, the two explanations are strongly linked together. Let me say that* MATHEON *is a very smart name and you should congratulate the person who proposed it!"*

## The first year of MATHEON

Starting a center like this just three weeks after its formal creation was decided is not an easy task. The infrastructure had to be set up, the internal organization to be designed, the legal structure to be worked out, responsibilities to be distributed, administration persons to be hired, rooms to be found, openings for 6 new chairs and 7 leaders of junior research groups (Nachwuchsgruppenleiter) to be formulated and advertised, and about 70 new research positions to be filled. Not everything went as smoothly as hoped, but almost everything went much better than feared.

Of course, all individual projects started their research agenda right from the beginning. And after about one year of hard work, the MATHEON began operating as a joint research center with a common vision and not just as a collection of independent individual projects. The goal and promise was that the center should be more than the sum of its parts. And indeed, the added value became soon visible.

True, there were (fortunately very few) colleagues who viewed the MATHEON as a means to get more research money and who made no effort to contribute to the whole operation. The executive board monitored the development closely and in a few cases decided to stop projects because of the unwillingness of project leaders to cooperate – even though good mathematics was produced. Such decisions are unpleasant but seem to be inevitable if one wants to keep a large activity like the MATHEON on track.

## Statistics

I will not go into the details of the development of MATHEON, but I want to summarize some of the important facts.

### Finances

The annual financial support from the Deutsche Forschungsgemeinschaft is about 5 million Euros. The Berlin institutions provide additional annual support of about 3 million Euros.

### Legal Aspects

MATHEON is not a legal entity. It exists via a cooperation agreement between its participating institutions, which are

- FU Berlin (Fachbereich Mathematik und Informatik)
- HU Berlin (Institut für Mathematik and Institut für Informatik)
- TU Berlin (Institut für Mathematik)
- Weierstrass-Institut für Angewandte Analysis und Stochastik (WIAS)
- Konrad-Zuse-Zentrum für Informationstechnik (ZIB).

The TU is the leading university (Sprecherhochschule). All legal and financial actions of a MATHEON member for MATHEON are made via the institution to which the acting MATHEON member belongs. All central operations go through TU. This sounds complicated but has not lead to any difficulties so far.

### Organization

Here is a brief overview of the internal organization. There is, of course, some fluctuation. Whenever persons and numbers are mentioned, the state as of July 1, 2005 is reflected.

– MATHEON has

- a *Chair* (Sprecher), Martin Grötschel (TU, ZIB) and a *Deputy Chair*, Volker Mehrmann (TU),
- an *Executive Board* (Vorstand) consisting of 6 professors and 1 junior scientist: Peter Deuflhard (FU, ZIB), Martin Grötschel (TU, ZIB), Peter Imkeller (HU), Volker Kaibel (ZIB), Volker Mehrmann (TU), Christof Schütte (FU), Jürgen Sprekels (HU, WIAS) (The Executive Board "runs" MATHEON and is responsible for all its activities.),
- a *Council* (Rat) consisting of up to 20 members (with at least 3 junior scientists), see `http://www.matheon.de/about_us/organization/list_bodies.asp` (The

Council coordinates and directs the activities within the seven application areas and three mathematical fields of MATHEON and fosters cooperation between these topics.),

- a *General Assembly* (Mitgliederversammlung) that, for instance, elects all the other bodies,
- a *Scientific Advisory Board* (Wissenschaftlicher Beirat) consisting of 10 members meeting once a year, see `http://www.matheon.de/about_us/organization/advisory.asp` (It advises and supports MATHEON in various matters, such as national and international cooperation with other sciences and industry, organizational aspects, and long term perspectives.).

– The central management and administration of MATHEON, `http://www.matheon.de/about_us/organization/management.asp`, is located on the third floor of the mathematics building of TU. This floor, completely made available to MATHEON by TU, also hosts the central rooms (seminar and guest rooms, lounge) and the two TU MATHEON chairs. Otherwise the MATHEON members reside at their institutions, and every researcher hired from MATHEON funds is integrated in the research environment of the project head.

– MATHEON has 200 members, 42 of whom are professors. 94 of the MATHEON members are paid from the DFG MATHEON grant, the others from their home institutions, grants from industry, or other sources.

## New chairs

Six new C4-professor positions, two at each of the participating universities, have been created from the DFG funds, and additional professors (on regular and open positions) have been hired by the three universities in areas related to the center. The 6 new MATHEON professors are:

- Carsten Carstensen (HU), chair for "Numerical Solution of Differential Equations" (formerly TU Wien, Austria)
- Andreas Griewank (HU), chair for "Nonlinear Optimization" (formerly TU Dresden)
- John Sullivan (TU), chair for "Mathematical Visualization" (formerly U Illinois at Urbana-Champaign, USA)
- Harry Yserentant (TU), chair for "Numerical Solution of Partial Differential Equations" (formerly U Tübingen)
- Alexander Bockmayr (FU), chair for "Mathematics in Life Sciences" (formerly U Nancy, France)
- The FU chair "Mathematical Geometry Processing" will hopefully be filled within a few weeks.

For each of the six MATHEON C4-positions, a currently filled professor position has been designated to which the MATHEON professor will move as soon as the current position holder retires. At that moment in time a new MATHEON chair can be opened.

MATHEON also created 7 junior research groups (Nachwuchsgruppen), 3 at TU and 2 each at FU and HU. The decision to establish such research groups is just one of many actions of MATHEON to particularly promote young researchers. The leader of a junior research group (Nachwuchsgruppenleiter) is – within the area to which the group has been assigned – free to choose his or her particular research and application activity and to fill the positions that have been granted to the group. The chance to follow one's own research tracks independently very early in an academic career seems to be attractive and to produce particular visibility. That is one of the reasons why there has been quite some fluctuation on these positions – just as MATHEON hoped would be the case.

## Vision & Mission

A research center such as MATHEON has to specify a vision that is supposed to guide its future developments. And we have to convince colleagues in other fields that mathematics is an important ingredient in technological progress. What can mathematics offer to those who invent and develop high tech? I have already indicated the MATHEON vision above. Our vision statement in our application was as follows:

*"Key technologies become more complex, innovation cycles get shorter. Flexible mathematical models open new possibilities to master complexity, to react quickly, and to explore new smart options. Such models can only be obtained via abstraction. This line of thought provides our global vision: Innovation needs flexibility, flexibility needs abstraction, the language of abstraction is mathematics. But mathematics is not only a language, it adds value: theoretical insight, efficient algorithms, optimal solutions. Thus, key technologies and mathematics interact in a joint innovation process.*

*The mission of the center is to give a strong push to the role of mathematics in this interactive process. The center's research program is application-driven. Its implementation will have a strong impact on the development of mathematics itself and will define a new stage of inter- and transdisciplinary cooperation."*

## Mathematical Fields

Of course, there is no way to support every high tech subject. And not all mathematical fields potentially useful in these areas are present in Berlin. Building upon the special strengths of mathematics in Berlin, we decided to involve the following fields:

I Optimization and discrete mathematics
II Numerical analysis and scientific computing
III Applied and stochastic analysis.

The mathematical fields have been structured along well-established cooperations between the participating institutions, for example, along already existing DFG-supported Research Units, Priority Programs, and Research Training Groups. The existing seminars, colloquia, etc. in these fields continue, of course. MATHEON successfully fosters the interaction between the fields through special lectures and workshops such as

„Stochastik/Numerik", „Nichtlineare Optimierung und Diskrete Geometrie", „Stochastische Aspekte kombinatorischer Netzplanung", or „Scheduling-Probleme und Hysterese-Operatoren". Such workshops try to bring together mathematicians from different fields (who usually do not talk to each other) with the aim to make important questions, results, and techniques known to the members of the other group. These workshops are also employed to describe questions from industry one research team tries to solve, where it hopes that another team may be able to provide some mathematical input or fresh ideas.

## Application areas

Just as the mathematical fields reflect the Berlin expertise in applied mathematics, the application areas chosen build upon cooperations that have proven to be fruitful in the past. Of course, the goal is to extend the applications considerably and open up further lines of cooperation with industrial and scientific partners. The following key technologies are addressed in the first 4-year period:

A Life sciences
B Traffic and communication networks
C Production
D Electronic circuits and optical technologies
E Finance
F Visualization.

In addition, MATHEON views

G Education
as an important area to which its activities should radiate.

We are, of course, aware of the fact that it would be outrageously arrogant to claim that MATHEON is vital for the scientific progress of huge and important areas such as life sciences or production. These labels simply indicate that MATHEON makes a considerable effort to develop certain mathematical aspects of these areas.

While these application areas seem quite disjoint in real terms, the mathematical approach to major questions in these diverse topics brings out common structures. Analysis, stochastics, and discrete mathematics supply conceptual frameworks in these areas. Numerical mathematics and optimization provide the algorithmic machinery that enables the quantitative solution of a wide range of real-world instances. Even though the existing expertise has been highly successful in quite a few practical applications, more research has to be initiated to cope with many of the challenges of current technology. The DFG Research Center MATHEON "Mathematics for key technologies" is built as an institutional basis to further increase this level of competence. Modelling, simulation, and optimization of real-world processes are highly important aspects of many activities in industry and society, MATHEON views itself as a major effort in this direction.

## MATHEON's application areas: an overview

It is beyond the space limit of this survey paper to outline all the research activities in the mathematical fields and projects in the application areas. The interested reader is invited to visit `http://www.matheon.de/research/research.asp` and check the more than 80 projects that have been carried out within MATHEON so far. What follows is a brief characterization of our application areas, a short description of their research activities together with some examples of project titles.

It is occasionally useful to employ the "matrix view" of the MATHEON activities shown in Figure 3. This helps spotting possible synergies and initiating research cooperations between various mathematical fields, application areas, and projects therein.

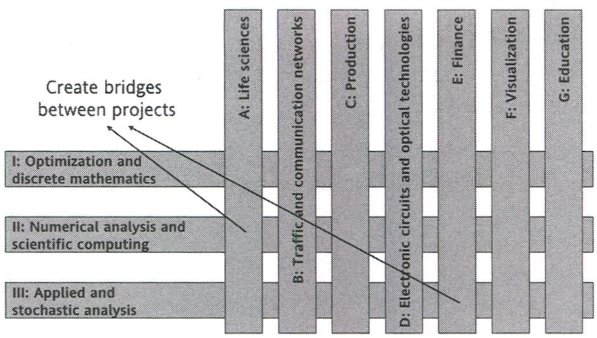



Figure 3: Matrix view

## Application Area A: Life sciences

(Scientists in Charge: Alexander Bockmayr, Peter Deuflhard, Hans Jürgen Prömel, Christof Schütte)

Laboratory work or surgery experiments are, of course, outside the expertise of MATHEON. Our center contributes to *computational aspects* of the life sciences. During the first four-year period the focus is on two fields: *computational medicine* and *computational biology*.

The long term vision in computational medicine is the development of mathematical models (and their mathematical analysis) that will make *quantitative individual medicine* possible, i.e., a patient-specific medical treatment on the basis of individual data and physiological models. In computational biology the vision is to understand molecular flexibility and function up to proteomic and systemic networks.

Most projects in Application Area A have to incorporate several, if not all, of the following aspects: *modelling* (in contrast to other fields, in biological or medical applications appropriate mathematical models are not at hand and need to be developed), *efficient simulation* (many of the resulting models couple several levels of description, and reliable and efficient discretization techniques are not available "from scratch", adequate numerical simulation techniques have to be designed), *optimization* (in patient-oriented medical treatment, e.g., the ultimate aim is the design of an optimal strategy or control of the treatment), *data acquisition and/or data analysis* (producing data by experiments is not the key issue, but getting the right data and understanding them), *visualization* (visualizations help analyzing data and computational results and are extremely important for the communication with partners from medicine and biology).

It is MATHEON's vision to incorporate simulation, optimization, and data analysis into virtual laboratories that support interfaces between the different algorithms, help to extract features by means of visualization, and finally make *conceptualization* possible.

Five project titles may indicate the concrete questions that are addressed in Application Area A (the project heads are mentioned in parentheses):

- A1: *Modelling, simulation, and optimal control of thermoregulation in the human vascular system* (Weiser, Deuflhard, Tröltzsch)
- A2: *Modelling and simulation of human motion for osteotomic surgery* (Deuflhard, Kornhuber), see Figure 4
- A4: *Towards a mathematics of biomolecular flexibility: Derivation and fast simulation of reduced models for conformation dynamics* (Deuflhard, Schütte)
- A6: *Stochastic modelling in pharmacokinetics* (Huisinga)
- A8: *Constraint-based modeling in systems biology* (Bockmayr)

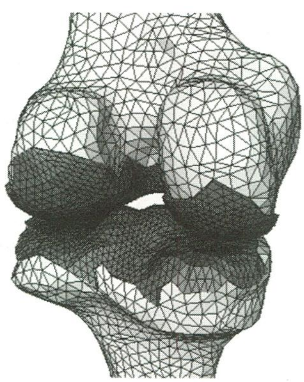


Figure 4: Modelling / simulation of human cartilage

## Application Area B: Traffic and communication networks

(Scientists in Charge: Martin Grötschel, Volker Kaibel, Rolf Möhring)

The guiding question for Application Area B is brief: *"What constitutes a good network?"*. We all want to be connected to a good telecommunication network or want to be served by a good public transport network. But what does *good* really mean here? Different "players" (customers, operators, politicians, tax payers) may value different aspects. Can we quantify quality in some way to come up with (somewhat) objective "quality measures" so that optimal networks can be designed?

Our society is becoming more and more "networked". Networks are crucial for the functioning of our daily life. But networks are also expensive to install, maintain, and control. Therefore, adequate network design and utilization is crucial for many activities. Application Area B focusses on traffic and communication networks. The vision is to ultimately develop theory, algorithms, and software for a new, advanced level of network analysis and design that addresses network planning problems as a whole. The "machinery" developed here has, of course, a much wider range of application than just transport and communication.

Linear and integer programming and combinatorial optimization are "key mathematical technologies" for the analysis and optimization of networks. There has been tremendous progress in the recent years in the algorithmic solution techniques so that there is now some hope to address not only single issues (such as vehicle circulation in transport networks or message routing in telecommunication networks) but integrated models that combine various network features. Moreover, the time is ripe to incorporate stochastic and nonlinear aspects to obtain models that more faithfully represent complicated scenarios.

The following project titles represent some of the issues studied in this application area:

- B1: *Strategic planning in public transport* (Borndörfer, Grötschel)
- B3: *Optimization in telecommunication: Integrated planning of multi-level/multi-layer networks* (Wessäly, Grötschel), see Figure 5
- B5: *Line planning and periodic timetabling in railway traffic* (Möhring)
- B6: *Origin destination control in airline revenue management by dynamic stochastic programming* (Römisch)
- B7: *Computation of performance measures of communication networks* (Scheutzow)


Figure 5: Integrated planning of multi-level / multi-layer networks

## Application Area C: Production

(Scientists in Charge: Carsten Carstensen, Jürgen Sprekels, Fredi Tröltzsch)

Condensed to one sentence, the long-term vision of Application Area C is to make a significant step towards the ultimate goal of production control and planning: fully automatic (and optimized) online control of the processes involved. Of course, there is a long way to go before a sound mathematical understanding of the effects of modern production processes is achieved and production innovations can be guided and optimized via mathematical models.

Production is a vast field with many facets. Application Area C focusses in particular on selected *multi-functional materials* and various aspects of *power generation.*

For multi-functional materials, for instance, we study a variety of phenomena that occur on a whole hierarchy of different space and time scales, ranging from microscopic changes of crystal lattice configurations, over the effects of mesoscopic thermo-stresses, to macroscopic hysteresis. Examples include the formation of liquid droplets in GaAs-crystals, martensitic phase transitions in shape memory alloys, the occurrence of spatial patterns in thin magnetic films, and structural phase transitions in the crystal lattices of modern steels. The mathematical tools required for such studies range from stochastics, asymptotic and multiscale analysis, thermodynamic modelling and phase-field theories, to numerics and optimization.

Here are a few example projects:

- C1: *Coupled systems of reaction-diffusion equations and application to the numerical solution of direct methanol fuel cell (DMFC) problems* (Fuhrmann)
- C8: *Shape optimization and control of curved mechanical structures* (Sprekels)
- C9: *Optimal control of sublimation growth of SiC bulk single crystals* (Klein, Sprekels, Tröltzsch), see Figure 6
- C10: *Modelling, asymptotic analysis and numerical simulation of the dynamics of thin film nanostructures on crystal surfaces* (Münch, Wagner)
- C11: *Modeling and optimization of phase transitions in steel* (Tröltzsch, Hömberg)
- C18: *Analysis and numerics of multidimensional models for elastic phase transformations in shape-memory alloys* (Mielke)

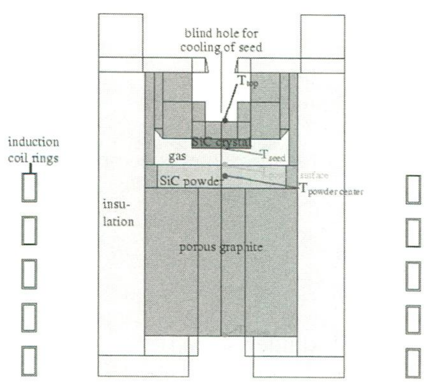



Figure 6: Optimal control of sublimation growth of SiC crystals

## Application Area D: Electronic circuits and optical technologies

(Scientists in Charge: Volker Mehrmann, Frank Schmidt, Caren Tischendorf)

By contributing to the development of electronic circuits and opto-electronic devices, MATHEON gets to the core of most current key technologies. The innovation cycles and the life cycles of products get shorter implying that new products have to be developed in ever shorter time periods. This requires increased design automation, new simulation, control, and optimization techniques, as well as new verification tools.

To cope with many of the difficulties coming up in this area (e. g., fast-growing density of components on a chip, parasitic and thermal effects, crosstalk, noise effects due to small signal-to-noise ratio) MATHEON follows a new paradigm: Modelling, analysis, simulation, optimization, and software development are carried out in an integrated way by joining the mathematical fields of MATHEON: applied and stochastic analysis, numerical analysis and scientific computing, optimization and discrete mathematics. This unique way of combining almost all areas of applied mathematics makes MATHEON an ideal partner for research institutions and industry.

Here are a few example projects of this application area:

- D1: *Model reduction for large-scale systems in control and circuit simulation* (Mehrmann)
- D4: *Quantum mechanical and macroscopic models for optoelectronic devices* (Hünlich, Rehberg)
- D7: *Numerical simulation of integrated circuits for future chip generations* (Tischendorf, März)
- D8: *Nonlinear dynamical effects in integrated optoelectronic structures* (Wolfrum, Recke)
- D9: *Design of nano-photonic devices* (Schmidt), see Figure 7

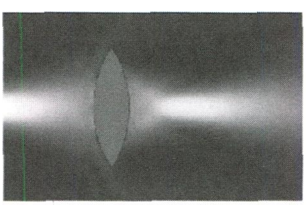


Figure 7: Gaussean beam

## Application Area E: Finance

(Scientists in Charge: Anton Bovier, Peter Imkeller, Alexander Schied)

The topic of this application area is the mathematical modelling of risk with a particular emphasis on modelling and understanding financial risk. A slightly longer title could, thus, be *Mathematics of financial risk*.

Mathematics has become a major driving force in finance and insurance. The central problem this industry faces is the management of *risk* in its various forms. Advanced probabilistic and statistical methods are being applied to qualify and quantify financial risk. Mathematics dominates all levels of approach: from the conceptual challenge of *modelling* and measuring risk in terms of stochastic processes, to the challenge of statistical-numerical *simulation* in the *optimization* of investment by minimizing risk.

MATHEON aims at moving towards a new generation of market models, in which a multitude of highly interdependent risk factors (e.g., volatility clusters, long-term memory effects, or external factors such as climate events) become integrated into an adequate model and MATHEON emphasizes the implementation of more sophisticated measures for the financial downside risk. Constructing hedging strategies that are risk-minimal in terms of these risk measures leads, for instance, to new mathematical optimization problems.

Examples of projects in this application area are:

- E1: *Microscopic modelling of complex financial assets* (Bovier)
- E2: *Hedging of external risk factors: Weather and climate* (Deuschel, Imkeller)
- E4: *Beyond value at risk: Quantifying and hedging the downside risk* (Schied, Föllmer)
- E5: *Statistical and numerical methods in modelling of financial derivatives and valuation of risk* (Schoenmakers, Spokoiny)

## Application Area F: Visualization

(Scientists in Charge: Konrad Polthier, John M. Sullivan, Günter M. Ziegler)

Visualization is a field that does not have a “natural home”. There are, for instance, engineering, biology, or computer science aspects. The emphasis on *Visualization* as a separate discipline and application area within a mathematical research center is a unique selling point of MATHEON. Visualization techniques open an informative (and attractive) window into mathematical research, scientific simulations, and industrial applications.

Application Area F concentrates on key problems in the fields of geometry processing, medical image processing, and virtual reality. Geometric algorithms are key to many industrial technologies, including computer-aided design, image and geometry processing, computer graphics, numerical simulations and animations involving large-scale data sets. For all these technologies, a deep understanding of the underlying abstract mathematical concepts is essential. Notable has been the effort in recent years by the CAD industry to rethink the foundations of their work and put it on a firmer mathematical footing. Today, progress in the development and use of new mathematical concepts is often what characterizes state-of-the-art applications. Mathematical knowledge then becomes a key resource within industry to stay competitive.

Application Area F operates the PORTAL, a new virtual reality installation at TU Berlin, to do interactive, immersive visualizations of various topics from geometry and other sciences. Needless to say, the PORTAL is in high demand for presentations.

Some of the projects in Application Area F are:

- F1: *Discrete differential geometry* (Bobenko, Pinkall, Polthier, Ziegler)
- F2: *Atlas-based 3d image segmentation* (Deuflhard, Hege)

- F4: *Geometric shape optimization* (Bobenko, Polthier), see Figure 8
- F5: *Mathematics in virtual reality* (Pinkall, Sullivan).

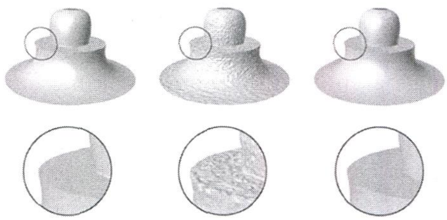


Figure 8: Geometric shape optimization

The MATHEON buddy bear standing in front of the TU mathematics building, see Figure 9, was designed by the MATHEON visualization group. The bear carries the three-dimensional version of the MATHEON logo on one of his paws. The body painting is the image of a conformal mapping of a cylinder surface around which the MATHEON logo has been wrapped several times. The picture in Figure 9 was taken at the inauguration of the MATHEON bear on June 11, 2005.

Figure 9: MATHEON bear

## Application Area G: Education

(Scientists in Charge: Ulrich Kortenkamp, Jürg Kramer)

The basic motivation for the scientists of MATHEON to engage themselves in educational activities is the need for more qualified young people in the MINT fields (Mathematik, Informatik, Naturwissenschaften, Technik), in particular in mathematics. To achieve this goal, the basis for a positive attitude towards mathematics has to be built already in school. Furthermore, the unbalanced transitions from school to university, and later on to the working life, have to be smoothened out by integrating these phases more strongly with each other, specifically in mathematical education. As a consequence, the mathematical education for teachers and engineers must become more practice- and problem solving-oriented. The scientists of Application Area G – with the support of many other members of MATHEON – provide various activities in this direction. Based on their application-oriented research and their teaching experience, they develop concepts for teaching in a more application- and problem-driven way. At the same time, through the close cooperation with schools, in particular the four mathematically profiled schools of the Berlin Network, prototypical examples for a smooth transition from school to university have been set up. This, in turn, leads to a fruitful cooperation between teachers, teacher students, and teacher educators at the universities.

Examples of projects in application Area G are:

- G2: *Current mathematics at schools* (Kramer)
- G3: *Teachers at universities* (Kramer)
- G5: *Discrete mathematics for high school education* (Grötschel)
- G6: *Visualization of algorithms* (Kortenkamp)

In addition, the scientists of MATHEON help – with the support of the teachers of the Network Schools – bridging the gap between mathematicians and the public at large via MATHEON's public relations program. A particularly successful activity is the Urania lecture series directed towards high school students and teachers. Three to four times a year, three mathematical lectures are presented in the Berlin Urania on one morning. The lecture topics are MATHEON mathematics and applications and, of course, the presentations are tuned towards high school students.

MATHEON has had (most likely) its highest public visibility through its „Mathematischer Adventskalender", see `http://www.mathekalender.de/` and Figure 10. More than 6000 persons participated in the Adventskalender 2004. Over 500 pupils attended the awards ceremony in January 2005. The fact that a report on MATHEON in the ZDF „heute journal" opened up with the Adventskalender certainly contributed to its success. Of course, Education is not really an application area. In the next funding period we will merge this area with our public relations activities and call it our *Outreach* program.

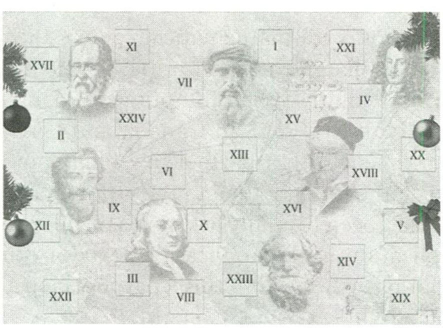



Figure 10: Mathematischer Adventskalender

## Links and cross fertilization

What I cannot indicate here, because of lack of space, are the many links that have been established in the meantime. There are links between different application areas; colleagues from different mathematical fields communicate mathematically much more than before the beginning of MATHEON; and – due to the close contact with industry – all mathematical fields involved find many more challenges in the real world than they dreamt of before we started the "MATHEON experiment". Mathematics, at our own universities, at related scientific institutions, and in industry has become much more visible. The added value of a large scale endeavor such as MATHEON shows up via this cross fertilization more than anywhere else.

## Industry cooperation and scientific networking

MATHEON is not just a mathematical research institution. It aims at cooperation with other sciences, engineering, management science and economics, and in particular, with partners in commerce and industry that are active in the key technologies the center is addressing. MATHEON is open for discussion in its range of expertise and invites proposals for cooperation.

MATHEON is not just waiting for partners. We are actively seeking industry cooperations. The contact management is very individual. There may be merely one general lecture in a company to high ranking representatives or potential partners, a series of presentation to members of the research department of a company, a meeting with "the boss", or direct contacts to persons in the "production line". Such efforts are made on all levels of MATHEON, but of course, the executive board and the leading figures whose names are known in the industrial world play a particular role here.

MATHEON by now has collaborations and joint projects with major firms such as DaimlerChrysler, Volkswagen, Siemens, NEC, Infineon, Deutsche Bank, Deutsche Telekom, Schering, BASF, Bayer, Lufthansa, etc., but there are also medium size and small companies with which MATHEON cooperates.

Former and some new spin-off companies of the participating institutions play a particular role in the distribution of mathematical software that is developed at MATHEON. Mentioning everything that is going on here is also beyond the scope of this article.

MATHEON also seeks international cooperation with mathematical centers that are active in a similar scientific domain. A cooperation agreement with MASCOS in Australia is already working in real practice. We are jointly trying to optimize a container terminal in Sydney. Cooperation agreements with MITACS in Canada and CMM in Chile have just been signed. A similar arrangement with a partner in China is in preparation. The MATHEON concept has influenced the foundation and the vision of several other centers around the world. We have, in fact, been formally asked in several cases to endorse the use of certain parts of our application in the application texts of others. (Some have not asked but just copied from our pages.) We view this a particular success. If you are copied you must be doing something right!

## MATHEON and the mathematical community

This article is not the place to "show off". But I would like to mention that MATHEON mathematicians are not only greedily cashing grant money. Many colleagues spend considerable time and energy for the general scientific community. At present, two MATHEON members are members of the Executive Board of DMV and two of the Executive

Board of GAMM, one colleague is president of the Mathematical Programming Society, one a member of the Executive Committee of the International Mathematical Union, and one will become DMV president in 2006. Four members of MATHEON have been elected by the German mathematical community to the DFG mathematics review board (Fachkolleg). Many of us serve in scientific advisory boards, engage in activities for developing countries, electronic information and communication, etc., on national and international levels.

Quite a number of scientific awards have been bestowed upon MATHEON mathematicians, the younger ones received a lot of recognitions for their PhD and Master theses. And of course, MATHEON did not only offer position to others. About twenty offers for professorships reached MATHEON members in the last three years. More than ten colleagues (in particular young scientists) accepted these offers. One project group consisting of four persons, for instance, was almost "wiped out". Three members (young scientists) followed the calls to other universities. We consider such events success and not disaster. MATHEON tends to let certain research and application topics move with the scientists leaving for other universities (so that they have a good chance to create their own research environment) and to take up new challenges and redirect the MATHEON work.

To our regret, not all offers MATHEON made for professor or head of junior research group positions materialized. In these cases the home institutions made counter offers that MATHEON somehow could not match. This typically resulted in a strengthening of the local research group, an upgrade in position or salary and, in particular, in a reassessment of the value certain persons or groups have for the local academic community. In this way MATHEON indirectly influenced the research focus and the structural development of a number of institutions outside of Berlin.

## Further information

MATHEON maintains an extensive Web presentation. Since we were not happy with the first version, we recently launched a new home page. Have a look, if you want to learn more about MATHEON and its activities, see `http://www.matheon.de/`. The new Web site is supposed to be completed by November 2005.

## The future of MATHEON, excellence clusters

MATHEON has been existing for more than three years. Its members are currently writing their reports on the achievements during this period of time. And we are also writing the reapplication for the second 4-year period. All projects went through a thorough internal screening period. Almost one third of the projects will not be continued and replaced by new ones. No project has a survival guarantee. Even for a running project, whenever a MATHEON financed member leaves, the project head has to reapply for this position.

The full report and the reapplication are due by November 15, 2005, while the onsite evaluation by a DFG nominated group of referees will be at the end of January 2006. We hope that we will be successful.

This summer the German government announced the creation of a new support program: *excellence clusters*. All university presidents in Germany are lining up their "armies", and at the time of writing this article, "tons" of colleagues are writing and rewriting preliminary applications for this new form of financial support. Will MATHEON become one of the new clusters of excellence?

Well, nobody knows in the long run. Not only the DFG will have difficulties to explain the difference between a DFG research center and a cluster of excellence. According to "trusted information sources", for the time being, the DFG research center program and the excellence cluster will run in parallel. The two programs have different financial sources and different time lines, although from their intentions there will not be much of a difference. In fact, whenever the DFG is asked how an excellence cluster should look like, the usual answer is "like MATHEON". We are happy to hear that, of course, though there is some irony here. The DFG research centers were conceived as centers at one university, possibly with support from research institutions in the vicinity. After our application won the competition, the DFG was not sure that the somewhat decentralized center in Berlin was what it wanted. It took the DFG a while before it granted the center to Berlin.

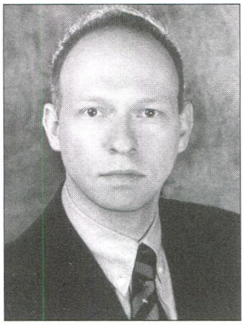
Reading the plans for the 7th Research Framework of the European Union, one can find that one of the main objectives will be the creation of *European centers of excellence*. They are supposed to be established through collaboration between laboratories with the aim to launch European technological initiatives, to stimulate the creativity of basic research, to make Europe more attractive to the best researchers, and to develop a research infrastructure of European interest. Nobody knows at present whether these plans will materialize. If they do, the mathematical community can't ignore such a development. To participate, strong mathematical partners in Germany are needed. In fact, we should start soon looking for German mathematical centers that may be able to play a role in this "game". The experience we had with creating MATHEON may help in this process.

## A personal final remark

Not every mathematical colleague, as I learned, was happy that the Berlin group won a DFG Research Center. Leaving aside a few statements indicating jealousy, the critique was that now "big science" is entering mathematics. Shouldn't we rather keep the successful small scale individual approach? And why do we need all this emphasis on applications? Let's, so the suggestion, just develop mathematics as before, whatever comes out will be used some day. I personally believe that these are dangerous misconceptions of the role of mathematics. It is my opinion that mathematics, when overemphasizing its independence as a scientific endeavor, looses much of its vitality. The real world, at any time of our history, has presented a lot of new challenges to mathematics. And the masters of our field took them up. Just remember that Gauss computed the trajectory of the planetoid Ceres (which made him famous worldwide and resulted in the invention of the least squares method). Gauss accepted to carry out the geodesic survey of the state of Hannover (which lead to the normal distribution and to differential geometry) and he made risk calculations and asset management for the *Witwen- und Waisenkasse* of Göttingen University. Fortunately, he did not know that there is one type of mathematics called pure and another called applied. One of our problems in mathematics (in our professional work and in our student education) is that pure and applied are defined not by work and attitude but by subject: numerics and stochastics are applied, number theory and differential geometry are pure, for example. But I do know quite a few number theorists and differential geometers who are much more into real applications than certain colleagues in stochastics. And there are numerical analysts who have not touched real data. For me, doing applied mathematics means interaction with real problems (from business, industry, society, or other sciences), the willingness to dig into details, to cope with unclear specifications and possibly fuzzy data, all this with the aim to help others solve their problems.

Problem solving of this type is, today, best done in larger scientific environments with a mix of people ranging from the "computer wiz" to the totally pure, but where there are overlaps of interest and respect for the work the other person does. It, moreover, needs the readiness to talk to users of mathematics, to learn their language and to try to formulate their questions mathematically. We cannot stay away from modelling issues, we cannot leave simulations to the engineers, we cannot reduce our work to the final step of mathematical analysis. MATHEON tries to create such a work and research environment and to teach its students and junior scientists this approach to mathematical research. MATHEON makes its offsprings fit for an industrial and an academic career. And that is not a contradiction. Remember, Gauss did the high tech mathematics of his time and got his fingers dirty with data. And MATHEON is trying to give this approach to doing mathematics a strong push. It has selected key technological topics of our present day as application areas where there is hope that the interaction with practitioners leads to particular progress in the application, but also in mathematics.

MATHEON is not the only institution doing this, of course. Let me just mention, in Germany, the Institut für Wissenschaftliches Rechnen (IWR) in Heidelberg, the Fraunhofer-Institut für Techno- und Wirtschaftsmathematik (ITWR) in Kaiserslautern, the Max-Planck-Institut für Mathematik in den Naturwissenschaften (MIS), Leipzig, or the Zentrum Mathematik at TU München where the philosophy of mathematical work is related to the MATHEON approach. And there are beginnings of such moves visible worldwide. If the mathematicians in Germany (and the world over) find the right balance between theory and application, experimental mathematics and rigorous approaches, there is every reason to be optimistic about the future of mathematics. The world and not only the scientific world needs good mathematics, and it needs mathematicians ready to get involved.



# Evolutionary Game Dynamics under Stochastic Influences

Lorens A. Imhof

## Abstract

- Keywords and Phrases: asymptotic stability, diffusion process, evolutionarily stable strategy, Nash equilibrium, stochastic replicator dynamics, war of attrition
- Mathematics Subject Classification: 60 H 10, 60 J 70, 91 A 22, 92 D 25

Evolutionary game theory links non-cooperative game theory and population dynamics. Recently, diffusion models have played an increasingly important role in this area, as they explicitly describe the far-reaching consequences of stochastic disturbances. The present paper reviews several current developments, compares deterministic and stochastic dynamics, and discusses the role of evolutionarily stable strategies.

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## 1 Introduction

The concept of a Nash equilibrium is fundamental in game theory and has been used in a wide range of applications. It is however connected with two major difficulties. Traditional game theory does not explain how a population playing a particular game evolves towards a Nash equilibrium. Moreover, if a game has several equilibria, it is not clear which one will be selected. To address the equilibrium selection problem various refinements of the definition of a Nash equilibrium have been suggested. The Nash equilibrium concept is typically based on the assumption that all players behave rationally, and many refinements are obtained by elaborate definitions of rational behaviour. However, strong rationality assumptions may often be hard to justify, particularly in biological applications. Starting with the seminal paper by Maynard Smith and Price [35], attention has shifted from increasingly complicated definitions of rationality to evolutionary game dynamics. Here explicit dynamical processes are studied to understand the evolution of the population, and stability results can be obtained without rationality assumptions. The recent surveys by Hofbauer and Sigmund [22] and Nowak and Sigmund [37] are devoted to deterministic dynamics describing populations driven by games. For an extensive treatment of stability results, see Cressman [10]. Fudenberg and Levine [19], Hofbauer and Sigmund [21] and Weibull [46] provide detailed discussions from economic and biological perspectives. Bomze [6] explains some results of the winners of the Nobel Memorial Prize in economics 1994, Nash, Harsanyi and Selten, which are closely related to the dynamical models considered in the present paper. Samuelson [40] gives a book-length treatment of the equilibrium selection problem. Cressman [11] extends the study of deterministic game dynamics to extensive form games.

A central role in evolutionary game theory is played by the deterministic replicator equations by Taylor and Jonker [45]. These equations describe in very general terms population growth under selection, and have found many diverse applications ranging from predator-prey models in ecology to population genetics and chemical networks. In most of these applications, there is an inherent stochastic component, which leads quite naturally to stochastic processes to model the evolution of populations under selection. It is only relatively recent, though, that explicitly stochastic replicator equations have been introduced. Foster and Young [15] add stochastic terms to the deterministic replicator equations and analyse the behaviour of the resulting stochastic differential equations when the size of the stochastic elements becomes small. Fudenberg and Harris [17] introduce the stochastic disturbances in a different way and derive stochastic differential equations with a more natural boundary behaviour. It turns out that the stochastic and the deterministic models may behave rather differently even if the stochastic effects are small. For example, suppose the underlying game is the prisoner's dilemma game with the pure strategies 'defect' and 'cooperate'. The strategy 'defect' is a strict Nash equilibrium and under the deterministic replicator dynamics the cooperators become extinct. Under the stochastic replicator model of Fudenberg and Harris, however, it is possible that the defectors become extinct, see [23], page 8, so that this model may explain cooperation. Nowak, Sasaki, Taylor and Fudenberg [36] investigate the emergence of cooperation in a discrete Markov chain model. Specifically, they introduce a frequency dependent Moran process to describe game dynamics in a finite population and derive conditions for selection to favour cooperation. Further related results on stochastic game dynamics in finite populations are given in [18], [25], [26] and [44].

The present paper gives a review of some recent results for stochastic replicator models for infinite populations. Section 2 presents the dynamics of Taylor and Jonker as a deterministic starting point and discusses three stochastic versions, including those of Foster and Young and of Fudenberg and Harris. In the next section deterministic and stochastic dynamics are compared for a three-strategy game, the rock-scissors-paper game. Section 4 deals with evolutionarily stable strategies, a concept of central importance to evolutionary game dynamics which paves the way for an analysis of games with an arbitrary finite set of pure strategies. By way of illustration, a game is studied that describes a discrete war of attrition. As a further motivation of the stochastic processes considered here, a family of discrete Markov chains that gives rise to the stochastic replicator dynamics via a diffusion approximation is presented in Section 5. Section 6 concludes with some open problems: possible extensions to games with infinitely many strategies and to more general types of game dynamics.

In view of the development of deterministic evolutionary game theory and the variety of areas to which it has been successfully applied, it seems probable that stochastic evolutionary game theory will continue to be a vibrant field of research. New impetus is likely to come from both mathematical biology and economics. The present paper is not meant to be an extensive survey of this theory. Rather its aim is to explain several aspects that are related to the author's habilitation thesis [23]. Occasionally, some ideas of proofs are sketched to give a flavour of typical arguments.

## 2 Stochastic Replicator Dynamics

Consider a symmetric two-player game with pure strategies $1, \dots, n$ and payoff matrix $A = (a_{jk})_{j,k=1}^{n}$. The entry $a_{jk}$ is the payoff to a player using strategy $j$ against an opponent playing strategy $k$. Consider a large population and suppose that each individual of the population is programmed to play one fixed pure strategy. For every point of time $t \geq 0$, let $\zeta_j(t)$ denote the size of the subpopulation of $j$-players. Then $\bar{\zeta}(t) = \sum_{j=1}^{n} \zeta_j(t)$ is the size of the whole population and $x_j(t) = \zeta_j(t)/\bar{\zeta}(t)$ is the population share programmed to play strategy $j$. If the population is in state $\mathbf{x}(t) = (x_1(t), \dots, x_n(t))^T$, then, under random pairwise matching, the average payoff to $j$-players is given by $\{A\mathbf{x}(t)\}_j$. The possibility that an individual plays against itself is not explicitly excluded here, but the probability that this happens is negligible in the present large population model. Suppose that the payoff is identified with the increase of fitness, measured as the number of offspring per unit of time. Suppose also that strategies breed true and that generations blend continuously into each other. This leads to the differential equation

$$(1) \quad \frac{d\zeta_j(t)}{dt} = \zeta_j(t)\{A\mathbf{x}(t)\}_j, \qquad j = 1, \dots, n,$$

and it follows that

$$\text{(2)} \quad \frac{dx_j(t)}{dt} = x_j(t)\left[\{A\mathbf{x}(t)\}_j - \mathbf{x}(t)^T A\mathbf{x}(t)\right], \qquad j = 1, \ldots, n.$$

This is the deterministic replicator equation of Taylor and Jonker [45]. Note that the growth rate of the subpopulation of players programmed to strategy $j$ is given by the difference between that strategy's payoff and the average payoff.

To discuss stochastic extensions of (2) we focus first on the case $n = 2$. Then $x_2(t) = 1 - x_1(t)$ for all $t \geq 0$, so that it is sufficient to consider the one-dimensional differential equation

$$\text{(3)} \quad \frac{dx_1(t)}{dt} = x_1(t)[1 - x_1(t)][\alpha + \beta x_1(t)],$$

where $\alpha = a_{12} - a_{22}$ and $\beta = a_{11} - a_{12} - a_{21} + a_{22}$. Three stochastic versions will be presented now. They are given by Ito diffusions with different diffusion coefficients and drift coefficients equal to or similar to the right-hand side of (3). In the seminal paper by Foster and Young [15], the stochastic disturbances are added directly to the deterministic equation describing the evolution of the relative sizes of the subpopulations. They study in particular a stochastic differential equation of the form

$$\text{(4)} \quad dX_1(t) = X_1(t)[1 - X_1(t)][\alpha + \beta X_1(t)]\, dt + \sigma dW(t),$$

where $W(t)$ is a standard Brownian motion and $\sigma$ is a positive constant that reflects the size of the stochastic shocks. Fudenberg and Harris [17] argue that it is preferable to add the stochastic effects to the equations for the absolute sizes of the subpopulations. They then derive an equation of the form

$$\begin{aligned}\text{(5)} \quad dX_1(t) = {} & X_1(t)[1 - X_1(t)]\left\{[\alpha + \beta X_1(t)] + \sigma^2[1 - 2X_1(t)]\right\}\, dt \\ & + \sqrt{2}\sigma X_1(t)[1 - X_1(t)]dW(t).\end{aligned}$$

Note that the ordinary differential equation obtained from (5) by deleting the diffusion term is different from (3). Beggs [1] considers, among other models, the equation

$$\text{(6)} \quad dX_1(t) = X_1(t)[1 - X_1(t)][\alpha + \beta X_1(t)]\, dt + \sigma\sqrt{X_1(t)[1 - X_1(t)]}dW(t).$$

Over a short finite time horizon, the solutions to (4), (5) and (6) behave similarly and with high probability, they do not deviate much from the solution to (3) when the stochastic terms are small. To make this precise, let $X_1(t) = X_1^{(\sigma)}(t)$ be given by (4) with deterministic initial condition $X_1(0) = \xi_1$, $\xi_1 \in (0, 1)$. Denote the probability measure corresponding to the initial value $\xi_1$ by $P_{\xi_1}$. Let $x_1(t)$ be the solution of (3) with $x_1(0) = \xi_1$. Then according to [16], Theorem 1.2, page 45, for every finite time horizon $T > 0$ and every $\epsilon > 0$, there exists $\sigma^* = \sigma^*(T, \epsilon) > 0$ such that

$$P_{\xi_1}\left\{\max_{0 \leq t \leq T}\left|X_1^{(\sigma)}(t) - x_1(t)\right| \leq \epsilon\right\} > 1 - \epsilon,$$

provided $\sigma < \sigma^*$. The same is true for the solutions to (5) and (6). However, over an infinite time horizon, the stochastic solutions may differ substantially. Moreover, it is not clear whether the solution of the deterministic replicator equation can be regarded as a reasonable approximation to any of the solutions of (4), (5) or (6).

There are essentially three possible types of long-run behaviour of the deterministic dynamics, apart from borderline cases.

(i) If $a_{11} > a_{21}$ and $a_{12} > a_{22}$, then strategy 1 is dominant, and $x_1(t) \to 1$ for every initial condition $x_1(0) = \xi_1 \in (0, 1)$. If $a_{11} < a_{21}$ and $a_{12} < a_{22}$, then strategy 2 is dominant, and $x_1(t) \to 0$ for every initial condition $x_1(0) = \xi_1 \in (0, 1)$.

(ii) If $a_{11} < a_{21}$ and $a_{12} > a_{22}$, then there is a unique stationary point $x^* \in (0, 1)$ and $x_1(t) \to x^*$ for every initial condition $x_1(0) = \xi_1 \in (0, 1)$. Thus there is stable coexistence.

(iii) If $a_{11} > a_{21}$ and $a_{12} < a_{22}$, then there is a unique stationary point $x^* \in (0, 1)$, which is unstable, and $x_1(t) \to 0$ if $x_1(0) < x^*$ and $x_1(t) \to 1$ if $x_1(0) > x^*$. This is the coordination case.

In contrast to (3), none of the stochastic differential equations (4), (5) and (6) can have an interior stationary point in $(0, 1)$. Convergence as in case (ii) is therefore not to be expected. In fact, if $X_1(t)$ is a solution to (4), (5) or (6), then

$$P_{\xi_1}\{X_1(t) \text{ converges to a point in } (0, 1)\} = 0.$$

One can even show that for any given compact subset $K \subset (0, 1)$,

$$P_{\xi_1}\{X_1(t_j) \notin K \text{ for a sequence of finite random times } t_j \text{ increasing to } \infty\} = 1.$$

This is true for any initial value $\xi_1 \in (0, 1)$ and any underlying payoff matrix $A$. The assertion continues to hold, with obvious modifications, for games with any finite number of pure strategies. This suggests that the long-run behaviour of the stochastic dynamics is closely related to its behaviour near the boundary points 0 and 1.

For one-dimensional diffusion processes an accurate description of the boundary behaviour can be obtained by means of the scale function and the speed measure. Consider the dynamics (6). The associated scale function and the Lebesgue density of the speed measure take the forms

$$\phi(y) = \int_{\frac{1}{2}}^{y} \exp\left\{\frac{-2}{\sigma^2}\int_{\frac{1}{2}}^{z} (\alpha + \beta\zeta)\, d\zeta\right\} dz, \qquad m(y) = \frac{2}{\phi'(y)\sigma^2 y(1-y)},$$

respectively. Let $\tau$ denote the first time that $X_1(t)$ hits the boundary, that is, $\tau = \inf\{t : X_1(t) \notin (0, 1)\}$. Both boundary points are absorbing, so that $X_1(t) = X_1(\tau)$ for all $t \geq \tau$. Set

$$\psi(y) = \int_{\frac{1}{2}}^{y} (\phi(y) - \phi(z))m(z)\, dz.$$

As $\psi(0+)$ and $\psi(1-)$ are finite, it follows by an extension of Feller's test for explosion, see e.g. [29], page 350, that $E_{\xi_1}(\tau) < \infty$, where $E_{\xi_1}$ denotes expectation with respect to $P_{\xi_1}$. In particular, $P_{\xi_1}\{\tau < \infty\} = 1$. Moreover, $\phi(0)$ and $\phi(1)$ are finite, and so

$$P_{\xi_1}\{X_1(\tau) = 0\} = 1 - P_{\xi_1}\{X_1(\tau) = 1\} = \frac{\phi(1) - \phi(\xi_1)}{\phi(1) - \phi(0)},$$

see [29], page 345. This shows that the difference between the deterministic solution $x_1(t)$ of (3) and the stochastic process $X_1(t)$ given by (6) is particularly pronounced in the coexistence case (ii). While $x_1(t)$ converges to an interior point, $X_1(t)$ converges to a boundary point. Similarly, in case (i), $x_1(t)$ converges always to the same boundary point, independent of the initial state, whereas $X_1(t)$ hits either boundary point with positive probability.

The boundary behaviour of the solution to (5) is quite different: the boundary points will not be reached in finite time. In case (i), the stochastic process shows the same convergence properties as $x_1(t)$. In the coexistence case (ii), there exists a unique stationary measure that concentrates near the interior equilibrium. The following theorem is a special case of a result proved in Fudenberg and Harris [17].

**Theorem 1.** *Let $X_1(t)$ be given by (5).*

*(i) If $a_{11} > a_{21}$ and $a_{12} > a_{22}$, then $P_{\xi_1}\{X_1(t) \to 1\} = 1$ for every $\xi_1 \in (0,1)$. If $a_{11} < a_{21}$ and $a_{12} < a_{22}$, then $P_{\xi_1}\{X_1(t) \to 0\}$ for every $\xi_1 \in (0,1)$.*

*(ii) If $a_{11} < a_{21}$ and $a_{12} > a_{22}$, then, for every $\xi_1 \in (0,1)$,*

$$P_{\xi_1}\left\{\liminf_{t\to\infty} X_1(t) = 0\right\} = P_{\xi_1}\left\{\limsup_{t\to\infty} X_1(t) = 1\right\} = 1$$

*and the distribution of $X_1(t)$ converges to the unique ergodic distribution.*

*(iii) If $a_{11} > a_{21}$ and $a_{12} < a_{22}$, then for every $\xi_1 \in (0,1)$,*

$$0 < P_{\xi_1}\{X_1(t) \to 0\} < 1, \qquad 0 < P_{\xi_1}\{X_1(t) \to 1\} < 1$$

*and*

$$P_{\xi_1}\{X_1(t) \to 0\} + P_{\xi_1}\{X_1(t) \to 1\} = 1.$$

Explicit formulas for the ergodic distribution in (ii) and the probabilities in (iii) are also given in [17]. Based on these formulas, a complete description of the behaviour of the system as the variance of the stochastic components tends to zero is obtained. See Finnoff [14] for some recent large deviation results tailored to the analysis of evolutionary games.

The stochastic process $X_1(t)$ given by (4) would leave the interval [0,1] in finite time, so that negative population shares would occur. To prevent this Foster and Young [15] modify the state space and introduce reflecting boundaries. Again, of particular interest is the behaviour as $\sigma \to 0$.

## 3 A rock-scissors-paper game

If the underlying game has more than two pure strategies, then the corresponding replicator dynamics become rather more complicated. If $n = 3$, there exist already 33 different generic phase portraits for the deterministic dynamics (2), and if $n \geq 4$, a complete classification of possible orbits appears to be beyond reach, see [22], Section 2.4 and references cited there. Moreover, in analysing the stochastic replicator dynamics, one cannot use the elegant approach based on scale functions when $n \geq 3$.

To compare the deterministic and the stochastic dynamics for a specific three-strategy scenario, consider the rock-scissors-paper game with payoff matrix

$$(7) \quad A = \begin{bmatrix} 0 & -a_1 & a_2 \\ a_2 & 0 & -a_1 \\ -a_1 & a_2 & 0 \end{bmatrix}, \qquad a_1, a_2 > 0.$$

Thus strategy 3 beats strategy 2, strategy 2 beats strategy 1, which in turn beats strategy 3. Let $\mathbf{p} = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})^T$ and $\Delta = \{\mathbf{x} \in [0,1]^3 : x_1 + x_2 + x_3 = 1\}$. Assume that $a_1 < a_2$. Then, in the deterministic model (2), $\mathbf{p}$ is globally stable, so that, for any initial condition $\mathbf{x}(0) = \xi \in \mathrm{int}(\Delta)$, $\mathbf{x}(t) \to \mathbf{p}$ as $t \to \infty$, see [21], page 80.

The stochastic replicator dynamics of Fudenberg and Harris now takes the form

$$dX(t) = b(X(t))\,dt + C(X(t))\,dW(t),$$

where $W(t)$ is a three-dimensional Brownian motion and

$$b(\mathbf{x}) = \left[\mathrm{diag}(x_1, x_2, x_3) - \mathbf{x}\mathbf{x}^T\right](A\mathbf{x} - \sigma^2 \mathbf{x}),$$
$$C(\mathbf{x}) = \sigma\left[\mathrm{diag}(x_1, x_2, x_3) - \mathbf{x}\mathbf{x}^T\right].$$

In this model, for any initial value $\xi \in \mathrm{int}(\Delta)$, $P_\xi\{\lim_{t\to\infty} X(t) = \mathbf{p}\} = 0$. In fact, one even has

$$P_\xi\left\{\liminf_{t\to\infty} X_j(t) = 0\right\} = 1, \qquad j = 1, 2, 3.$$

Thus the solution $X(t)$ is far from being path-wise globally stable in the sense indicated above, and the well-known concept of stochastic stability, see e.g. [41], Chapter III, is not appropriate here either.

To study the asymptotic behaviour of $X(t)$ consider the associated second-order partial differential operator given by

$$Lf(\mathbf{x}) = \sum_{j=1}^{3} b_j(\mathbf{x}) \frac{\partial f(\mathbf{x})}{\partial x_j} + \frac{1}{2} \sum_{j,k=1}^{3} \{C(\mathbf{x}) C^T(\mathbf{x})\}_{jk} \frac{\partial^2 f(\mathbf{x})}{\partial x_j \partial x_k}, \qquad f \in C^2.$$

Let $g(\mathbf{x}) = -(\log x_1 + \log x_2 + \log x_3)$. Note that $g(\mathbf{x}) > 0$ for all $\mathbf{x} \in \mathrm{int}(\Delta)$. A brief calculation shows that

$$Lg(\mathbf{x}) = \frac{3}{2}\left[(x_1 - p_1)^2 + (x_2 - p_2)^2 + (x_3 - p_3)^2\right](a_1 - a_2) + \frac{3\sigma^2}{2}\left(1 - x_1^2 - x_2^2 - x_3^2\right).$$

Hence if $a_1 < a_2$ and $\sigma$ is small enough, then there exist some $\epsilon > 0$ and some compact set $K \subset \mathrm{int}(\Delta)$ such that $Lg(\mathbf{x}) \le -\epsilon$ for all $\mathbf{x} \in \mathrm{int}(\Delta) \setminus K$. From this it follows that the process $X(t)$ is recurrent, see [2], [31]. Moreover, there exists a unique stationary probability measure $\mu$ on $\mathrm{int}(\Delta)$, and the transition probabilities converge to $\mu$ in total variation, that is, for any initial value $\xi \in \mathrm{int}(\Delta)$, $\lim_{t\to\infty} P_\xi\{X(t) \in B\} = \mu(B)$ uniformly in $B$, where $B$ ranges over all Borel subsets of $\Delta$, see [12], Chapter 7. In this sense the distribution of $X(t)$ stabilizes when the corresponding solution of the deterministic replicator equations is globally stable. Furthermore, by the ergodic theorem,

$$P_\xi\left\{\lim_{T\to\infty} \frac{1}{T}\int_0^T h(X(t))\,dt = \int h(\mathbf{x})\,d\mu(\mathbf{x})\right\} = 1,$$

provided $h$ is integrable with respect to $\mu$.

## 4 Evolutionarily stable strategies

This section is devoted to the long-run behaviour of stochastic dynamics for games with an arbitrary finite number of pure strategies.

Some terminology will be introduced first. Let $A$ be an $n \times n$ payoff matrix of a symmetric two-player game with pure strategies $1, \ldots, n$. A mixed strategy is a probability distribution on the set of pure strategies written as a vector $\mathbf{p} = (p_1, \ldots, p_n)^T$, where $p_i$ is the probability of using pure strategy $i$. Let $\Delta = \{\mathbf{p} \in [0,1]^n : p_1 + \ldots + p_n = 1\}$ denote the set of mixed strategies. The vertices of $\Delta$ are identified with the pure strategies. The (expected) payoff to a player using strategy $\mathbf{p} \in \Delta$ against an opponent using $\mathbf{q} \in \Delta$ is given by $\mathbf{p}^T A\mathbf{q}$. A Nash equilibrium is a mixed strategy which is a best reply to itself, and a strict Nash equilibrium is a strategy which is the unique best reply to itself. That is, $\mathbf{p} \in \Delta$ is a Nash equilibrium if $\mathbf{p}^T A\mathbf{p} \geq \mathbf{q}^T A\mathbf{p}$ for all $\mathbf{q} \in \Delta$, and $\mathbf{p} \in \Delta$ is a strict Nash equilibrium if the inequality is strict for all $\mathbf{q} \neq \mathbf{p}$.

According to the folk theorem of evolutionary game theory, strict Nash equilibria are asymptotically stable rest points of the deterministic replicator dynamics (2). It is shown in [23] that, under some mild conditions, strict Nash equilibria are also asymptotically stochastically stable for the replicator dynamics of Fudenberg and Harris [17], which in the $n$-strategy case is given by

$$(8) \quad dX(t) = b(X(t))\,dt + C(X(t))\,dW(t).$$

Here $W(t)$ is an $n$-dimensional Brownian motion and

$$b(\mathbf{x}) = \left[\operatorname{diag}(x_1, \ldots, x_n) - \mathbf{x}\mathbf{x}^T\right]\left[A - \operatorname{diag}(\sigma_1^2, \ldots, \sigma_n^2)\right]\mathbf{x},$$
$$C(\mathbf{x}) = \left[\operatorname{diag}(x_1, \ldots, x_n) - \mathbf{x}\mathbf{x}^T\right]\operatorname{diag}(\sigma_1, \ldots, \sigma_n).$$

The state space is $\Delta$. The requirements for a strict Nash equilibrium are so restrictive that only pure strategies can be strict equilibria. The concept of a Nash equilibrium on the other hand is not strong enough to imply convergence or stability, as is suggested by the fact that for every underlying payoff matrix there exists a Nash equilibrium, while in general the composition of the population may vary strongly with time. An intermediate concept, which is of fundamental importance in evolutionary game theory, is that of an evolutionarily stable strategy (ESS), introduced by Maynard Smith and Price [35].

A strategy $\mathbf{p} \in \Delta$ is said to be an evolutionarily stable strategy if the following two conditions hold.

(i) $\mathbf{p}^T A\mathbf{p} \geq \mathbf{q}^T A\mathbf{p}$ for all $\mathbf{q} \in \Delta$

and

(ii) if $\mathbf{q} \neq \mathbf{p}$ and $\mathbf{p}^T A\mathbf{p} = \mathbf{q}^T A\mathbf{p}$, then $\mathbf{p}^T A\mathbf{q} > \mathbf{q}^T A\mathbf{q}$.

In other words, if $\mathbf{p}$ is an ESS, then $\mathbf{p}$ is a best reply to itself, and if there is another best reply $\mathbf{q}$, then $\mathbf{p}$ is a strictly better reply to $\mathbf{q}$ than $\mathbf{q}$ is to itself. For instance, in the coexistence case (ii) described in Section 2, that is, in the case where $n = 2$, $a_{11} < a_{21}$ and $a_{12} > a_{22}$, the strategy $(x^*, 1 - x^*)^T$ is an ESS, where $x^* = (a_{12} - a_{22})/\ (a_{12} - a_{22} + a_{21} - a_{11})$ is the interior stationary point of (3). In the coordination case (iii), that is, if $a_{11} > a_{21}$ and $a_{12} < a_{22}$, the strategy $(x^*, 1 - x^*)^T$ is still a Nash equilibrium but not an ESS, whereas both pure strategies are strict Nash equilibria. In case (i), the dominant strategy is also a strict Nash equilibrium. In the rock-scissors-paper game (7) with $a_1 < a_2$, the strategy $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})^T$ is an ESS. If $a_1 \geq a_2$, the strategy is a Nash equilibrium but not an ESS.

Under the deterministic replicator dynamics (2) every ESS is an asymptotically stable state, and if $\mathbf{p}$ is an ESS that lies in $\mathrm{int}(\Delta)$, then for any given initial condition $\mathbf{x}(0) = \xi \in \mathrm{int}(\Delta)$, the solution $\mathbf{x}(t)$ of (2) converges to $\mathbf{p}$, see, e.g., [46], Section 3.5.2. This is not true in the stochastic setting. In view of Theorem 4.3 in [23] it is not possible that the solution $X(t)$ of the stochastic replicator dynamics (8) converges with positive probability to any point $\mathbf{p} \in \Delta$ unless $\mathbf{p}$ is a vertex of $\Delta$, that is, unless $\mathbf{p}$ is one of the $n$ pure strategies. In fact, almost every trajectory comes infinitely often arbitrarily close to one of the vertices. This difference between the deterministic and the stochastic dynamics has a strong influence on the analysis of ESSs in the stochastic set-up. To illustrate this point, the following characterization of evolutionary stability is useful. A strategy $\mathbf{p} \in \Delta$ is an ESS if and only if for every $\mathbf{q} \in \Delta$, $\mathbf{q} \neq \mathbf{p}$, there is some $\epsilon^* = \epsilon^*(\mathbf{q}) \in (0,1)$ such that

$$(9) \quad \mathbf{p}^T A[\epsilon\mathbf{q} + (1-\epsilon)\mathbf{p}] > \mathbf{q}^T A[\epsilon\mathbf{q} + (1-\epsilon)\mathbf{p}] \qquad \text{for all } 0 < \epsilon < \epsilon^*,$$

see [46], pages 36–37. Intuitively, if a population plays an ESS, $\mathbf{p}$, and is invaded by a small group of players using one and the same deviating strategy, $\mathbf{q}$, the payoff to the deviating players is strictly smaller than that of the ESS-players. Hence the invaders should become extinct. In this sense, an evolutionarily stable population is expected to be proof against invasion. Inequality (9) shows that the ESS condition yields a local stability concept. Considering that the solutions $X(t)$ of (8) do not remain in a small neighbourhood of an ESS, the relevance of the local ESS concept to the long-run behaviour of $X(t)$ it not at all clear in the first place.

It turns out that a useful link between the static stability concept and the dynamic stochastic process is given by the Kullback-Leibler distance, which is defined as $\kappa(\mathbf{x}, \mathbf{y}) = \sum_{j: y_j > 0} y_j \log(y_j / x_j)$, $\mathbf{x} \in \mathrm{int}(\Delta)$, $\mathbf{y} \in \Delta$. Although $\kappa$ is not a metric, $\kappa$ can be regarded as a function that measures how much $\mathbf{x}$ differs from $\mathbf{y}$, see [9], Chapter 2. For example, according to the information inequality, $\kappa(\mathbf{x}, \mathbf{y}) \geq 0$ with equality if and only if $\mathbf{x} = \mathbf{y}$. The Kullback-Leibler distance, also known as cross entropy, plays a decisive role in the analysis of the ordinary differential equation for the deterministic replicator dynamics, see e.g. [5]. In the present stochastic context, $\kappa$ serves to define (a variant of) a stochastic Lyapunov function. Based on this approach, it can be shown that, under certain conditions, the expected Euclidean distance between $X(t)$ and an ESS $\mathbf{p}$ is small most of the time. To be more specific, consider the expected time average

$$(10) \quad E_\xi \frac{1}{t} \int_0^t \|X(s) - \mathbf{p}\|^2 \, ds,$$

where $\|\cdot\|$ denotes the Euclidean norm. Theorem 2 below gives an upper bound of this quantity in terms of $\sigma_1, \ldots, \sigma_n$ and the Kullback-Leibler distance between the initial point $\xi$ and $\mathbf{p}$. Roughly, when $t$ is large, the bound is of the order of magnitude of $\sigma_1^2, \ldots, \sigma_n^2$. It is shown in Theorem 5.3 in [23] that the expected value in (10) is in fact bounded away from zero, provided that $\mathbf{p}$ is not a pure strategy. If the ESS does not lie in the interior of $\Delta$, then it will be assumed in Theorem 2 that the payoff matrix meets a certain definiteness condition. The fact that some additional assumption is made is not surprising and substantiates the remark of Foster and Young [15] that the concept of an ESS, on its own, "does not capture the notion of long-run stability when the system is subjected to stochastic effects." The definiteness condition is automatically satisfied if the ESS is an interior point of $\Delta$.

Let $\mathbf{1}$ denote the $n \times 1$ vector all of whose entries are equal to 1. The $n \times n$ matrix $A$ is said to be conditionally negative definite if

$$\mathbf{y}^T A \mathbf{y} < 0 \quad \text{for all } \mathbf{y} \in \mathbb{R}^n \text{ such that } \mathbf{1}^T \mathbf{y} = 0,\ \mathbf{y} \neq \mathbf{0}.$$

For $t \geq 0$, $\xi \in \Delta$ and every Borel subset $G \subset \Delta$ let $P(t, \xi, G) = P_\xi\{X(t) \in G\}$.

**Theorem 2.** *Let $X(t)$ be given by the stochastic replicator dynamics (8) with initial condition $X(0) = \xi \in \operatorname{int}(\Delta)$ and underlying payoff matrix $A$. Let $\mathbf{p} \in \Delta$ be an ESS for $A$, and if $\mathbf{p} \notin \operatorname{int}(\Delta)$, suppose in addition that $A$ is conditionally negative definite. Then, for every $t > 0$,*

$$E_\xi \frac{1}{t} \int_0^t \|X(s) - \mathbf{p}\|^2 \, ds \leq \frac{1}{|\lambda_2|} \left[ \frac{\kappa(\xi, \mathbf{p})}{t} + \frac{1}{2} \max\left\{\sigma_1^2, \ldots, \sigma_n^2\right\} \right],$$

*where $\lambda_2$ is the second largest eigenvalue of*

$$\overline{A} - \frac{1}{n} \overline{A} \mathbf{1}\mathbf{1}^T - \frac{1}{n} \mathbf{1}\mathbf{1}^T \overline{A} + \frac{\mathbf{1}^T \overline{A} \mathbf{1}}{n^2} \mathbf{1}\mathbf{1}^{\mathrm{T}}$$

and $\overline{A} = \frac{1}{2}[A + A^T]$.

*If $\mathbf{p} \in \operatorname{int}(\Delta)$ and $\frac{1}{2}\max\{\sigma_1^2, \ldots, \sigma_n^2\} < |\lambda_2| \min\{p_1^2, \ldots, p_n^2\}$, then $X(t)$ is recurrent, there is a unique stationary distribution on $\operatorname{int}(\Delta)$ and as $t \to \infty$, the transition probability $P(t, \xi, \cdot)$ converges to this distribution in total variation.*

For a proof of Theorem 2 and more detailed results on the stationary distribution, see [24]. In particular, it is shown there that the stationary distribution puts most mass close to the ESS. These results tie in with the general discussion of stability in randomly fluctuating environments given in Chapter 5 of [33]. Note that $|\lambda_2|$ can be regarded as a measure of how strongly the ESS attracts $X(t)$. The assumptions of Theorem 2 ensure that $\lambda_2 \neq 0$.

In the rock-scissors-paper game (7) with $a_1 < a_2$, the strategy $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})^T$ is an ESS in $\operatorname{int}(\Delta)$, so that Theorem 2 yields the recurrence and convergence results described in Section 3. Theorem 2 also yields the assertion of Theorem 1 in the coexistence case, at least when $\sigma_1$ and $\sigma_2$ are small enough. Note that in Theorem 1, $\sigma_1 = \sigma_2 = \sigma$, but $\sigma$ was not required to be small.

Returning to the general case, suppose that $\mathbf{p}$ is an ESS for $A$. Then, for some constant $c \in \mathbb{R}$, $\{A\mathbf{p}\}_j = c$ for every $j \in \{1, \ldots, n\}$ with $p_j > 0$. Consequently,

$$\left[\operatorname{diag}(p_1, \ldots, p_n) - \mathbf{p}\mathbf{p}^T\right] A\mathbf{p} = \mathbf{0}.$$

This shows that the drift vector $\mathbf{b}(\mathbf{y})$ of the stochastic differential equation (8) will in general not be zero at $\mathbf{y} = \mathbf{p}$. However, if $\mathbf{q}$ is an ESS for the modified payoff matrix $B = A - \operatorname{diag}(\sigma_1^2, \ldots, \sigma_n^2)$, then $\mathbf{b}(\mathbf{q}) = \mathbf{0}$. Therefore, it is also natural to investigate the distance between $X(t)$ and $\mathbf{q}$. This leads to results similar to, but slightly better than, Theorem 2, see [24].

The deterministic replicator equation in $n$ variables is equivalent to the deterministic Lotka-Volterra equation in $n-1$ variables, see [21], Section 7.5. The long-run behaviour of solutions to Lotka-Volterra systems with random disturbances has recently been studied by Khasminskii and Klebaner [32] and Skorokhod, Hoppensteadt and Salehi [42]. Cabrales [7] shows that under the stochastic replicator dynamics iteratively strictly dominated strategies are eliminated in the long-run, provided the stochastic disturbances are sufficiently small.

We now turn to an example where the ESS can be calculated explicitly: a game that describes a discrete variant of the symmetric war of attrition, introduced by Maynard Smith and Price [35]. One application of games of this type is to model animal conflicts that are settled by display rather than violence. For further discussion, see [34]. In a war of attrition, a pure strategy corresponds to the maximum time span for which the player is prepared to display. The contest terminates as soon as one of the players has reached his chosen limit. The player who has been willing to persist longer collects a reward, and both players incur a cost determined by the length of the contest. The value of the reward is either constant or a decreasing function of the length of the contest. If both players have chosen the same strategy, they share the reward equally. Specifically, if $c_j$ denotes the cost of display and $v_j$ denotes the value of the reward, then the payoff matrix $A = (a_{jk}), j, k = 0, \ldots, n$, of the discrete war of attrition is of the form

$$a_{jk} = \begin{cases} v_k - c_k, & j > k, \\ \dfrac{v_k}{2} - c_k, & j = k, \\ -c_j, & j < k, \end{cases}$$

where

$$c_0 < c_1 < \ldots < c_n \qquad \text{and} \qquad v_0 \geq v_1 \geq \ldots \geq v_n.$$

Various general results for discrete and continuous wars of attrition have been established by Bishop and Cannings [3].

The following theorem gives an explicit expression for the ESS in the case where the value of the reward is constant and $c_j = j$ for all strategies $j$. The ESS is given in terms of linear combinations of Chebyshev polynomials of the second kind evaluated along the imaginary axis. Let $U_m(x)$ denote the $m$th Chebyshev polynomial of the second kind,

$$U_m(x) = \frac{\sin(m+1)\theta}{\sin\theta}, \qquad \cos\theta = x.$$

Let $U_{-1}(x) \equiv 0$.

**Theorem 3.** *Consider the discrete war of attrition with payoff matrix $A = (a_{jk})$, $j, k = 0, \ldots, n$, where*

$$a_{jk} = \begin{cases} v - k, & j > k, \\ \dfrac{v}{2} - k, & j = k, \\ -j, & j < k, \end{cases}$$

*and $v > 0$. The unique ESS $\mathbf{p} = (p_0, \ldots, p_n)^T$ is given as follows. If the reward $v$ satisfies $v \geq 2n$, then*

$$p_0 = \ldots = p_{n-1} = 0, \quad p_n = 1.$$

*Otherwise there is a unique index* $s \in \{0, \dots, n-1\}$ *such that*

$$n - 1 \leq \frac{v}{2} + s < n,$$

*and*

$$\begin{aligned} p_k &= \frac{1}{c}\Big(-\frac{v}{2}\Big)^k \Big\{ u_{s-k+1} + \Big(s + 1 - n + \frac{v}{2}\Big) u_{s-k} \\ &\qquad\qquad + (s+1-n)\frac{v}{2} u_{s-k-1} \Big\}, && 0 \leq k \leq s, \\ p_k &= 0, && s+1 \leq k \leq n-1, \\ p_n &= \frac{1}{c}\Big(-\frac{v}{2}\Big)^{s+1}, \end{aligned}$$

*where*

$$u_k = \Big(-\frac{iv}{2}\Big)^k U_k\Big(-\frac{i}{v}\Big)$$

*and*

$$c = u_{s+1} - (n - s - 1)u_s.$$

Theorem 3 is a special case of Theorem 5.2 in [24]. To obtain Theorem 3, set $\rho = 0$ in Theorem 5.2 and observe that, for every $m \geq -1$, $U_{m+2}(x) + U_m(x) = 2xU_{m+1}(x)$, so that $u_{m+2} - (v^2/4)u_m = -u_{m+1}$.

Combining Theorems 2 and 3 one obtains a rather precise picture of the long-run behaviour of the stochastic replicator dynamics in the case where the conflicts between the individuals are modelled by a discrete war of attrition. According to Lemma 5.1 in [24], the payoff matrix is conditionally negative definite, which ensures that Theorem 2 is indeed applicable.

A related resource allocation problem for a multiple-trial conflict is solved in [23], Section 7. Here the players are involved in several successive wars of attrition and have to allocate a fixed amount of resource to the individual trials. The ESS is again given explicitly in terms of Chebyshev polynomials. It is shown, in particular, that if the reward to the winner is sufficiently large, then all pure strategies in the ESS will indeed make use of all resources available. This substantiates a conjecture of Whittaker [47].

## 5 A discrete stochastic replicator model

Not only are there several ways to introduce stochastic effects into the replicator equations that lead to diffusions, there are also various quite natural approaches to justify particular choices such as (5) or (6). Beggs [1] considers a learning process and arrives at a class of stochastic differential equations that includes (6). Corradi and Sarin [8] study certain discrete matching and mimicking processes and take suitable limits as the population size grows to obtain both ordinary and stochastic differential equations similar to (5) and (6). The frequency dependent Wright-Fisher process examined by Imhof and

Nowak [26] converges, if suitably scaled, to the solution of (6). The generalized Moran process investigated by Fudenberg, Imhof, Nowak and Taylor [18] can be approximated by the solution of a stochastic differential equation similar to (6).

In this section we consider a family of discrete-time replicator dynamics with randomly perturbed payoffs. As the size of the time steps shrinks to zero, the stochastic processes converge to a certain diffusion process, which in turn leads directly to the stochastic replicator equations (8). A similar construction has recently been given in [27] in support of a stochastic chemostat model. For further related examples, see [30], Chapter 15.

Let $A = (a_{jk})_{j,k=1}^{n}$ be the payoff matrix of a symmetric two-player game with $n$ pure strategies. Consider a large population of agents, each playing one pure strategy. For every time step size $\delta > 0$ construct a discrete-time Markov chain $\{Z^{(\delta)}(\nu\delta) : \nu = 0, 1, 2, \ldots\}$ as follows. Here $Z^{(\delta)}(\nu\delta) = (Z_1^{(\delta)}(\nu\delta), \ldots, Z_n^{(\delta)}(\nu\delta))^T$, and $Z_j^{(\delta)}(\nu\delta)$ represents the size of the subpopulation of $j$-players at time $\nu\delta$. For $\mathbf{z} = (z_1, \ldots, z_n)^T \in \mathbb{R}^n$ with $z_1, \ldots, z_n > 0$ set $\pi_j(\mathbf{z}) := \{A\mathbf{z}\}_j/(z_1 + \ldots + z_n)$, $j = 1, \ldots, n$. Thus $\pi_j(\mathbf{z})$ is the expected payoff to a player that uses strategy $j$ against a population that is in state $\mathbf{z}$. Assume that during the time period from $\nu\delta$ to $(\nu+1)\delta$, the growth rate of the $j$th subpopulation is $\pi_j[Z^{(\delta)}(\nu\delta)] + \epsilon_j^{(\delta)}(\nu)$, where $\epsilon_j^{(\delta)}(\nu)$ is a random variable with $E\epsilon_j^{(\delta)}(\nu) = 0$ and $0 < E[\epsilon_j^{(\delta)}(\nu)]^2 < \infty$. This random variable describes the effect of stochastic disturbances on $j$-players in the time interval $[\nu\delta, (\nu+1)\delta]$, aggregate shocks in the words of Fudenberg and Harris [17]. All the variables $\epsilon_j^{(\delta)}(\nu)$ are assumed to be independent, and for every $j = 1, \ldots, n$, the variables $\epsilon_j^{(\delta)}(0), \epsilon_j^{(\delta)}(1), \ldots$ are assumed to be identically distributed. The Markov chain is thus determined by an initial condition $Z^{(\delta)}(0) = \zeta$ and the relation

$$Z_j^{(\delta)}((\nu+1)\delta) = Z_j^{(\delta)}(\nu\delta) + Z_j^{(\delta)}(\nu\delta)\delta\Big\{\pi_j[Z^{(\delta)}(\nu\delta)] + \epsilon_j^{(\delta)}(\nu)\Big\}, \qquad \begin{array}{l} j = 1, \ldots, n, \\ \nu = 0, 1, \ldots \end{array}$$

To examine the limit behavior of the chains $\{Z^{(\delta)}(\nu\delta) : \nu = 0, 1, \ldots\}$ as $\delta \to 0$, assume that, for some constants $\sigma_1^2, \ldots, \sigma_n^2 > 0$, $\lim_{\delta\to 0} \delta E[\epsilon_j^{(\delta)}(\nu)]^2 = \sigma_j^2$ for all $j$ and $\nu$. Note that this condition means that for any time interval $[t_1, t_2]$ the accrued variance of the corresponding random terms satisfies

$$\mathrm{Var}\left\{\sum_{\nu:\mathrm{t}_1\le\nu\delta\le\mathrm{t}_2} \delta\epsilon_{\mathrm{j}}^{(\delta)}(\nu)\right\} = |\{\nu : \mathrm{t}_1 \le \nu\delta \le \mathrm{t}_2\}|\delta^2\mathrm{E}[\epsilon_{\mathrm{j}}^{(\delta)}(0)]^2 \to \sigma_{\mathrm{j}}^2(\mathrm{t}_2 - \mathrm{t}_1), \qquad \delta \to 0.$$

It follows that

$$\lim_{\delta\to 0}\frac{1}{\delta}E\left[Z_j^{(\delta)}((\nu+1)\delta) - Z_j^{(\delta)}(\nu\delta)\middle|Z^{(\delta)}(\nu\delta) = \mathbf{z}\right] = z_j\,\pi_j(\mathbf{z})$$

and

$$\lim_{\delta\to 0}\frac{1}{\delta}E\left[\Big\{Z_j^{(\delta)}((\nu+1)\delta) - Z_j^{(\delta)}(\nu\delta)\Big\}\Big\{Z_k^{(\delta)}((\nu+1)\delta) - Z_k^{(\delta)}(\nu\delta)\Big\}\middle|Z^{(\delta)}(\nu\delta) = \mathbf{z}\right]$$

$$= \lim_{\delta\to 0}\delta z_j z_k\Big\{\pi_j(\mathbf{z})\pi_k(\mathbf{z}) + E[\epsilon_j^{(\delta)}(\nu)\epsilon_k^{(\delta)}(\nu)]\Big\} = \begin{cases} \sigma_j^2 z_j^2, & j = k, \\ 0, & j \ne k. \end{cases}$$

The convergence is uniform on compact sets. Extend the Markov chains to continuous-time processes by setting $Z^{(\delta)}(t) = Z^{(\delta)}(\delta\nu)$ if $\nu\delta \le t < (\nu+1)\delta$. Under some mild higher moment conditions it now follows that, as $\delta \to 0$, $Z^{(\delta)}(t)$ converges weakly to the solution $Z(t) = (Z_1(t), \ldots, Z_n(t))^T$ of the stochastic differential equation

$$\begin{aligned} dZ_j(t) &= Z_j(t)\pi_j(Z(t))dt + \sigma_j Z_j(t)dW_j(t), \qquad j = 1, \ldots, n, \\ Z(0) &= \zeta, \end{aligned}$$

where $W_1(t), \ldots, W_n(t)$ are independent standard Brownian motions, see [43], Chapter 11. Define $X_j(t)$ to be the proportion of individuals playing strategy $j$, thus $X_j(t) = Z_j(t)/(Z_1(t) + \ldots + Z_n(t))$. Then an application of Ito's formula shows that $X(t) = (X_1(t), \ldots, X_n(t))^T$ satisfies (8).

## 6 Some open problems

Although many results and applications of the stochastic replicator dynamics are available by now, the literature on the deterministic replicator dynamics is far more extensive. A substantial part of the problems that have been studied in the deterministic case lend themselves for investigations which take random effects explicitly into account, leading to a wide field for future research. In closing this paper it seems appropriate to outline some open problems that, in the authors view, would be particularly challenging.

### Infinite strategy space

Most papers in evolutionary game theory have concentrated on finite strategy sets. In the standard replicator dynamics, every member of the population is programmed to play one of finitely many pure strategies. The infinite set of mixed strategies is used only to represent possible states of the population. It is not difficult to extend the model so that the agents are allowed to play one of finitely many mixed strategies, but if players are allowed to use any mixed strategy, new concepts are required. Moreover, in many games of interest, for example in games of timing, an infinite set of pure strategies seems the most natural setting. Extending the replicator equations to cover such games gives rise to differential equations on an infinite dimensional space. In the deterministic case, several results have been established by (among others) Hines [20], Bomze [4] and recently by Oechssler and Riedel [38], [39]. It turns out that the condition of an evolutionarily stable strategy is no longer sufficient for dynamic stability and new static stability concepts have to be considered. This led to the notion of evolutionary robustness, introduced by Oechssler and Riedel. It would thus be of interest to extend the stochastic replicator dynamics (8) to games with an infinite strategy space, to study the role of evolutionary robustness and to find further suitable static stability concepts.

## Payoff monotonic dynamics

The derivation of the replicator dynamics is based on the assumption that the growth rate of a subpopulation playing a particular strategy is equal to or at least proportional to the difference between the current payoff to that strategy and the average payoff. While this assumption seems justifiable in biological contexts when reproduction is asexual, the assumption may be too restrictive in applications in economics or social sciences. Here it may still be reasonable to regard different types of behaviour as pure strategies in a symmetric two-player game and to relate the success of a particular behaviour to its current payoff. However, it is not the number of offspring that drives the evolution of the population. Rather, the players are people who choose strategies and may change their choice with more successful strategies being more likely to be chosen. A fairly general class of game dynamics tailored to this type of selection mechanisms is the class of payoff monotonic dynamics, see e.g. [21], Chapter 8, or [46], Chapter 4. Equation (2) is replaced by more general differential equations of the form

$$\frac{dx_i(t)}{dt} = g_i(\mathbf{x})x_i, \qquad i = 1, \ldots, n,$$

where $\sum_{i=1}^{n} g_i(\mathbf{x})x_i \equiv 0$, and for all $\mathbf{x}$ and all $j$, $k$, $\{A\mathbf{x}\}_j > \{A\mathbf{x}\}_k$ if and only if $g_j(\mathbf{x}) > g_k(\mathbf{x})$. Kandori, Mailath and Rob [28] use discrete-time Markov chains with finite state space to model payoff monotonic dynamics with stochastic effects. A diffusion approximation developed along the lines of Section 5 may be tractable enough to give further insight into this type of models, particularly when the underlying game has more than two strategies, cf. Chapters 4 and 5 in [13].

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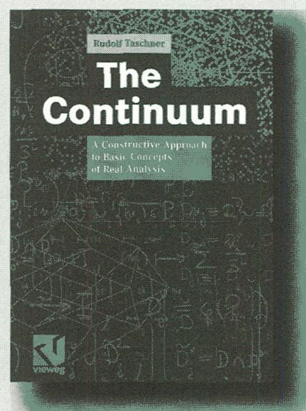
# Das Continuum - der Inbegriff der Reellen Zahlen

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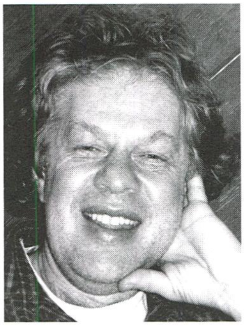

# Directions in Combinatorial Geometry

**János Pach***

By indirections find directions out.
(*William Shakespeare: Hamlet*)

## Abstract


- Mathematics Subject Classification: 52C10, 52C35, 05B10, 05B25, 11Pxx
- Keywords and Phrases: incidence, direction, additive number theory, pseudoline, allowable sequence


This mini-survey concentrates on some recent developments in combinatorial geometry related to the distribution of directions determined by a finite point set. It is based on the material of my invited address at the Jahrestagung der Deutschen Mathematiker Vereinigung in Rostock on September 19, 2003.

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## 1 Introduction, apology, directions, incidences

I would not dare to hazard any judgment or prediction concerning the most important *directions* of research in combinatorial geometry. During the past couple of decades the subject has gone through a growth spurt that is far from being over. It it very difficult to identify the most important trends. Many of the changes have been stimulated by the "geometrization" of other parts of mathematics and by the theoretical and practical demands of computer science and industry (including computer graphics, robotics, computer-aided design).

I will concentrate on a few open problems in discrete geometry related to the concept of "direction", used as a technical term. The *direction* determined by a pair of points $p_1 = (x_1, y_1)$ and $p_2 = (x_2, y_2)$ in the (affine or Euclidean) plane is the ratio $\frac{y_2-y_1}{x_2-x_1}$, that is, the slope of the line $p_1p_2$. Two pairs determine the same direction if the corresponding ratios coincide.

We get another possible interpretation of this concept, by completing the plane with a "line at infinity," $\ell_\infty$, and saying that two point pairs determine the same direction if their connecting lines intersect $\ell_\infty$ at the same point. In this latter context, it is apparent that the directions determined by a point set depend only on the structure of *incidences* between points and the lines. Problems of this type have been extensively investigated ever since Euclid proposed his system of axioms based entirely on these notions. Although the parallel postulate was scrutinized for well over two thousand years, and by the end of the nineteenth century projective geometry had become one of the most developed mathematical disciplines, a number of exciting simple questions concerning incidences were completely overlooked. One such question that Euclid would have certainly liked was asked by Sylvester [46] in 1893: is it true that any finite set $P$ of points, not all in a line, determines at least one *ordinary line*, that is, a line passing through precisely two elements of $P$? Forty years later the question was rediscovered by Erdős and shortly thereafter answered by Gallai [25].

**Sylvester-Gallai theorem.** *Every noncollinear set of n points in the plane determines an ordinary line.*

In fact, the minimum number of ordinary lines determined by such a point set is known to be at least $\lceil \frac{6}{13}n \rceil$, for $n > 7$, but the conjectured minimum is $\lceil \frac{1}{2}n \rceil$, if $n$ is sufficiently large [15], [13], [8].

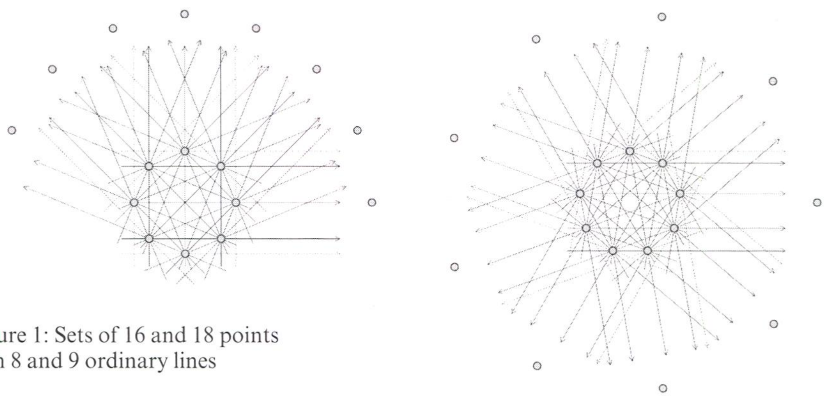

Figure 1: Sets of 16 and 18 points
with 8 and 9 ordinary lines

Erdős pointed out the following immediate corollary of the Sylvester-Gallai theorem.

**Corollary 1.1.** *Any set of n noncollinear points in the plane determines at least n distinct connecting lines. Equality is attained if and only if all but one of the points are collinear.*

We can argue by induction. The corollary is trivially true for $n = 3$. Suppose that we have already verified it for $(n-1)$-element sets, where $n > 3$. Consider a noncollinear set $P$ of $n$ points. Let $pq$ be an ordinary line, $p, q \in P$. At least one of the sets $P \setminus \{p\}$ or $P \setminus \{q\}$ is not collinear. Applying the induction hypothesis to this set, we conclude that it determines at least $n-1$ connecting lines, and all of them are different from $pq$. The cases of equality can be obtained by a similar argument.

The question naturally arises: can the corollary be strengthened to guarantee the existence of $n$ connecting lines with distinct slopes? The answer is yes if $n$ is even, as was conjectured by Scott [44] in 1970 and proved by Ungar [49] twelve years later.

**Ungar theorem.** *The minimum number of different directions assumed by the connecting lines of $n \geq 4$ noncollinear points in the plane is $2\lfloor n/2 \rfloor$.*

In contrast to Corollary 1.1, here there is an overwhelming diversity of extremal configurations, for which equality is attained. Four infinite families and more than one hundred sporadic configurations were catalogued by Jamison and Hill [35] (see also [34] for an excellent survey).

The main difficulty in studying the distribution of directions determined by a finite point set is that, although the problem is invariant under affine transformations of the plane, it seems likely that one has to analyze the *algebraic* relations between the slopes of the connecting lines. This would "smuggle" some *metric* elements into our investigations – and perhaps Euclid would be not so enthusiastic about such a development. We mention some algebraic aspects of these problems in Section 4 of this paper.

Ungar's brilliant proof uses the method of *allowable sequences*, invented by Goodman and Pollack [26], [27], for coding the angular information by a sequence of permutations. This enables him to translate the problem into a combinatorial one, and solve it in an elegant and much more general setting, for "pseudolines." This approach, suggested independently by Goodman and Pollack and by Cordovil [12], is outlined at the beginning of Section 2. In the rest of the section, we discuss a number of generalizations of Ungar's theorem, including a recent three-dimensional version, found by Pinchasi, Sharir, and myself [39], [40]. Section 3 contains some related results and open problems on repeated angles.

Over the years Erdős [19] raised a number of innocent looking questions on incidences between points and lines (or other curves) that turned out to be notoriously difficult. One of the first significant accomplishments in this respect was the proof of the following result conjectured by Erdős.

**Szemerédi-Trotter theorem [47].** *The maximum number of incidences of $n$ points and $l$ lines in the plane is $O(n^{2/3}l^{2/3} + n + l)$. The order of magnitude of this bound cannot be improved.*

The Szemerédi-Trotter theorem is one of the very few asymptotically tight results in this field. One may wonder why such a "natural" question on incidences did not occur to anyone, say, in the nineteenth century? I believe that the explanation is simple: no matter how natural these problems may sound today, they must have appeared quite "exotic" to "mainstream" mathematicians a hundred years ago, before combinatorial optimization became a separate subject.

In the past two decades research in this field has gained considerable momentum. The Szemerédi-Trotter theorem has found several applications in additive number theory [16], [17], [45], in Fourier analysis [32], [33], and in measure theory [2], [5], [51]. It is also related to Kakeya's problem [50]: A *Kakeya set* (or *Besicovitch set*) is a subset of $\mathbf{R}^d$ that contains a unit segment in every direction. Besicovitch was the first to construct such sets with zero measure. Kakeya's problem is to decide whether the Hausdorff dimension of a Kakeya set is always at least $d$. The planar version of this question was answered in the affirmative by Davies [14] and, in a stronger form, by Córdoba [12] and by Bourgain [6]. For $d \geq 3$, this is a major unsolved problem.

## 2 Allowable sequences, Ungar-type theorems

Fix a noncollinear set $P$ of $n$ points in the plane such that no two points have the same $x$-coordinate. Label the elements of $P$ by $1, 2, \ldots, n$ in the order of increasing $x$-coordinates. Following Goodman and Pollack [26], [27], we define a circular sequence of permutations. We take a horizontal line $\ell$ and start turning it around one of its points in the counterclockwise direction. In each position, we record the order of the orthogonal projections of the elements of $P$ into $\ell$. The original order is represented by the permutation $\pi = 12\ldots n$. As we turn $\ell$, changes occur in this permutation if and only if $\ell$ passes through a position perpendicular to one of the slopes determined by two (or more) points of $P$. In such a case, we obtain a new permutation $\pi'$ that can be obtained from $\pi$ by "flipping" some of its substrings: namely those corresponding to subsets of elements lying on parallel lines orthogonal to $\ell$. Thus, as we turn $\ell$ through 180 degrees, the number of changes in the permutation will be equal to the number of different slopes determined by point pairs in $P$. Finally, we end up with the permutation $n, n-1, n-2, \ldots, 1$. If we continue turning $\ell$, we obtain the same sequence of permutations as before, except that now each of them is reversed. After a full turn, we get back $\pi = 12\ldots n$.

Ungar's idea was the following. Suppose $n$ is even, and mark the middle of each permutation by an imaginary barrier separating the first $\frac{n}{2}$ elements from the last $\frac{n}{2}$. To estimate the number of permutations in the sequence, Ungar first classified the "moves" transforming one permutation into the next one. If a move involves flipping a string containing (resp. touching) the barrier, he called it a *crossing* (resp. *touching*) move. If a move is neither crossing nor touching, it is called *ordinary*. The basic observation is that between any two crossing moves there must be a touching one. Indeed, in a crossing move the order of the two elements on opposite sides of the barrier will change, and if the next nonordinary move is again a crossing move, then the order of these two elements would change back. However, as we turn $\ell$ through 180 degrees, the order of any two points can (and must) reverse only once. Another elegant argument allows us to give a lower bound on the number of ordinary moves between a touching move and a crossing move, leading to a proof of Ungar's theorem. In fact, the proof applies to a more general situation. Suppose that we have a sequence of permutations starting with $1, 2, \ldots, n$ and ending with $n, n-1, \ldots, 1$, with the property that the order of any two elements changes precisely once. In each move, we are allowed to flip a collection of nonoverlapping proper subsegments of the permutation. A sequence of permutations satisfying this condition is called an *allowable sequence*. It follows that the length of any allowable sequence on $n$ elements is at least $2\lfloor\frac{n}{2}\rfloor$.

Scott [44] also conjectured that in three-dimensional space the minimum number of different directions assumed by the connecting lines of $n$ points, not all in a plane, is $2n - O(1)$. For instance, if $n$ is odd, consider the set obtained from the vertex set of a regular $(n-3)$-gon $P_{n-3}$ (or from any other centrally symmetric extremal configuration for Ungar's theorem) by adding its center $c$ and two other points whose midpoint is $c$ and whose connecting line is orthogonal to the plane of $P_{n-3}$.

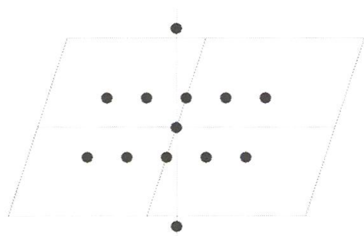

Figure 2: $n$ noncoplanar points in 3-space with $2n-5$ directions

At first glance it appears that Ungar's approach is doomed to fail in higher dimensions, because it is based on the linear (or rather the circular) ordering of all critical directions. This may well be the case in higher dimensions. However, somewhat surprizingly, Scott's three-dimensional conjecture can be settled by reducing it to a planar statement, which is a far-reaching generalization of Ungar's theorem.

**Theorem 2.1 [40].** *Any noncoplanar set of $n \geq 6$ points in $\mathbf{R}^3$ determines at least $2n-5$ different directions if $n$ is odd and at least $2n-7$ different directions if $n$ is even. This bound is sharp for every odd $n$.*

Ungar's theorem can be rephrased as follows: from all closed segments whose endpoints belong to a noncollinear set of $n$ points in the plane, one can always select at least $2\lfloor n/2 \rfloor$ such that no two of them are parallel. To formulate our generalization of Ungar's result, we need to relax the condition of two segments being *parallel*.

Two *closed* segments in the plane (or in $\mathbf{R}^d$) are called *convergent* if

(1) they do not belong to the same line, and

(2) their supporting lines intersect, and their intersection point does not belong to either of the segments.

An alternative definition is that two segments are convergent if and only if they are disjoint and their convex hull is a nondegenerate planar quadrilateral. (Two parallel segments that lie on distinct lines are also considered convergent, by regarding their lines to meet at infinity.)

**Theorem 2.2 [39].** *From all closed segments determined by a set of $n$ noncollinear points in the plane, one can always select at least $2\lfloor n/2 \rfloor$ pairwise nonconvergent ones, lying in distinct lines.*

It is easier to handle the $d$-dimensional problems ($d \geq 4$) under the assumption that no three points of the set are collinear. For this case, Blokhuis and Seress [4] conjectured that any set of $n$ points determines at least $(d-1)n - d(d-2)$ distinct directions. For $d = 4$, this conjecture was verified in [40] up to an additive constant. Perhaps asymptotically the same bound holds under the weaker assumption that not all of the points lie in the same hyperplane.

Ungar's theorem states that every noncollinear point set in the plane determines many directions. Dirac [15] asked whether one can always find a *point* belonging to at least roughly $\frac{n}{2}$ connecting lines of distinct slopes.

**Dirac's conjecture.** *There is a constant $c$ such that any set $P$ of $n$ points, not all on a line, has an element incident to at least $\frac{n}{2} - c$ lines spanned by $P$.*

Putting the same number of points on two lines shows that this bound, if true, is asymptotically tight. Many small examples listed by Grünbaum [28] show that the conjecture is false with $c = 0$. An infinite family of counterexamples was constructed by Felsner (personal communication). The "weak Dirac conjecture," first proved by Beck [3], states that there exists $\epsilon > 0$ such that one can always find a point incident to at least $\epsilon n$ lines spanned by $P$. This statement also follows from the Szemerédi-Trotter theorem (see Section 1).

According to a beautiful result of Motzkin [37], Rabin, and Chakerian [10], any set of $n$ noncollinear points in the plane, colored with two colors, *red* and *green*, determines a monochromatic line. Motzkin and Grünbaum [29] initiated the investigation of *biased* colorings, i.e., colorings without monochromatic red lines. Their motivation was to justify the intuitive feeling that if there are many red points in such a coloring and not all of them are collinear, then the number of green points must also be rather large. Denoting the sets of red and green points by $R$ and $G$, respectively, it is a challenging unsolved question to decide whether the "surplus" $|R| - |G|$ of the coloring can be arbitrarily large. We do not know any example where this quantity exceeds six [30]. It is another important ingredient of the proof of Theorem 2.1 that under some special restrictions the surplus is indeed bounded.

The problem of biased colorings was rediscovered by Erdős and Purdy [22], who formulated it as follows. What is the smallest number $m(n)$ of points necessary to represent (i.e., stab) all lines spanned by $n$ noncollinear points in the plane, if the generating points cannot be used? An $\Omega(n)$ lower bound follows immediately from the weak Dirac conjecture.

## 3 Repeated angles

In an important paper [18] published in the *American Mathematical Monthly*, Erdős asked the following twin questions. Consider a set $P$ of $n$ points in the plane (or in a higher-dimensional space).

(1) At most how many point pairs $\{p,q\} \subset P$ can determine the same distance?

(2) At least how many distinct distances must be determined by the point pairs in $P$?

In the same spirit, one can raise a number of interesting questions for *triples* of points. This line of research was initiated by Erdős and Purdy [20], [21].

(1') At most how many triples $(p,q,r) \subset P$ can determine the same angle?

(2') At least how many distinct angles must be determined by triples of points in $P$?

Concerning question (1'), Pach and Sharir [41] proved the following result.

**Theorem 3.1.** *For any $\gamma \in (0,\pi)$, there are at most $O(n^2 \log n)$ triples among n points in the plane that determine angle $\gamma$. Moreover, this order of magnitude is attained for a dense set of angles.*

We do not know whether this order of magnitude can indeed be reached for every $\gamma$. In three-dimensional space, Apfelbaum and Sharir [1] showed that among $n$ points the same angle can occur at most $O(n^{\frac{7}{3}})$ TG and that for right angles this bound can be attained. In this case, it is not even clear whether there exists any other angle for which the bound is asymptotically tight.

Purdy [42] noticed that in four-dimensional space the right angle can occur $\Theta(n^3)$ TG, since the points $p_x = (\cos x, \sin x, 0, 0)$, $q_y = (1,0,y,0)$, and $r_z = (-1,0,0,z)$ always determine a right angle at $p_x$. For all other angles, there is an upper bound of $O(n^{\frac{5}{2}}\beta(n))$, where $\beta(n)$ is an extremely slowly growing function related to the inverse Ackermann function [1]. However, the best known lower bound for angles different from $\frac{\pi}{2}$ is the same as in the plane: $\Omega(n^2)$ and $\Omega(n^2 \log n)$ for some special values.

In spaces of dimensions six and higher, any given angle can be represented by $\Theta(n^3)$ triples taken from an $n$-element set. According to a well known construction of Lenz (see e.g. [8], [38]), the number of mutually congruent triangles with an angle $\gamma$ can be $\Omega(n^3)$. The analogous statement in five-dimensional space is not known to be true.

**Problem 3.2.** *Can every angle $0 < \gamma < \pi$ different from $\frac{\pi}{2}$ occur $\Omega(n^3)$ TG among n points in five-dimensional space?*

Almost nothing is known about problem (2').

**Corrádi-Erdős-Hajnal conjecture [23].** *Given n points in the plane, not all on a line, they always determine at least $n-2$ distinct angles in $[0,\pi)$.*

The number of distinct angles determined by a regular $n$-gon is precisely $n-2$, but there are several other configurations for which the conjectured lower bound is tight. It easily follows from the "weak Dirac conjecture" (mentioned in the previous section) that there is a constant $c>0$ such that any noncollinear set of $n$ points in the plane determines at least $cn$ distinct angles.

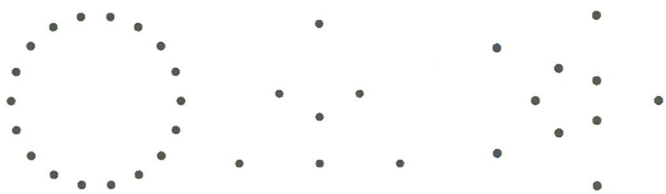

Figure 3: Sets of $n$ points with $n-2$ distinct angles

## 4 Finite planes, algebraic aspects

So far most of the results concerning directions, slopes, angles, and incidences have been established using combinatorial arguments. But the use of certain algebraic tools may turn out to be inevitable.

The following classical result, which is a far-reaching generalization of Corollary 1.1, can be obtained by an elegant application of the so-called "linear algebra method." This was the starting point of many investigations in the theory of block designs and finite projective planes.

**De Bruijn-Erdős theorem [9].** *Let $\mathcal{L}=\{L_1,L_2,\ldots\}$ be a family of proper subsets of an n-element set with the property that each pair $\{p,q\}\subset P$ belongs to precisely one member of $\mathcal{L}$. Then we have $|\mathcal{L}|\geq n$ with equality if and only if (1) one of the sets contains all but one elements of P and the others are two-element sets containing the remaining element; or (2) $\mathcal{L}$ is the system of lines of a finite projective plane defined on P.*

Rédei [43] used lacunary polynomials to prove an analogue of Ungar's theorem for finite affine planes.

**Rédei-Megyesi theorem.** *Let p be an odd prime. Then any noncollinear set of p points in the affine plane $AG(2,p)$ determines at least $\frac{p+3}{2}$ different directions.*

Rédei's analysis was completed by Lovász and Schrijver [36], who characterized the extremal configurations. These considerations turned out to be intimately related to the structure of blocking sets in finite projective planes. (A *blocking set* is a set of points intersecting every line.) See [48], for a survey.

As we have mentioned in Section 1, the Szemerédi-Trotter theorem has some exciting number-theoretic consequences.

**Erdős-Szemerédi theorem [24].** *There exists $\varepsilon > 0$ such that for any set $A$ of $n$ reals either the set of sums $A + A = \{a + b \mid a, b \in A\}$ or the set of products $A \cdot A = \{ab \mid a, b \in A\}$ has at least $\Omega(n^{1+\varepsilon})$ elements.*

The best known value of $\varepsilon$ (roughly $\frac{3}{11}$) was established by Solymosi [45], but it is conjectured that the theorem remains true for every $\varepsilon < 1$.

The following elegant argument due to Elekes [16] proves that the result holds with $\varepsilon = \frac{1}{4}$. Apply the Szemerédi-Trotter theorem to the set of points $P = (A + A) \times (A \cdot A) \subseteq \mathbf{R}^2$ and to the set $\mathcal{L}$ of $n^2$ lines of the form $y = a(x - b)$, where $a, b \in A$. Observe that the line $y = a(x - b)$ passes through at least $n$ elements of $P$, namely, all points of the form $(c + b, ac)$ for $c \in A$. Therefore, the number of incidences between the elements of $P$ and $\mathcal{L}$ is at least $n^3$. On the other hand, this quantity is at most $O(|P|^{2/3}|\mathcal{L}|^{2/3} + |P| + |\mathcal{L}|) = O(|P|^{2/3}n^{4/3} + |P| + n^2)$. Comparing these two bounds, we obtain $|P| = |A + A| \times |A \cdot A| = \Omega(n^{5/2})$, as required.

According to the above results, any finite subset $A$ of the field of real numbers is very far from being closed either under addition or under multiplication. The same question can be asked for other fields $F$. If $F$ has a subfield $A$, then we cannot expect such a result. However, for finite fields $F$ of prime order, we have the following.

**Bourgain-Katz-Tao theorem [7].** *Let $F$ be a finite field of prime order. For any $\delta > 0$, there exists $\varepsilon = \varepsilon(\delta) > 0$ such that, whenever $|F|^{\delta} < |A| < |F|^{1-\delta}$, we have*

$$\max\{|A + A|, |A \cdot A|\} = \Omega(|A|^{1+\varepsilon}).$$

The proof is based on the following Szemerédi-Trotter-type result. Let $F^2 = F \times F$ be a finite field plane, where $F = \mathbf{Z}/p\mathbf{Z}$ and $p$ is a prime. For any $0 < \alpha < 2$, there exists $\varepsilon = \varepsilon(\alpha) > 0$ such that the number of incidences between $n \le p^{\alpha}$ points and $l \le p^{\alpha}$ lines in $F^2$ is at most $O(n^{\frac{3}{2}-\varepsilon})$.

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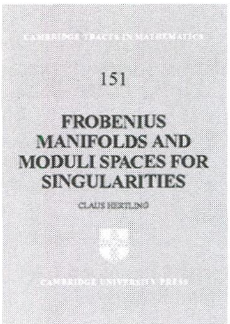


C. Hertling
**Frobenius Manifolds and Moduli Spaces for Singularities**
Cambridge Tracts in Math. 151

Cambridge University Press, 2002, 280 S., £ 50,–

The rich differential geometric structure that arises on the parameter space of topological field theories was axiomatized by B. Dubrovin, [D] into the notion of a *Frobenius manifold*. Its main ingredients are a *flat metric* and a *multiplication of tangent vectors*, making each tangent space into a Frobenius-algebra. The structure 3-tensor of the multiplication should be given (at least locally) by the partial third derivatives of a *potential* $\Phi$. The standard reference is the book by Manin, [M]. Among the most important examples of Frobenius manifolds are those that arise from what is now called the genus zero part of Gromov-Witten theory on a symplectic manifold $X$. The total cohomology space $H^*(X)$ carries the structure of Frobenius manifold, where the flat metric is given by Poincare duality, the multiplication of tangent vectors is quantum multiplication of cohomology classes, whose 3-tensor encodes the counting rational curves mapping three points to three prescribed cycles in $X$. The creation of Gromov-Witten theory was one of the major developments in symplectic and algebraic geometry in the last decade and led to a revival of interest in enumerative geometry. A particular lucid account can be found in the lecture notes by Givental, [G1].

Another class of Frobenius manifolds arises from the theory of critical points of functions, or singularity theory. The rich structure on the base space of the miniversal deformation of a singularity first described in the eighties by K. Saito is nowadays interpreted as giving the historically first examples of Frobenius manifolds. The book under review by Claus Hertling is the first book that is exclusively devoted to this topic, and is therefore complementary to the book of Manin. The book of C. Sabbah, [S] describes the Frobenius manifold structure in the global situation of a (tame) polynomial map an a smooth affine variety.

In the first part of the book Hertling introduces the notion of an F-manifold which is a more general structure introduced in [HM], where the metric plays no role. An F-manifold is a triple $(M, \circ, e)$, where $\circ$ is an $\mathcal{O}_M$-linear commutative and associative multiplication on the tangent sheaf $\Theta_M$, $e$ is a global unit vector field, such that the following integrability condition

$$\mathrm{Lie}_{X\circ Y}(\circ) = X \circ \mathrm{Lie}_Y(\circ) + Y \circ \mathrm{Lie}_X(\circ)$$

holds for all pairs of vector fields $X$ and $Y$. Such F-manifolds arise as follows in singularity theory. Let $f : X \longrightarrow \mathbb{C}$ a germ of an isolated hypersurface singularity, and let $F : X \times M \longrightarrow \mathbb{C} \times M$ a miniversal deformation, with parameter space $M$. It has dimension $\mu$, the Milnor number of $f$. Let

$$C = \{(x, \lambda) \in X \times M \ |\partial F/\partial x = 0\} \subset X \times M$$

the critical space of the map $F$. The miniversality is equivalent to the statement that the Kodaira-Spencer map

$$ks : \Theta_M \longrightarrow \rho_* \mathcal{O}_C$$

is an isomorphism. Here $\rho$ is the composition of $F$ and the projection to $M$, that presents the critical space $C$ as a finite $\mu$-fold cover of $M$. As $\rho_*\mathcal{O}_C$ has an obvious ring structure, one immediately gets a multiplication of tangent vectors on $M$. This can be described in terms of lagrangian geometry as follows. Consider the map $\phi : C \longrightarrow T^*M$ which maps $(x, \lambda)$ to the covector $\partial/\partial\lambda \mapsto \partial F(x, \lambda)/\partial\lambda$. By Schwarz theorem on the second derivatives, it follows that the image $L := \phi(C)$ is a Lagrange submanifold. Let $\pi : T^*M \longrightarrow M$ be the bundle projection. Its restriction exhibits $L$ as $\mu$-fold cover of $M$. As $\rho_{|C} = \pi \circ \phi$, it follows that $\rho_* \mathcal{O}_C = \pi_* \mathcal{O}_L$. Composing with the Kodaira-Spencer map then gives an isomorphism

$$a : \Theta_M \longrightarrow \pi_* \mathcal{O}_L$$

Givental [G2] abstracted from this situation, and proposed to forget about the function $f$ we started with and study general $L \subset T^*M$, now possibly singular, for which $a$ is an isomorphism. One gets a multiplication of tangent vectors, one can define a caustic, and can start to develop all further notions from singularity theory. It was noticed by Hertling and Manin, [HM], that the lagrangian condition on $L$ translates into the above mentioned identity defining an F-manifold. The author gives full proofs here and the main result is the proof of a conjecture of Dubrovin characterizing the Frobenius manifolds associated to reflection groups, thereby extending earlier results of Givental.

The second part of the book gives a description of the full structure of a Frobenius manifold on the miniversal base of an isolated hypersurface singularity, in particular the construction of a flat metric. Its existence in general was conjectured by K. Saito, who described it in the case of the simple singularities in terms of reflection groups. His idea was as follows: the residue pairing is intrinsic when formulated in terms of (top-dimensional) differential forms

$$\Omega_F \times \Omega_F \longrightarrow \mathcal{O}_M$$

The choice of such a (parameter dependent) differential form gives an identification

$$\Omega_F \approx \mathcal{O}_C$$

and via the earlier identifications a pairing on $\Theta_M$, that is, we obtain an holomorphic metric. It was conjectured by K. Saito that one can make a special choice (primitive form) such that, among other things, the resulting metric is flat. The claim of K. Saito was proven by M. Saito and used in an essential way the mixed Hodge structure in the vanishing cohomology as constructed by Steenbrink-Scherk and Varchenko. The book of Hertling gives a reformulation of M. Saito's results in terms of the connections on the space $M \times \mathbb{P}^1$. The upshot is that such flat structures on $M$ are basically parameterized by opposite filtration to the Hodge filtration. The text is at places difficult to read, as material here is highly technical in itself, and the text is not completely self-contained. For many basic and more advanced aspects of the construction of the mixed Hodge structure on the vanishing cohomology the reader is referred to the literature. (One alternative proof given in the text of a known result is wrong (p. 169)). The text is basically Hertling's Habilitationsschrift and is certainly not intended to be an introduction or a text book on the subject.

The main new results can be found in chapter 13 and 14. In chapter 13 the existence of moduli spaces for singularities is proven. Here the Frobenius structure is used in an essential way and the result seems to be out of reach of conventional methods. In chapter 14 the a surprising conjecture on the variation of the spectral numbers of a singularity is obtained. The conjecture is verified in special cases. This conjecture and further conjectural properties of the spectrum of an isolated singularity belong the most intriguing open problems in the theory of isolated hypersurface singularities.

Summarizing one can say this book is a *must* for workers in the field of singularity theory. It will be of interest for the broader audience of people working on Gromov-Witten theory, mirror symmetry or Frobenius manifolds who want have a deeper understanding of the singularity theory aspects.

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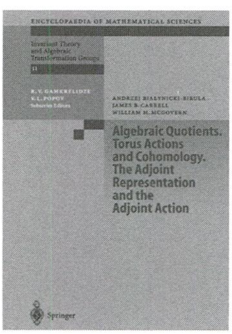



A. Białynicki-Birula, J. Carrell, W.M. McGovern
**Algebraic Quotients. Torus Actions. The Adjoint Representation and the Adjoint Action**
Encycl. of Math. Sciences 131

Berlin u. a., Springer, 2002, 247 S., € 103,74

This volume of the Encyclopaedia of Mathematical Sciences contains three contributions on actions of algebraic groups. The first one, by A. Białynicki-Birula, is concerned with the general concept of a quotient, while the other two, by J. B. Carrell and W. M. McGovern, are on more specific topics.

The ultimate foundation for the theory of quotients in algebraic geometry is 19th century invariant theory. The problem considered by Clebsch, Gordan, Aronhold and others was a very specific one (invariants of $n$-ary forms, for which the existence of a finite basis was proved by Gordan for $n = 2$ and by Hilbert in general). The analytic theory for the classical groups was developed by Hermann Weyl in the 1920s, but the working out of Hilbert's ideas in algebro-geometric terms had to await the introduction of the concept of stability by Mumford. This allowed Mumford to develop his geometric invariant theory (GIT), which solves the existence problem for quotients by actions of reductive groups and allows many applications to the construction of moduli spaces (that is, spaces classifying objects in algebraic geometry). Since then the major developments have been further applications of geometric invariant theory, links with symplectic geometry through the use of moment maps and construction methods for quotients by actions for which Mumford's GIT does not give good results.

Although Białynicki-Birula discusses all of these developments, his article is mainly concerned with the construction of quotients. The initial problem is that one cannot work within the theory of schemes. Mumford's theory produces quasi-projective quotient schemes (or even varieties) with natural compactifications. In general, this procedure doesn't work and one needs to enlarge the category in which one looks for quotients. There are at least two possibilities for this, algebraic spaces and stacks, both of which were designed to solve the problem of constructing quotients. If we use stacks the quotients exist by definition, but the geometric properties are not very clear. Algebraic spaces provide a somewhat more geometrical approach to the problem. The second major problem is to determine what is meant by a quotient; in the reductive case the various possible definitions are compatible, but this does not extend to the non-reductive case.

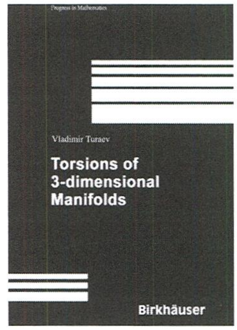
The author begins by widening the problem to consider not just group actions but groupoids and pre-equivalence relations. He then considers various notions of quotient, ranging from categorical to geometric, before settling on Seshadri's concept of a "good" quotient as the most useful. He then describes some standard examples, before discussing the affine case as a preliminary to a chapter on Mumford's GIT. This is followed by a discussion of good quotients, ending with a paragraph on the variation of stability, a topic which has become of great importance in recent years. Chapter 8 covers proper actions, and stacks are introduced in Chapter 9. The following chapter includes results for the complex analytic case (including moment maps). This is followed by a lengthy chapter on quotients by torus actions and shorter chapters on Hilbert-Mumford type theorems, Chow and Hilbert quotients, categorical quotients, sections, slices and reductions, local and global properties of quotients and applications to the theory of moduli, before the author concludes with some final remarks. As will be seen from this list, the scope of the article is immense. Moreover, as the author says, the theory is not a unified theory, but rather a conglomeration of various examples and theorems. This is essentially a very useful summary and an important point of reference, but is not intended to be a definitive account; the time for this has not yet come. A detailed study of the article is also a demanding exercise, since the reader requires a good understanding of many aspects of algebraic geometry in order to appreciate what is happening.

Carrell's article covers a much narrower problem, but nevertheless one which is of great importance and has far-reaching implications. The aim is to deduce global properties of a complex projective variety admitting an action by an algebraic torus from the local behaviour at the fixed points of the action. Classical examples of such results are the Atiyah-Bott fixed point formula, the Bott residue formula and equivariant cohomology and K-theory. Recent applications include the computation of Gromov-Witten invariants and mirror symmetry for the general quintic 3-fold (work of Kontsevich, Givental and others). Following an introductory chapter, Chapter 2 contains some general comments on varieties with torus actions, including the important topic of toric varieties. Chapter 3 covers torus actions in Lie theory and in particular the closures of nilpotent orbits (a topic taken up in more detail in McGovern's article). In Chapter 4, the author turns first to the Białynicki-Birula decomposition of a smooth projective variety $X$ under an action of the multiplicative group $G_m$. This splits such a variety into a union of "plus" cells (or similarly a union of "minus" cells). For each component $X_i$ of the fixed point set, there corresponds a cell in each of the decompositions. If one denotes by $T_i^+$ (respectively, $T_I^-$) the subbundle of the restriction to $X_i$ of the tangent bundle of $X$ on which $G_m$ acts with positive (respectively, negative) weights, then the "plus" cell is isomorphic to the total space of $T_i^+$ and the "minus" cell is isomorphic to the total space of $T_i^-$. The next section covers applications of the Białynicki-Birula decomposition theorem to homology, including the homology of Schubert varieties, minimal nilpotent orbits, Hilbert schemes, toric varieties and complete symmetric varieties. The final section of Chapter 4 considers the possible generalisation of the Bialinicki-Birula decomposition to varieties which may be singular or not admit a torus action. In Chapter 5, the author considers the more subtle problem of computing the cohomology ring of $X$ from fixed point data. He begins with some examples and then derives the Bott residue formula, before returning to a result which determines the cohomology ring of a projective variety with a regular action of a Borel subgroup of $SL(2, \mathbf{C})$. The final chapter covers the cohomology of invariant subvarieties of $X$, equivariant cohomology and rational smoothness. The article contains throughout many examples and has a comprehensive bibliography.

Finally, McGovern's article is a detailed survey of the action of a semisimple Lie or algebraic group on its Lie algebra by the adjoint representation and on itself by the adjoint action. He begins by stating the basic facts about the structure of semisimple Lie algebras and semisimple Lie groups and about orbits, classes and centralizers. The study of nilpotent orbits starts in Chapter 3 with the statement and proof of the finiteness theorem, which states that there are only finitely many nilpotent orbits in any semisimple Lie algebra or algebraic group. This is proved first in the classical case over an algebraically closed field and then extended to arbitrary Lie algebras in the algebraically closed case and finally to real Lie algebras. Chapter 4 introduces the principal nilpotent orbit (the unique nilpotent orbit of maximal dimension whose closure is the entire nilpotent variety). In Chapter 5 the author describes a construction of Lusztig and Spaltenstein which produces a uniquely determined nilpotent orbit in a reductive Lie algebra starting from a nilpotent orbit in a Levi subalgebra; several applications are given, including the Bala-Carter classification of nilpotent orbits. The chapter finishes with a table listing the nilpotent orbits for the exceptional Lie algebras. In Chapter 6 the closures of nilpotent orbits are studied, together with the partial ordering induced by inclusion of closures; again detailed tables are included. Chapter 7 is concerned with the nilpotent variety and the flag variety, while Chapter 8 discusses Springer's Weyl group representations. The article concludes with a brief survey of recent work and an extensive bibliography.

These three articles are all of value, but have somewhat different natures. Białynicki-Birula's is a survey of a very wide area of research and requires much prior knowledge on the part of the reader. It is certainly not something one could suggest as reading for a starting graduate student, but it is a useful reference for those who already have some knowledge and want to know about some aspect of the theory of quotients. The other two articles are much more accessible, especially that of McGovern, but could also be used as a source of general reference in the (more limited) areas they study.

Liverpool P. Newstead




V. Turaev
**Torsions of 3-dimensional Manifolds**
Progr. Math. 208

Basel u. a., Birkhäuser Verlag, 2003, 206 S., € 87,74

In der dreidimensionalen Topologie gibt es verschiedene zentrale Fragestellungen. Eine von ihnen beschäftigt sich mit Invarianten, die sich aus kombinatorischen Darstellungen von 3-Mannigfaltigkeiten ableiten lassen, zum Beispiel aus CW-Strukturen oder Darstellungen durch Chirurgie an Knoten. In diesem Buch geht es um eine solche Invariante, die maximal-abelsche Torsion.

Um zu erklären, was diese maximal-abelsche Torsion ist, betrachtet man zunächst die Reidemeister-Torsion. Reidemeister versuchte in [3], eine CW-Darstellung einer dreidimensionalen Mannigfaltigkeit $M$ durch geeignete Transformationen in eine möglichst einfache Form zu bringen; das geht bis zum 2-Skelett gut. Bei der Wahl der top-dimensionalen Zelle stößt man auf ein Hindernis im Gruppenring der Fundamentalgruppe $\pi_1(M)$, der Torsion. Reidemeister fand heraus, dass diese neue Invariante verschiedene homotopie-äquivalente Linsenräume voneinander unterscheiden kann, dazu benutzte er Darstellungen von $\pi_1(M)$, um von Elementen des Gruppenringes zu Zahlen zu gelangen. Turaev fasst diese Zahlen zu allen eindimensionalen Darstellungen von $\pi_1(M)$ zusammen zur sogenannten maximal-abelschen Torsion von $M$.

Die Reidemeister-Torsion ist leider keine eindeutige Invariante, sondern hängt von Wahlen ab. Man benötigt eine Basis des vorliegenden CW-Komplexes und muss für jede

Zelle einen Lift in der universellen Überlagerung festlegen. Außerdem benötigt man eine Basis der Kohomologie. Bei der Konstruktion der maximal-abelschen Torsion einer orientierten, geschlossenen Mannigfaltigkeit $M$ sind diese Wahlen äquivalent zur Festlegung einer sogenannten Euler-Struktur $e$ auf $M$. Eine solche Euler-Struktur lässt sich etwa als eine Klasse von Vektorfeldern ohne Nullstellen darstellen. Interessanterweise entsprechen die Euler-Strukturen einer 3-Mannigfaltigkeit $M$ genau den Spin$^c$-Strukturen auf $M$. Etwas allgemeiner werden in dem Buch auch kompakte 3-Mannigfaltigkeiten betrachtet, deren Rand aus einer Vereinigung von Tori besteht.

Sei $\tau(M,e)$ die maximal-abelsche Torsion einer geschlossenen, orientierten 3-Mannigfaltigkeit $M$ mit Euler-Struktur $e$. Aus $\tau(M,e)$ lassen sich andere Invarianten von 3-Mannigfaltigkeiten ableiten. Beispielsweise bestimmt $\tau$ das erste elementare Ideal der Fundamentalgruppe, und damit auch ihr Alexander-Polynom. Stellt man $M$ dar durch Chirurgie an einem Link in der Sphäre $S^3$, so kann man nach Milnor aus der maximal-abelschen Torsion entsprechend das Alexander-Polynom des Links ablesen.

Die Thurston-Norm einer 3-Mannigfaltigkeit ist eine Seminorm $\|\cdot\|_T$ auf $H^1(M)$, die eine untere Schranke für die Eulerzahl einer zu $s \in H^1(M)$ dualen Fläche in $M$ angibt. Turaev definiert mit Hilfe von $\tau(M,e)$ eine Torsions-Seminorm $\|\cdot\|_\tau$ auf $H^1(M)$ mit $\|\cdot\|_T \geq \|\cdot\|_\tau$ und erhält so untere Abschätzungen für Eulerzahlen von Flächen in $M$.

Die abelsche Torsion nimmt Werte im Quotientenring $Q(H)$ des Gruppenrings $\mathbb{Z}[H]$ der ersten Homologiegruppe $H = H_1(M)$ an. Dieser Ring lässt sich mit Hilfe des Augmentationsideals $I \subset \mathbb{Z}[H]$ filtrieren. Der führende Term von $\tau(M,e)$ hängt nicht von $e$ ab und lässt sich mit Hilfe des Cup-Produktes auf $H^1(M)$, oder falls dieses verschwindet, mit Hilfe des ersten nichttrivialen Massey-Produktes beschreiben. Ähnliche Beziehungen gelten auch in der Kohomologie modulo $r \in \mathbb{Z}$.

Turaev beschreibt darüberhinaus das Verhalten der Torsion unter Chirurgie an Knoten in $M$. Nimmt man all diese Informationen zusammen, so erhält man ein Axiomensystem, dass $\tau(M,e)$ zumindest für Mannigfaltigkeiten mit $b_1(M) \geq 1$ bis auf das Vorzeichen festlegt.

Die ursprünglich für $\mathbb{Q}$-Homologie-Sphären definierte Casson-Invariante besitzt eine Verallgemeinerung $\lambda(M)$ für beliebige 3-Mannigfaltigkeiten. Falls die erste Betti-Zahl $b_1(M)$ nicht verschwindet, lässt sich $\lambda(M)$ ebenfalls aus $\tau(M,e)$ berechnen. Im Falle $b_1(M) = 0$ sind maximal abelsche Torsion und Casson-Invariante voneinander unabhängig, und man fasst beide Invarianten zur modifizierten Torsion $\tau'(M,e) = \tau(M,e) + \lambda(M)/|H|$ zusammen.

Eines der tiefsten Resultate im vorliegenden Buch ist die Äquivalenz der maximal-abelschen Torsionen von $M$ auf der einen und der Seiberg-Witten-Invarianten $\mathrm{SW}_M$ für alle Spin$^c$-Strukturen auf der anderen Seite im Falle $b_1(M) \geq 1$, die zuerst von Meng und Taubes in [1] gefunden wurde. Diese Beziehung ergibt sich daraus, dass $\tau(M,e)$ und $\mathrm{SW}_M(e)$ den gleichen Axiomen genügen. Nach Nicoalescu [2] existiert im Falle $b_1(M) = 0$ eine ähnliche Beziehung zwischen $\mathrm{SW}_M$ und $\tau'(M,e)$, die hier aber leider nur kurz erwähnt wird.

Das Buch wendet sich an Leser mit einem ausführlichen Hintergrund in kombinatorischer dreidimensionaler Topologie. Die Auswahl des Inhalts erfolgt im Hinblick auf mögliche Anwendungen in der dreidimensionalen Topologie, andere Aspekte der Reidemeister-Torsion wurden leider stark vernachlässigt. Beispielsweise werden Whitehead-Torsion, analytische Torsion und Zusammenhänge mit geschlossenen Orbits von Vektorfeldern nur am Rande erwähnt. Auch auf das Thurston-Programm wird nicht eingegangen. Interessant wäre hier eine Formel, die die Torsion einer Vereinigung geometrischer Mannigfaltigkeiten zu Daten der einzelnen Stücke und der Verklebungen in Beziehung setzt.

Eine gewisse Vertrautheit mit der Franz-Reidemeister-Torsion und ihren Verfeinerungen wird vorausgesetzt, und an vielen Stellen verweist Turaev auf seine Original-Artikel. Ansonsten ist die Darstellung technisch sehr präzise und nachvollziehbar. Das Buch eignet sich nicht unbedingt als Einführung in die behandelten Fragestellungen. Wir empfehlen, parallel das Buch [2] von Nicolaescu zu lesen, das das gleiche Thema aus einem informelleren Blickwinkel heraus betrachtet, und mehr Wert auf Beispiele und Anschaulichkeit legt.

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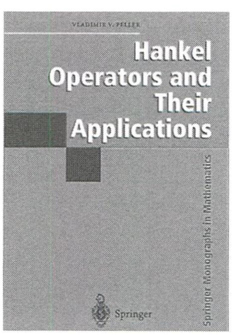
Regensburg S. Goette




V. Peller

**Hankel Operators and Their Applications**

Berlin u. a., Springer, 2003, 799 S., € 82,34

Hankeloperatoren treten in vielen Verkleidungen auf. In ihrer einfachsten Form begegnen sie uns als Operatoren, die im Folgenraum $\ell^2$ über den nichtnegativen ganzen Zahlen durch eine unendliche Matrix der Gestalt

$$(a_{j+k})_{j,k=0}^{\infty} = \begin{pmatrix} a_0 & a_1 & a_2 & \dots \\ a_1 & a_2 & \dots & \dots \\ a_2 & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \end{pmatrix}$$

gegeben sind. Die Determinanten der endlichen Abschnitte solcher Matrizen wurden von Hermann Hankel in seiner Dissertation (1861) betrachtet.

Die Frage, wann obige Matrix einen beschränkten Operator ergibt, wird durch einen berühmten Satz von Nehari (1957) beantwortet: dies ist genau dann der Fall, wenn eine Funktion $a$ in $L^\infty$ auf dem Einheitskreis existiert, deren Fourierkoeffizienten $a_n$ mit $n \geq 0$ gerade die erste Zeile der Matrix bilden. Man bezeichnet den Hankeloperator dann mit $H_a$. Sind $a$ und $b$ Funktionen aus $L^\infty$, die sich nur in Fourierkoeffizienten mit negativen Indizes unterscheiden, so ist natürlich $H_a = H_b$. Nehari hat außerdem gezeigt, dass die Norm von $H_a$ gleich dem Infimum von $\|b\|_\infty$ über alle $b$ mit $H_a = H_b$ ist.

Das Problem, wann ein Hankeloperator kompakt ist, wurde von Hartman (1958) gelöst. Er zeigte, dass $H_a$ genau dann kompakt ist, wenn es eine Funktion aus $C + H^\infty_-$ gibt, deren Fourierkoeffizienten $a_n$ $(n \geq 0)$ die Einträge der ersten Zeile sind. Hierbei ist $C$ die Menge der stetigen Funktionen auf dem Einheitskreis, und $H^\infty_-$ steht für die Funtionen aus $L^\infty$, deren Fourierkoeffizienten mit positiven Indizes alle verschwinden. Adamjan, Arov und Krein (1968) haben dann die Distanz eines beliebigen Hankeloperators zur Menge der kompakten Operatoren bestimmt, und Sarason (1967) hat wohl als erster erkannt, dass Hartmans Kriterium impliziert, dass $C + H^\infty_-$ eine abgeschlossene Unteralgebra von $L^\infty$ ist. Schließlich wusste bereits Kronecker (1881), dass $H_a$ genau dann ein Operator mit endlichdimensionalem Bildraum ist, wenn $a$ als rationale Funtion gewählt werden kann.

Damit sind im wesentlichen die Helden der klassischen Theorie der Hankeloperatoren genannt. Peller betrat die Bühne 1979 mit der Lösung eines damals seit 20 Jahren offenen Problems, nämlich des Problems, wann ein Hankeloperator zur Spurklasse gehört. Peller bewies, dass $H_a$ genau dann ein Spuroperator ist, wenn $a \in L^\infty$ aus dem Besovraum $B_1^1$ genommen werden kann. Dieses Resultat hat eine Renaissance des Interesses an Hankeloperatoren eingeleitet, und das vorliegende Werk ist eine Bilanz der Entwicklung auf dem Gebiet vom klassischen Stadium bis hin zu ihrer jüngsten Epoche während der letzten 25 Jahre.

Die Reichhaltigkeit der Theorie der Hankeloperatoren rührt von den eingangs erwähnten Verkleidungen dieser Operatoren her. Man kann zum Beispiel von $\ell^2$ zum Hardyraum $H^2$ von in der Einheitskreisscheibe analytischen Funktionen oder zum Raum $L^2$ auf $(0, \infty)$ übergehen. Dadurch erscheinen Hankeloperatoren in neuem Gewand, und es werden einerseits Eigenschaften sichtbar, die beim Arbeiten mit Fourierkoeffizienten im Dunkeln bleiben, und andererseits offenbart sich eine Fülle von Querverbindungen zu scheinbar gänzlich verschiedenen Welten. Dieser geschickte Wechsel des Kontexts durchzieht das gesamte Buch und macht einen großen Teil der Faszination der Thematik aus.

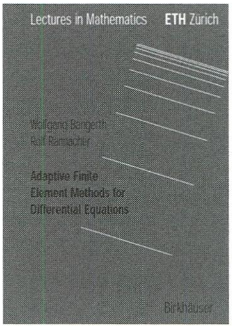
Das Buch ist inhaltlich außerordentlich breit gefächert und hält, was ein Opus von nahezu 800 Druckseiten verspricht. Peller beginnt mit der klassischen Theorie der Hankeloperatoren, diskutiert vektorielle Verallgemeinerungen (die von zahlreichen Anwendungen diktiert werden), geht ausführlich auf das fruchtbare Zusammenspiel von Toeplitz- und Hankeloperatoren ein, analysiert die Singulärwerte von Hankeloperatoren (Ideenkreis von Adamjan, Arov und Krein), beschreibt die Hankeloperatoren in den Schatten – von Neumann – Klassen $S_p$ und gibt schließlich eine sorgfältige Einführung in die Spektraltheorie von Hankeloperatoren. Zwei Kapitel des Buches beschäftigen sich mit der Anwendung von Hankeloperatoren auf stationäre Prozesse, und ein Kapitel ist der Rolle von Hankeloperatoren in der Kontrolltheorie gewidmet.

Verschiedene Probleme der Approximationstheorie sind im Buch allgegenwärtig. Zwei Beispiele mögen genügen, um den Zusammenhang zwischen Approximationsfragen und Hankeloperatoren zu illustrieren. Zum einen kann der oben zitierte Satz von Nehari so interpretiert werden, dass die Norm von $H_a$ gerade der $L^\infty$-Abstand von $a$ zur Menge aller beschränkten Funktionen ist, die sich analytisch in das Außengebiet des Einheitskreises fortsetzen lassen. Zum anderen gibt Pellers Kriterium für Hankeloperatoren in den Klassen $S_p$ Einsichten in das Verhalten der Singulärwerte her, die zu traumhaft scharfen Resultaten über die Güte der besten Approximation durch rationale Funktionen führen.

Das letzte Kapitel ist ein glanzvoller Höhepunkt: Peller zeigt, wie vektorielle Hankeloperatoren einen eleganten Zugang zu dem berühmten und erst 1997 von Pisier gelösten Problem liefern, ob ein polynomial beschränkter Operator in einen Hilbertraum zu einer Kontraktion ähnlich sein muss.

Die vorliegende Monographie ist konkurrenzlos. Es gibt zwei schöne dünne Bücher von Power und Partington über Hankeloperatoren, einige hervorragende Übersichtsartikel, sowie einführende Darstellungen gewisser Aspekte der Theorie der Hankeloperatoren in Büchern von N. Nikolskii und von Silbermann und dem Rezensenten. Peller hat nun ein Monumentalwerk geschaffen, das für lange Zeit die Standardreferenz werden wird. Wer immer sich mit Operatortheorie und deren vielfältigen Anwendungen beschäftigt, ist gut beraten, dieses Buch in greifbarer Nähe zu haben.

Chemnitz A. Böttcher




W. Bangerth,
R. Rannacher
**Adaptive Finite Element Methods for Differential Equations**
Lectures in Mathematics
ETH Zürich

Basel, Birkhäuser, 2003, 216 S., € 23,54

Gegenstand des Buches ist die *Dual Weighted Residual method* (DWR), ein sehr effizientes numerisches Verfahren zur Behandlung einer großen Klasse von variationell formulierten Differentialgleichungen. Das numerische Verfahren ist adaptiv, d. h. es konstruiert eigenständig eine Folge von Approximationen für eine gegebene Fragestellung. Typische Fragestellungen sind die Bestimmung gewichteter Mittelwerte der Lösung oder ihrer Ableitungen, die Bestimmung von Randintegralen über Lösungskomponenten (relevant z. B. für die Berechnung von strömungsmechanischen Kenngrößen) oder die Bestimmung von Spannungsintensitätsfaktoren (z. B. in der Bruchmechanik). Das Verfahren basiert auf Projektionsmethoden wie z. B. der Finiten Elemente Methode (FEM). Dort wird die Approximationsgüte durch die Wahl der Gitter gesteuert. Der Kern jeder adaptiven FEM ist deshalb die Art, wie die Gitter gewählt werden. Typischerweise geschieht dies in einer adaptiven Schleife, in der in mehreren Durchgängen schrittweise das Gitter verbessert wird, bis eine gewünschte Genauigkeit erreicht ist. Bei der DWR wird in jedem Schleifendurchgang ein lineares Hilfsproblem – das sog. duale Problem, welches von der vorliegenden Fragestellung abhängt – (näherungsweise) gelöst. Weiterhin wird, basierend auf dem aktuellen Gitter, eine Approximation der Lösung der Differentialgleichung bestimmt. Aus diesen nun vorliegenden Daten wird dann herausdestilliert, wo das Gitter verfeinert werden sollte bzw. vergröbert werden kann, um eine genauere Lösung zu erhalten. Ziel eines adaptiven Algorithmus ist, das gewünschte Ergebnis möglichst effizient zu bestimmen, d. h. mit möglichst geringem Bedarf an Resourcen (Rechenzeit, Speicherbedarf etc.). Mit zahlreichen Beispielen belegt das Buch, daß die DWR dieses Ziel erreicht. Es sei hier hervorgehoben, daß eine Kosten-Nutzen-Betrachtung für die DWR besonders bei nichtlinearen Problemen günstig ausfällt, da die Kosten für die Lösung des linearen Hilfsproblems vergleichbar mit denen eines Newtonschrittes sind und somit nur einen kleinen Teil der Gesamtkosten ausmachen.

Das Buch gibt einen sehr guten Überblick über die Technik und die Möglichkeiten der DWR. In einleitenden Kapiteln wird die DWR an gewöhnlichen Differentialgleichungen und dann an einfachen linearen, elliptischen partiellen Differentialgleichungen sehr klar und verständlich vorgeführt. Anschließend wird die DWR in einem abstrakten funktionalanalytischen Rahmen vorgestellt. Der Rest des Buches illustriert auf eindrucksvolle Weise die Leistungsfähigkeit und Breite der Anwendungsfähigkeit des Konzeptes an Hand von Fallbeispielen: Es werden Eigenwertprobleme, Optimierungsaufgaben mit Zwangsbedingungen, die durch eine partielle Differentialgleichung gegeben sind, Strukturmechanikprobleme (lineare Elastizität, Plastizität), Strömungsmechanik (hydrodynamische Stabilitätsanalyse, Berechnung von Strömungskennwerten) behandelt. Auch zeitabhängige Probleme wie die Lösung der Wellengleichung werden mit der DWR erfolgreich bearbeitet. Insgesamt wird klar ersichtlich, daß die DWR eine sehr flexible und vielseitig anwendbare Technik ist. Die ausgewählten numerischen Beispiele, die vor allem aus umfangreichen numerischen Untersuchungen der Gruppe von Rolf Rannacher aus den letzten 10 Jahren ausgewählt wurden, sind sehr illustrativ. Die Erläuterungen zu den Beispielen sind auch deshalb interessant, weil eine Menge zusätzlicher Informationen über die numerische Behandlung des vorliegenden Problems quasi nebenbei einfließen.

Das Buch entstand aus einer fortgeschrittenen Spezialvorlesung, die an der ETH Zürich gehalten wurde. Einen Lehrbuchcharakter erhält das Buch dadurch, daß Übungsaufgaben (mit detailierten Lösungen im Anhang) jedes Kapitel abschließen. Die Aufgaben enthalten zahlreiche praktische Übungen, die mit der Softwarebibliothek Deal II zu lösen sind. Sowohl die Bibliothek als auch die Lösungen werden im Internet bereitgestellt. Das Buches ist jedoch kein Lehrbuch für „Anfänger" im Bereich der FEM. Notationen folgen den auf dem Gebiet der FEM üblichen Konventionen und werden deshalb nur knapp erklärt. Hin und wieder werden fortgeschrittene mathematische Aussagen ohne Referenz gemacht. Bei den anspruchsvolleren Beispielen im hinteren Teil des Buches wird zugunsten eines klaren Herausarbeitens der Spezifika der DWR auf eine detailierte Diskussion zahlreicher Diskretisierungsaspekte wie z. B. Stabilisierung bei Strömungsproblemen verzichtet. Obwohl auf Literatur verwiesen wird, sind für ein vertieftes Verständnis dieser Beispiele gute Kenntnisse der FEM-Literatur notwendig.

Insgesamt stellt das Buch für Forscher einen sehr guten Überblick über die Leistungsfähigkeit der DWR dar. Auch für eine Spezialvorlesung bietet das Buch dem Vortragenden reichlich Material zur Auswahl.

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